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SUM CONNECTIVITY MATRIX AND ENERGY OF A T_2 HYPERGRAPH

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Abstract: Let H be a T_2 hypergraph with $n \geq 4$. The sum connectivity matrix of H, denoted by $SC(H)$ is defined as the square martix of order n, whose $(i,j)^{th}$ entry is $\frac{1}{\sqrt{d_i+d_j}}$ if x_i and x_j are adjacent and zero for other cases. The sum connectivity energy $SCE(H)$ of H is the sum of the absolute values of the eigenvalues of $SC(H)$. It is shown that, for a T_2 hypergraph $\lfloor SCE(H) \rfloor \leq \left\lfloor 1 + n - \sqrt{\frac{n}{\delta}} \right\rfloor$,where δ is the minimum degree of H.

Keywords and Phrases: T_2 hypergraph, sum connectivity matrix, sum connectivity energy.

2020 Mathematics Subject Classification: 05C65, 05C50.

1. Introduction

The basic definitions and terminologies of a hypergraph are not given here and we refer to it [1] and [5]. The concept of hypergraph was introduced by Berge in 1967. In 2017, Seena V and Raji Pilakkat introduced Hausdorff hypergraph,

T_0 hypergraph and T_1 hypergraph [2] and [3]. Based on [2] and [3] S. Sujitha and D. Sharmila introduce T_2 hypergraph and studied Adjacency matrix, Randic matrix, Zagreb matrix and its corresponding energies [4]. In 2010, Bo zhou and Nenad Trinajstic studied the sum connectivity energy of a graph [6] and later the same concept was studied by many authors. In this article, we determine the sum connectivity matrix and sum connectivity energy of a T_2 hypergraph. Throughout this article, H is a connected T_2 hypergraph with order n and size m, where the order and size are the minimum number of vertices and edges needed to define a T_2 hypergraph. The degree of a vertex $x \in X$ denoted by $d(x)$ is the number of edges that contain the vertex x. The maximum degree of the hypergraph H is denoted by $\Delta(H)$ or Δ. The minimum degree of the hypergraph H is denoted by $\delta(H)$ or δ. The following definitions and theorems are used in sequel.

Definition 1.1. [4] *A hypergraph $H = (X, D)$ is said to be a T_2 hypergraph if for any three distinct vertices u, v and w in X, there exist a hyperedge containing u and v but not w and another hyperedge containing w but not u and v.*

Example 1.2.

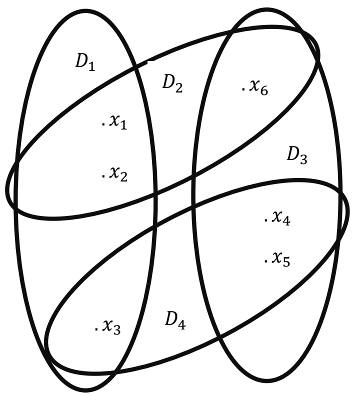

Figure 1: T_2 Hypergraph

Figure 1 is a T_2 Hypergraph with vertices $x_1, x_2, x_3, x_4, x_5, x_6$ and hyperedges D_1, D_2, D_3, D_4. It is easily seen that, for every three vertices x_i, x_j and x_k there exist a hyperedge containing x_i and x_j but not x_k and a hyperedge containing x_k but not x_i and x_j.

Result 1.3. [4]

(i) The minimum number of edges needed to define a T_2 hypergraph is $\left[\frac{2n+5}{4}\right]$

where n is the number of vertices.

(ii) For a T_2 hypergraph H, the minimum degree $\delta = \delta(H) = 2$.

(iii) For a T_2 hypergraph H, rank $r(H) = \left[\frac{2n+1}{4}\right]$ where $n \geq 5$. Here $r(H)$ is the largest cardinality of its edges.

Definition 1.4. [6] *The sum connectivity matrix is defined by*
$$SC(H) = \begin{cases} \frac{1}{\sqrt{d_i+d_j}} & if\ x_ix_j \in D \\ 0 & otherwise \end{cases}$$
where d_i and d_j are degrees of the vertices x_i and x_j.

Definition 1.5. [6] *The sum connectivity energy is defined by $SCE(H) = \sum\limits_{i=1}^{n} |\lambda_i|$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are the sum connectivity eigen values of H.*

2. Sum connectivity matrix and energy of a T_2 Hypergraph

In this section, we find the energy of a T_2 hypergraph using sum connectivity matrix.

Example 2.1. Consider a T_2 hypergraph given in Figure 2 with 12 vertices and 7 edges.

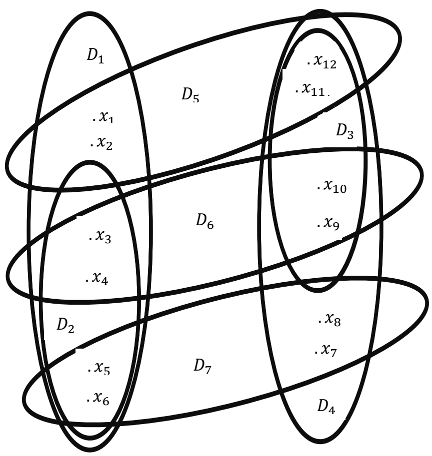

Figure 2: T_2 Hypergraph

Sum connectivity matrix of H is given by

$$SC(H) = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{4}} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

The sum connectivity eigen values of the above T_2 hypergraph is

$$\begin{pmatrix} 3.0583 & 1.3167 & 0.4483 & .4212 & -0.408 & -.5 & -1.2649 & -1.3475 \\ 1 & 1 & 1 & 1 & 4 & 2 & 1 & 1 \end{pmatrix}$$

Therefore, the sum connectivity energy $SCE(H) = \sum_{i=1}^{n} |\lambda_i| = 10.4889$

$\lfloor SCE(H) \rfloor = \lfloor 1 + n - \sqrt{\frac{n}{\delta}} \rfloor = 10$

The below table presents the sum connectivity energy of a T_2 hypergraph in relation with order.

Vertices	$SCE(H)$	$\lfloor 1 + n - \sqrt{\frac{n}{\delta}} \rfloor$
4	2.5	3
5	3.17	4
6	4.35	5
7	5.43	6
8	6.53	7
9	7.47	7
10	8.14	8
11	9.13	9
12	10.49	10
13	11.41	11
14	12.58	13
15	13.98	13

16	14.15	14
17	14.5	15
18	15.45	16
19	16.38	16
20	16.16	17
...	...	...
n	...	$\lfloor 1+n-\sqrt{\frac{n}{\delta}} \rfloor$

Table 1: Sum connectivity energy of a T_2 hypergraph

Result 2.2. Let H be a T_2 hypergraph with $n \geq 4$. Then $\left\lfloor \sum_{i=1}^{n} \lambda_i^2 \right\rfloor \leq \delta n - \delta^2$. Equality holds only if $n = 14$ and 18 in H.

Proof. From the below Table 2, we can see that $\left\lfloor \sum_{i=1}^{n} \lambda_i^2 \right\rfloor \leq \delta n - \delta^2$.

Vertices	$\sum_{i=1}^{n} \lambda_i^2$	$\left\lfloor \sum_{i=1}^{n} \lambda_i^2 \right\rfloor$	$\delta n - \delta^2$
4	3	3	4
5	3.5	3	6
6	4.9	4	8
7	6.1	6	10
8	8.73	8	12
9	10.4	10	14
10	13.2	13	16
11	13.03	13	18
12	16.03	16	20
13	18.39	18	22
14	24.04	24	24
15	24.5	24	26
16	26.17	26	28
17	28.32	28	30
18	32.32	32	32
19	31.6	31	34
20	35.74	35	36
n	...	...	...

Table 2: Value of $\delta n - \delta^2$

3. Bounds of the sum connectivity energy of a T_2 Hypergraph

In this section, we discover the upper and lower bounds of the T_2 hypergraph using sum connectivity matrix.

Result 3.1. Let H be a T_2 hypergraph with $n \geq 4$. Then $\lfloor \lambda_1 \rfloor \leq \left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil$ where λ_1 is the largest eigen value of H. Equality holds only if $n = 10$ in H.

Observation 3.2. Let H be a T_2 hypergraph with $n \geq 4$. Then $\lceil \lambda_1 \rceil = \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil$ where λ_1 is the largest eigen value of H. We can easily observe that, when n=10, $\lfloor \lambda_1 \rfloor = \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil$ and when n=16, $\lceil \lambda_1 \rceil = \left\lfloor \frac{n}{\sqrt{7\delta+1}} \right\rfloor$

Theorem 3.3. *Let H be a T_2 hypergraph with $n \geq 4$, $n \neq 10$ and 16. Then*
$SCE(H) > \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil - 1 - (n-\delta)\frac{(detSC(H))^{\frac{1}{n-\delta}}}{\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil}$

Proof. From the Cauchy - Schwarz inequality,

$$\sum_{i=2}^{n-1} \sqrt{\lambda_i} \leq \sqrt{(\sum_{i=2}^{n-1} \lambda_i)(n-2)}$$

$$\sum_{i=2}^{n-1} \sqrt{\lambda_i} \leq \sqrt{(SCE(H) - \lambda_1 - \lambda_n)(n-2)}$$

$$< \sqrt{(SCE(H) - \lceil \lambda_1 \rceil - \lceil \lambda_n \rceil)(n-2)}$$

$$\sqrt{(SCE(H) - \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil + 1} \geq \frac{\sum_{i=2}^{n-1} \sqrt{\lambda_i}}{\sqrt{n-\delta}}$$

$$\sqrt{(SCE(H) - \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil + 1} > \frac{(n-\delta)(\sqrt{\lambda_2\lambda_3 \dots \lambda_{n-1}})^{\frac{1}{n-2}}}{\sqrt{n-\delta}}$$

$$\text{SCE(H)} > \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil - 1 - (n-\delta)\frac{(detSC(H))^{\frac{1}{n-2}}}{\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil}$$

Illustration 3.4. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, and $\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil = 4$. Here, $SCE(H)$=10.4889 $> \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil - 1 - (n-\delta)\frac{(detSC(H))^{\frac{1}{n-2}}}{\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil} =$ $4 - 1 - 10 \times \frac{0.6242}{4} = 1.4395$.

Theorem 3.5. *Let H be a T_2 hypergraph with $n \geq 4$. Then*
$\sqrt{\delta(n-\delta)} < SCE(H) < \sqrt{n\delta(n-\delta)}$.

Proof. From the Cauchy - Schwarz inequality,

$$(\sum_{i=2}^{n} |\lambda_i|)^2 \leq n \sum_{i=2}^{n} |\lambda_i|^2 < n \left\lfloor \sum_{i=2}^{n} |\lambda_i|^2 \right\rfloor < n\delta(n-\delta)$$

$$SCE(H) < \sqrt{n\delta(n-\delta)}$$

$$(SCE(H))^2 = (\sum_{i=2}^{n} |\lambda_i|)^2 > \sum_{i=2}^{n} |\lambda_i|^2 > \left\lfloor \sum_{i=2}^{n} |\lambda_i|^2 \right\rfloor < n\delta(n-\delta).$$

Illustration 3.6. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, $\delta n - \delta^2 = 20$. Here, $\sqrt{20} = 4.472 < SCE(H) = 10.4889 < \sqrt{240} = 15.4919$.

Theorem 3.7. *Let H be a T_2 hypergraph with $n \geq 4, n \neq 5$ and 6. Then* $n(detSC(H))^{\frac{1}{n}} < SCE(H) < \frac{n(\delta n-\delta^2)^2}{(detSC(H))^{\frac{1}{n}}}$.

Proof. From an arithmetic and a geometric mean inequality,

$$\frac{(\sum_{i=1}^{n} |\lambda_i|)}{n} \geq (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}} = (detSC(H))^{\frac{1}{n}}$$

$$SCE(H) > n(detSC(H))^{\frac{1}{n}}$$

We have $|\lambda_1| > (detSC(H))^{\frac{1}{n}}$

$$|\lambda_1| \sum_{i=1}^{n} |\lambda_i| > (detSC(H))^{\frac{1}{n}} \sum_{i=1}^{n} |\lambda_i|$$

Since $|\lambda_i| < |\lambda_1| \forall i$

$$n |\lambda_1|^2 > (detSC(H))^{\frac{1}{n}} SCE(H)$$

$$SCE(H) < \frac{n|\lambda_1|^2}{(detSC(H))^{\frac{1}{n}}} < \frac{n\left\lfloor \sum_{i=1}^{n} \lambda_i^2 \right\rfloor^2}{(detSC(H))^{\frac{1}{n}}}$$

$$SCE(H) < \frac{n(\delta n-\delta^2)^2}{(detSC(H))^{\frac{1}{n}}}$$

$$n(detSC(H))^{\frac{1}{n}} < SCE(H) < \frac{n(\delta n-\delta^2)^2}{(detSC(H))^{\frac{1}{n}}}.$$

Illustration 3.8. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, $ndet(SC(H))^{\frac{1}{n}} = 12 \times 0.6752 = 8.1024 < SCE(H) = 10.4889 < \frac{n(\delta n-\delta^2)}{detSC(H)^{\frac{1}{n}}} = 355.45$.

Theorem 3.9. *Let H be a T_2 hypergraph with $n \geq 4$. Then* $SCE(H) < \sqrt{n} + \frac{13}{5} + (n-\delta)\sqrt{\delta}$.

Proof. From the Cauchy Schwarz inequality,

$$\sum_{i=3}^{n} \lambda_i \leq \sqrt{(\sum_{i=3}^{n} \lambda_i^2)(\sum_{i=3}^{n} 1)}$$

$$SCE(H)\text{-}(\lambda_1 + \lambda_2) \leq \sqrt{(n-2)[\sum_{i=1}^{n} \lambda_i^2 - \lambda_1^2 - \lambda_2^2]}$$

$$< \sqrt{(n-\delta)[\left\lfloor \sum_{i=1}^{n} \lambda_i^2 \right\rfloor - \lambda_1^2 - \lambda_2^2]}$$

$$SCE(H) < \sqrt{(n-\delta)[(\delta n-\delta^2) - \lambda_1^2 - \lambda_2^2]} + \lambda_1 + \lambda_2$$

Since $\lambda_1 + \lambda_2 \leq \sqrt{n} + \frac{13}{5}$,

$SCE(H) < \sqrt{n} + \frac{13}{5} + \sqrt{(n-\delta)[(\delta n - \delta^2) - \lambda_1^2 - \lambda_2^2]}$
Let h(s,t)=$\sqrt{n} + \frac{13}{5} + \sqrt{(n-\delta)[(\delta n - \delta^2) - s^2 - t^2]}$
Differentiate Partially with respect to s and t,
$h_s = \frac{-s\sqrt{n-\delta}}{\sqrt{(\delta n-\delta^2)-s^2-t^2}}$
$h_t = \frac{-t\sqrt{n-\delta}}{\sqrt{(\delta n-\delta^2)-s^2-t^2}}$
Stationary points are given by $h_s = 0$ and $h_t = 0$
$h_s = 0 \Rightarrow \frac{-s\sqrt{n-\delta}}{\sqrt{(\delta n-\delta^2)-s^2-t^2}} = 0 \Rightarrow s = 0$
$h_t = 0 \Rightarrow \frac{-t\sqrt{n-\delta}}{\sqrt{(\delta n-\delta^2)-s^2-t^2}} = 0 \Rightarrow t = 0$
$h_{ss} = -\frac{\sqrt{n-\delta}(\delta n-\delta^2-t^2)}{(\delta n-\delta^2-s^2-t^2)^{\frac{3}{2}}}$
$h_{tt} = -\frac{\sqrt{n-\delta}(\delta n-\delta^2-s^2)}{(\delta n-\delta^2-s^2-t^2)^{\frac{3}{2}}}$
$h_{st} = -\frac{\sqrt{n-\delta}(\delta n-\delta^2-t^2)}{(\delta n-\delta^2-s^2-t^2)^{\frac{3}{2}}}$
At(0,0), $h_{ss} = h_{tt} = -\frac{1}{\sqrt{\delta}} < 0, h_{st} = 0$
Also, $h_{ss}h_{tt} - (h_{st})^2 > 0$
Therefore,h(0,0)= $\sqrt{n} + \frac{13}{5} + (n-\delta)\sqrt{\delta}$
Hence, $SCE(H) < \sqrt{n} + \frac{13}{5} + (n-\delta)\sqrt{\delta}$.

Illustration 3.10. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, Here, $SCE(H) = 10.4889 < \sqrt{n} + \frac{13}{5} + (n-\delta)\sqrt{\delta} = \sqrt{12} + 2.6 + 10\sqrt{2} = 20.2062$.

Theorem 3.11. *Let H be a T_2 hypergraph with $n \geq 4$. Then*
$SCE(H) < \left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil + \frac{(n-1)(\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil)^2}{(detSC(H))^{\frac{1}{n}}}$.

Proof. We have $\lfloor \lambda_1 \rfloor \leq \left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil$
$\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil \geq \lfloor \lambda_1 \rfloor > [detSC(H)]^{\frac{1}{n}}$
$\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil \sum_{i=2}^{n} |\lambda_i| > [detSC(H)]^{\frac{1}{n}} \sum_{i=2}^{n} |\lambda_i|$
$\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil > |\lambda_i| \, \forall i = 2, 3, ...n$
$(n-1)(\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil)^2 > [detSC(H)]^{\frac{1}{n}}(SCE(H) - \lambda_1)$
$\frac{(n-1)(\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil)^2}{(detSC(H))^{\frac{1}{n}}} > (SCE(H) - \lambda_1) > (SCE(H) - \lfloor \lambda_1 \rfloor)$
SCE(H) $< \left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil + \frac{(n-1)(\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil)^2}{(detSC(H))^{\frac{1}{n}}}$.

Illustration 3.12. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$

and $\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil = 4$. Clearly, $SCE(H) = 10.4889 < \left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil + \frac{(n-1)(\left\lceil \frac{n}{\sqrt{7\delta}} \right\rceil)^2}{(detSC(H))^{\frac{1}{n}}} = 4 + \frac{11\times 4^2}{.6752} =$ 264.6635.

Theorem 3.13. *Let H be a T_2 hypergraph with $n \geq 4, n \neq 5$ and 6. Then* $SCE(H) < \frac{n\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil^{\delta}}{(detSC(H)^{\frac{1}{n}})}$.

Proof. From an arithmetic and a geometric mean inequality,

$\frac{(\sum\limits_{i=1}^{n} |\lambda_i|)}{n} \geq (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}} = detSC(H)^{\frac{1}{n}}$

$\lceil \lambda_1 \rceil > |\lambda_1| > detSC(H)^{\frac{1}{n}}$

$\lceil \lambda_1 \rceil \sum\limits_{i=1}^{n} |\lambda_i| > detSC(H)^{\frac{1}{n}} \sum\limits_{i=1}^{n} |\lambda_i|$

$\lceil \lambda_1 \rceil \sum\limits_{i=1}^{n} |\lambda_i| = \lceil \lambda_1 \rceil [|\lambda_1| + |\lambda_2| + ... |\lambda_n|] > n \lceil \lambda_1 \rceil^2 = n \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil^{\delta}$

$n \left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil^{\delta} > (detSC(H)^{\frac{1}{n}}) SCE(H)$

SCE(H) $< \frac{n\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil^{\delta}}{(detSC(H)^{\frac{1}{n}})}$.

Illustration 3.14. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, and $\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil = 4, detSC(H)^{\frac{1}{n}}$=.6752, Hence $SCE(H) = 10.4889 < \frac{n\left\lceil \frac{n}{\sqrt{7\delta+1}} \right\rceil^{\delta}}{(detSC(H)^{\frac{1}{n}})} = \frac{12\times 4^2}{.6752} = 284.3602$.

Theorem 3.15. *Let H be a T_2 hypergraph with $n \geq 4, n \neq 5$ and 6. Then* $SCE(H) < \sqrt{\delta n - \delta^2} + \frac{(n-1)(\delta n - \delta^2)}{(detSC(H))^{\frac{1}{n}}}$.

Proof. We have $|\lambda_1| \geq |detSC(H)|^{\frac{1}{n}}$

$|\lambda_1| \sum\limits_{i=2}^{n} |\lambda_i| > |detSC(H)|^{\frac{1}{n}} \sum\limits_{i=2}^{n} |\lambda_i|$

since $|\lambda_i| < |\lambda_1| \forall i$

$(n-1)|\lambda_1|^2 > |detSC(H)|^{\frac{1}{n}} [SCE(H) - |\lambda_1|]$

Let$|\lambda_1| = s$ and $S(s) = s + \frac{(n-1)s^2}{|detSC(H)|^{\frac{1}{n}}}$ where s=$|\lambda_1|$

$S'(s) = 0 \Rightarrow 1 + \frac{2s(n-1)}{|detSC(H)|^{\frac{1}{n}}} = 0 \Rightarrow s = -\frac{|detSC(H)|^{\frac{1}{n}}}{2(n-1)}$ and $S''(s) = \frac{2(n-1)}{|detSC(H)|^{\frac{1}{n}}} > 0$

minimum value=S(s)=$S(-\frac{|detSC(H)|^{\frac{1}{n}}}{2(n-1)}) = -\frac{|detSC(H)|^{\frac{1}{n}}}{4(n-1)}$

S(s) is increasing in $-\frac{|detSC(H)|^{\frac{1}{n}}}{2(n-1)} < s < \sqrt{B} < \sqrt{\lfloor B \rfloor} = \sqrt{\delta n - \delta^2}$

where B=$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{d_i+d_j}$,$S(s) < S(\sqrt{\delta n-\delta^2})$

Hence SCE(H)< $\sqrt{\delta n-\delta^2}+\frac{(n-1)(\delta n-\delta^2)}{(detSC(H))^{\frac{1}{n}}}$.

Illustration 3.16. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, n $det(SC(H))^{\frac{1}{n}} = 12\times 0.6752 = 8.1024$. Clearly $SCE(H) = 10.4889 < \sqrt{\delta n-\delta^2}+\frac{(n-1)(\delta n-\delta^2)}{(detSC(H))^{\frac{1}{n}}} = 4.4721 + 325.8294 = 330.301$.

Theorem 3.17. *Let H be a T_2 hypergraph with $n \geq 4$. Then*
$A(H) > \frac{n-1}{n-\delta}(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+\delta\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil$, *where* $A(H)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{d_i+d_j}}$.

Proof. From the Cauchy - Schwarz inequality,

$$(\sum_{i=2}^{n-1}\lambda_i)^2 \leq (\sum_{i=2}^{n-1}1)(\sum_{i=2}^{n-1}\lambda_i^2)$$

$$(-\lambda_1-\lambda_n)^2 \leq (n-2)(\sum_{i=1}^{n}\lambda_i^2-\lambda_1^2-\lambda_n^2) < (n-\delta)(A(H)-\lambda_1^2-\lambda_n^2)$$

$$(\lambda_1+\lambda_n)^2 < (\lceil\lambda_1\rceil+\lceil\lambda_n\rceil)^2 < (n-\delta)(A(H)-\lceil\lambda_1\rceil^2-\lceil\lambda_n\rceil^2)$$

$$(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta} < (n-\delta)[A(H)-\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1]$$

$$(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+(n-\delta)(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil+1) < (n-\delta)A(H)$$

$$(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+(n-\delta)[\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil+1-2\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil+2\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil] < (n-\delta)A(H)$$

$$(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+(n-\delta)[(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+\delta\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil] < (n-\delta)A(H)$$

$$< (n-1)(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+\delta(n-\delta)\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil < (n-\delta)A(H)$$

$$A(H) > \frac{n-1}{n-\delta}(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+\delta\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil.$$

Illustration 3.18. Consider a T_2 hypergraph with $n = 12$. $SCE(H) = 10.4889$, $\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil = 4$, and A(H)=36.5301.

Here, $A(H) = 36.5301 > \frac{n-1}{n-\delta}(\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil-1)^{\delta}+\delta\left\lceil\frac{n}{\sqrt{7\delta+1}}\right\rceil = \frac{11}{10}(4-1)^2+2\times 4 = 17.9$.

4. Conclusion

In this article, we established the sum connectivity matrix and its energy for the T_2 hypergraph. Also, we identified $n[detSC(H)]^{\frac{1}{n}} = 8.1024 < SCE(H) = 10.4889 < \sqrt{n\delta(n-\delta)} = 15.49$ gives the nearest upper and lower bounds of the sum connectivity energy of the T_2 hypergraph using the graph parameters δ and n.

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