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On Hypergeometric Proof of Certain Continued Fraction Results

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Abstract: In this paper, we provide hypergeometric proof of certain results and deduce a number of new and known results. This result is equivalent to Entry 12 of Chapter XVI of Ramanujan's second Notebook.
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1. Introduction, Notations and Definitions

Ramanujan's contribution to continued fractions associated with analytic functions is remarkable. His Notebooks contain a large number of beautiful results associated with hypergeometric functions (both, basic and ordinary) and continued fractions. Many of his continued fraction results can be provided with hypergeometric proof. In a recent publication Denis and Singh [2,3] provided hypergeometric proof of Entries 25 and 33 of Chapter XII of Ramanujan's [5] second Notebook, and also provided their basic analogues.

Motivated by the above results, we propose to provide hypergeometric proof of the following results.

$$\frac{[a^2q^3, b^2q^3; q^4]_\infty}{[a^2q, b^2q; q^4]_\infty} = \frac{1}{1-a^2q-}\frac{q(b^2-a^2q^2)}{1+q^2+}\frac{q(a^2-b^2q^2)}{(1-a^2q)(1+q^4)-}$$
$$\frac{q^5(b^2-a^2q^6)}{1+q^6+}\frac{q(a^2-b^2q^6)}{(1-a^2q)(1+q^8)-}\frac{q^9(b^2-a^2q^{10})}{1+q^{10}+...} \tag{1.1}$$

where

$$[\alpha, \beta; p]_\infty = [\alpha; p]_\infty [\beta; p]_\infty$$

and

$$(\alpha; q)_\infty = \prod_{r=0}^{\infty}(1-\alpha q^r), \quad |q| < 1.$$

Entry 12 of Chapter XVI of Ramanujan's second Notebook [5] mentions a different continued fraction representation for the left side of (1.1). The present result provides an equivalent representation for the infinite product on the left.

Before we proceed further, we introduce a basic hypergeometric function; For any number α and q ($|q| < 1$), real or complex we write

$$[\alpha]_n = (\alpha;q)_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)\dots(1-\alpha q^{n-1}) & \text{if } n > 0; \\ 1 & \text{if } n = 0. \end{cases} \tag{1.2}$$

Accordingly, we have

$$[\alpha;q]_{-n} = \frac{(-)^n q^{n(n+1)/2}}{\alpha^n [q/\alpha;q]_n}.$$

Also,

$$[a_1, a_2, ..., a_r; q]_n = [a_1;q]_n [a_2;q]_n ... [a_r;q]_n.$$

Now, we define a basic hypergeometric series,

$${}_r\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n} \left\{(-)^n q^{n(n-1)/2}\right\}^{1+s-r}, \tag{1.3}$$

where $0 < |q| < 1$ and $r < s + 1$.

We define a basic bilateral hypergeometric function as,

$${}_r\Psi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right] = \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[b_1, b_2, ..., b_s; q]_n} \left\{(-)^n q^{n(n-1)/2}\right\}^{s-r}, \tag{1.4}$$

where $|b_1 b_2 ... b_s / a_1 a_2 ... a_r| < |z| < 1$.

2. Proof of (1.1)

In a publication Singh [4] established the following result,

$$\frac{{}_3\Phi_2 \left[\begin{array}{l} a, b, c; q; de/abc \\ d, e \end{array} \right]}{{}_3\Phi_2 \left[\begin{array}{l} aq, b, c; q; de/abc \\ dq, e \end{array} \right]}$$

$$= 1 - \frac{(de/abc)(a-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{(1-e/aq)/(1-e)+}$$

$$\frac{(e/aq)(1-aq)(1-dq/b)(1-dq/c)/(1-e)(1-dq)(1-dq^2)}{1-}$$

$$\frac{(deq/abc)(a-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{(1-e/aq)/(1-eq)+}$$
$$\frac{(e/aq)(1-aq^2)(1-dq^2/b)(1-dq^2/c)/(1-eq)(1-dq^3)(1-dq^4)}{1-}$$
$$\frac{(deq^2/abc)(a-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{(1-e/aq)/(1-eq^2)+...} \tag{2.1}$$

Now, setting a=1 in (2.1) and then taking d=1 in it we get,

$$\frac{[e/b,e/c;q]_\infty}{[e,e/bc]_\infty}$$
$$=\frac{1}{1-}\frac{(e/bc)(1-b)(1-c)/(1-q)}{(1-e/q)+}$$
$$\frac{(e/q)(1-q)(1-q/b)(1-q/c)/(1-q)(1-q^2)}{1-}$$
$$\frac{(eq/bc)(1-q)(1-bq)(1-cq)/(1-q^2)(1-q^3)}{(1-e/q)+}$$
$$\frac{(e/q)(1-q^2)(1-q^2/b)(1-q^2/c)/(1-q^3)(1-q^4)}{1-}$$
$$\frac{(eq^2/bc)(1-q^2)(1-bq^2)(1-cq^2)/(1-q^4)(1-q^5)}{(1-e/q)+...} \tag{2.2}$$

The above can simplified to,

$$\frac{[e/b,e/c;q]_\infty}{[e,e/bc]_\infty}$$
$$=\frac{1}{(1-e/q)-}\frac{(e/bc)(1-b)(1-c)/(1-q)}{1+}$$
$$\frac{(e/q)(1-q/b)(1-q/c)/(1-q^2)}{(1-e/q)-}$$
$$\frac{(eq/bc)(1-q)(1-bq)(1-cq)/(1-q^2)(1-q^3)}{1+}$$
$$\frac{(e/q)(1-q^2)(1-q^2/b)(1-q^2/c)/(1-q^3)(1-q^4)}{(1-e/q)-}$$
$$\frac{(eq^2/bc)(1-q^2)(1-bq^2)(1-cq^2)/(1-q^4)(1-q^5)}{1+...} \tag{2.3}$$

Now, replacing q by q^4 and then replacing e, b and c by q^{x+n+5}, q^{2n+2} and q^2, respectively, in (2.3), we get,

$$\frac{[q^{x-n+3}, q^{x+n+3}; q^4]_\infty}{[q^{x+n+1}, q^{x-n+1}; q^4]_\infty} =$$

$$= \frac{1}{(1-q^{x+n+1})-} \frac{q^{x-n+1}(1-q^{2n+2})(1-q^2)/(1-q^4)}{1+}$$

$$\frac{q^{x+n+1}(1-q^{2-2n})(1-q^2)/(1-q^8)}{(1-q^{x+n+1})-}$$

$$\frac{q^{x-n+5}(1-q^4)(1-q^{2n+6})(1-q^6)/(1-q^8)(1-q^{12})}{1+}$$

$$\frac{q^{x+n+1}(1-q^8)(1-q^{6-2n})(1-q^6)/(1-q^{12})(1-q^{16})}{(1-q^{x+n+1})-} \tag{2.4}$$

Now, setting $q^{n+x} = a^2, q^{x-n} = b^2$ and $q^{2n} = a^2/b^2$ in (2.4), we get

$$\frac{[a^2q^3, b^2q^3; q^4]_\infty}{[a^2q, b^2q; q^4]_\infty} = \frac{1}{(1-a^2q)-} \frac{q(b^2-a^2q^2)/(1+q^2)}{1+}$$

$$\frac{q(a^2-b^2q^2)/(1+q^2)(1+q^4)}{(1-a^2q)-} \frac{q^5(b^2-a^2q^6)/(1+q^4)(1+q^6)}{1+}$$

$$\frac{q(a^2-b^2q^6)/(1+q^6)(1+q^8)}{(1-a^2q)-...} \tag{2.5}$$

which can be put in the form,

$$\frac{[a^2q^3, b^2q^3; q^4]_\infty}{[a^2q, b^2q; q^4]_\infty} = \frac{1}{(1-a^2q)-} \frac{q(b^2-a^2q^2)}{(1+q^2)+} \frac{q(a^2-b^2q^2)}{(1-a^2q)(1+q^4)-}$$

$$\frac{q^5(b^2-a^2q^6)}{(1+q^6)+} \frac{q(a^2-b^2q^6)}{(1-a^2q)(1+q^8)-...} \tag{2.6}$$

By an appeal to analytic continuation the result holds for general values of parameters. This proves (1.1)

In Chapter 16 of the second Notebook of Ramanujan [5], entry 12 states that,

$$\frac{[a^2q^3, b^2q^3; q^4]_\infty}{[a^2q, b^2q; q^4]_\infty} = \frac{1}{(1-ab)+} \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+}$$

$$\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)+}\frac{(a-bq^5)(b-aq^5)}{(1-ab)(1+q^6)+...} \tag{2.7}$$

The result (2.6) provides an equivalent continued fraction representation for the function on the left.

3. Special cases

In this section we shall discuss certain interesting special cases of our result (1.1).

Taking $a \to 0$ and $b \to 1$ in (1.1), we get the following known result (cf. Andrews and Berndt [1; p. 156]),

$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty} = \frac{1}{1-}\frac{q}{1+q^2-}\frac{q^3}{1+q^4-}\frac{q^5}{1+q^6-}\frac{q^7}{1+q^8-...} \tag{3.1}$$

Also, $a \to 1$ and $b \to 0$ in (1.1) leads to,

$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty} = \frac{1}{1-q+}\frac{q^3}{1+q^2+}\frac{q}{(1-q)(1+q^4)+}\frac{q^{11}}{1+q^6+}$$

$$\frac{q}{(1-q)(1+q^8)+}\frac{q^{19}}{1+q^{10}+}\frac{q}{(1-q)(1+q^{12})+...} \tag{3.2}$$

Next, if we take a=0 and $b=i$ and also $a=i$ and b=0 in (1.1), we get the following two equivalent continued fractions,

$$\frac{[-q^3;q^4]_\infty}{[-q;q^4]_\infty} = \frac{1}{1+}\frac{q}{1+q+}\frac{q^3}{1+q^4+}\frac{q^5}{1+q^6+...} \tag{3.3}$$

(a=0, b=i)

$$= \frac{1}{1+q-}\frac{q^3}{1+q^2-}\frac{q}{(1+q)(1+q^4)-}\frac{q^{11}}{1+q^6-}$$

$$\frac{q}{(1+q)(1+q^8)-}\frac{q^{19}}{1+q^{10}-...} \tag{3.4}$$

(a=i, b=0)

Next, with a=b=i, (1.1) yields,

$$\frac{[-q^3;q^4]^2_\infty}{[-q;q^4]^2_\infty} = \frac{1}{1+q+}\frac{q(1-q^2)}{1+q^2-}\frac{q(1-q^2)}{(1+q)(1+q^4)+}\frac{q^5(1-q^6)}{1+q^6-}$$

$$\frac{q(1-q^6)}{(1+q)(1+q^8)+}\frac{q^9(1-q^{10})}{1+q^{10}-....} \tag{3.5}$$

Further, setting $a = b = \sqrt{q}$ in (1.1), we get the following interesting result involving Ramanujan's Ψ-theta function,

$$\Psi^2(q^2) = \frac{[q^4;q^4]_\infty^2}{[q^2;q^4]_\infty^2} = \frac{1}{1-q^2-}\,\frac{q^2(1-q^2)}{1+q^2+}\,\frac{q^2(1-q^2)}{(1-q^2)(1+q^4)-}\,\frac{q^6(1-q^6)}{1+q^6+}$$
$$\frac{q^2(1-q^6)}{(1-q^2)(1+q^8)+}\,\frac{q^{10}(1-q^{10})}{1+q^{10}-\ldots} \tag{3.6}$$

where

$$\Psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty}$$

Again, taking $a = \omega i$ and $b = \omega^2 i (\omega = e^{2\pi i/3})$ in (1.1), we get the following interesting result

$$\prod_{n=1}^{\infty} \frac{(1-q^{4n-1}+q^{8n-2})}{(1-q^{4n-3}+q^{8n-6})} = \frac{1}{1+\omega^2 q+}\,\frac{\omega q(1-\omega q^2)}{(1+q^4)-}\,\frac{\omega^2 q(1-\omega^2 q^2)}{(1+\omega^2 q)(1+q^4)+}$$
$$\frac{\omega q^5(1-\omega q^6)}{1+q^6-}\,\frac{\omega^2 q(1-\omega^2 q^6)}{(1+\omega^2 q)(1+q^8)+}\,\frac{\omega q^9(1-\omega q^{10})}{1+q^{10}-}\,\frac{\omega q^2(1-\omega^2 q^{10})}{(1+\omega^2 q)(1+q^{12})+\ldots} \tag{3.7}$$

If we put $a = b = q$ in (1.1), we get

$$\frac{[q;q^4]_\infty^2}{[q^3;q^4]_\infty^2} = \frac{(1-q^2)^2}{(1-q^3)-}\,\frac{q^3(1-q^2)}{1+q^2+}\,\frac{q^3(1-q^2)}{(1-q^3)(1+q^4)-}\,\frac{q^7(1-q^6)}{1+q^6+}$$
$$\frac{q^3(1-q^6)}{(1-q^3)(1+q^8)-}\,\frac{q^{11}(1-q^{10})}{1+q^{10}+\ldots} \tag{3.8}$$

Further, for a=0 and b=q in (2.1), we get

$$\frac{[q;q^4]_\infty}{[q^3;q^4]_\infty} = \frac{1-q}{1-}\,\frac{q^3}{1+q^2-}\,\frac{q^5}{1+q^4-}\,\frac{q^7}{1+q^6-\ldots} \tag{3.9}$$

Lastly, if we replace q by q^2 in (1.1) and then set $a = q^{3/2}$ and $b = \sqrt{q}$ in it, we get

$$\frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty} = \frac{1-q}{1-q^5-}\,\frac{q^3(1-q^6)}{1+q^4+}\,\frac{q^5(1-q^2)}{(1-q^5)(1+q^8)-}$$
$$\frac{q^{11}(1-q^{14})}{1+q^{12}+}\,\frac{q^5(1-q^{10})}{(1-q^2)(1+q^{16})-\ldots} \tag{3.10}$$

A number of other special cases could easily be deduced.

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