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CONJUGATE S_3-MAGIC GRAPHS

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Abstract: In this paper, we introduce conjugate A-magic labeling of graphs where A is a finite non-abelian group and investigate the graphs that are conjugate S_3-magic.

Keywords and Phrases: A-magic labeling, Non-abelian group, conjugate S_3-magic.

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1. Introduction

Throughout this paper, we shall consider only connected, finite, simple, and undirected graphs. Let $G = (V(G), E(G))$ be any finite graph and let A be an abelian group under addition with the identity element 0. Let $A^* = A \setminus \{0\}$. Any mapping $\ell : E(G) \to A^*$ is called an edge labeling. Observe that any edge labeling induces a mapping $\ell^+ : V(G) \to A$ as follows: $\ell^+(u) = \Sigma\{\ell(uv) : uv \in E(G)\}$. A graph G is called A-magic, if there exists $a \in A$ such that $\ell^+(u) = a$, for all $u \in V(G)$. Several authors studied about V_4-magic graphs [6, 8, 9] and Z_k-magic graphs [5]. Recently, Anusha C. and Anil Kumar V. [2, 3, 4] introduced A-magic labeling of graphs where A is non-abelian and studied graphs that are S_3-magic, D_4-magic and Q_8-magic. In this paper, we introduce a new magic labeling of graphs using a non-abelian group namely, the conjugate A-magic labeling of graphs and investigate conjugate S_3-magic labeling of some graphs. Consider the set $X = \{1, 2, 3\}$. A permutation of X is a function from X to itself that is both

1-1 and onto. The permutations of X with the composition of functions as a binary operation is a non-abelian group, called the symmetric group S_3. The group S_3 is a non-abelian group of order 6 and its elements are given by

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$
$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

2. Main Results

Here we need the following definition due to Anusha C and Anil Kumar V [4].

Definition 2.1. [4] *Let $G = (V(G), E(G))$ be a finite graph with p vertices and q edges, and let $(A, *)$ be a finite non-abelian group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \to N_q \times A \setminus \{1\}$ by*

$$l(e) := (f(e), g(e)),\ e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \ \text{ if and only if } \ f(e) \leq f(e').$$

Then obviously, the relation $\leq$ is a partial order on the range of ℓ.
Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of this chain as follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V(G)$ and let $N^*(u)$ be the set of all edges incident with u. Consider the restriction of the function ℓ on $N(u)$, that is, $\ell|_{N^*(u)}$. Observe that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) = \prod_{i=1}^{n}(f(e_i), g(e_i)). \tag{2.1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A-magic. The map ℓ^* is called an A-magic labeling of G and the corresponding constant a is called the magic constant.

In the above definition if the elements $g(e_2), g(e_3), \ldots, g(e_n)$ belong to the conjugacy class determined by $g(e_1)$, then we say that the graph G is conjugate A-magic. Formally, we have the following:

Definition 2.2. *Let $G = (V(G), E(G))$ be a graph with p vertices and q edges, and A be a finite non-abelian group of order n with identity 1. The graph G is said to be a conjugate A-magic graph if*

(i) for all $u \in V(G)$,

$$\ell^*(u) = \prod_{i=1}^{n}(f(e_i), g(e_i)) = \text{constant in } A(\text{see definition 2.1}).$$

(ii) the elements $g(e_2), g(e_3), \ldots, g(e_n)$ belong to the conjugacy class determined by $g(e_1)$.

Definition 2.3. *If the map g in the definition 2.2 is a constant map then the conjugate A-magic labeling is said to be constant conjugate A-magic labeling otherwise it is said to be non-constant conjugate A-magic labeling.*

In this paper, we consider the nonabelian group S_3 and investigate the graphs which are conjugate S_3-magic.

Theorem 2.1. *Let G be a conjugate S_3-magic graph. If G has a vertex of degree 2, then the conjugate S_3-magic constant does not belong to $\{\mu_1, \mu_2, \mu_3\}$.*

Proof. Let G be a conjugate S_3-magic graph with magic constant a. Let v be the vertex of G having degree 2. Let u_1 and u_2 be the vertices adjacent to v. Then $\ell^*(v) = a = g(u_1v) * g(vu_2)$ or $\ell^*(v) = g(vu_2) * g(u_1v)$. Since the product of any two elements(need not be distinct) from a conjugacy class always belongs to $\{\rho_0, \rho_1, \rho_2\}$, we have $a \notin \{\mu_1, \mu_2, \mu_3\}$.

Corollary 2.1. *If G is conjugate S_3-magic determined by the functions $f : E(G) \to N_q$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$. Suppose G has a vertex of order 2 then the range of g is always belongs to the conjugacy class $\{\rho_1, \rho_2\}$.*

Theorem 2.2. *There does not exist a non-constant conjugate S_3-magic labeling for the cycle graph C_3.*

Proof. Let the vertices of C_3 be denoted by u_1, u_2 and u_3. Suppose to the contrary that there exist a non-constant conjugate S_3-magic labeling for C_3. Without loss of generality, we take $g(u_1u_2) = \rho_1$ then $g(u_2u_3) = \rho_2$ or $g(u_1u_3) = \rho_2$, since g is non constant. If $g(u_2u_3) = \rho_2$ then $\ell^*(u_2) = \rho_0$ then the magic constant must be ρ_0 so $\ell^*(u_3) = \rho_0 = g(u_1u_3) * g(u_2u_3)$ implies $g(u_1u_3) = \rho_1$ but then

$\ell^*(u_1) = g(u_1u_2) * g(u_1u_3) = \rho_1 * \rho_1 = \rho_2$ which is a contradiction. Hence we cannot label C_3 using ρ_1 and ρ_2 under the map g. Similarly, we can prove that there does not exist a non-constant mapping $g : E(G) \to \{\mu_1, \mu_2, \mu_3\}$ to make the cycle C_3 conjugate S_3-magic. This completes the proof of the theorem.

Theorem 2.3. *If $n > 3$, there exist a non-constant S_3-magic labeling for cycle C_n.*
Proof. Let the vertices of C_n be denoted by $u_1, u_2, \ldots, u_n$. We consider the following two cases:

Case(i) n is even.

Suppose n is even. Then take f as any bijective map from $E(C_n)$ to N_n and define the map g as follows: label the adjacent edges of C_n by ρ_1 and ρ_2 alternatively. Then clearly $\ell^*(u) = \rho_0, \forall u \in V(G)$.

Case(ii) n is odd and $n > 3$.

Suppose that n is odd and $n > 3$. we define a conjugate S_3-magic labeling of C_n with magic constant ρ_1. Define f and g as follows:

Subcase(a) n is odd and $n \equiv 2 (\text{mod } 3)$.

Let $g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1(\text{mod } 3), \\ \mu_2, & \text{if } i \equiv 2(\text{mod } 3), \\ \mu_3, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$ and $f(u_iu_{i+1}) = i,\ 1 \le i \le n,\ i+1$ is taken modulo n.

Subcase(b) n is odd and $n \equiv 1 (\text{mod } 3)$.

Let $g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1(\text{mod } 3), i < n, \\ \mu_2, & \text{if } i \equiv 2(\text{mod } 3) \text{ and } i = n, \\ \mu_3, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$

Now define $f(u_iu_{i+1}) = i,\ 1 \le i \le n-2,\ f(u_{n-1}u_n) = n,\ f(u_nu_1) = n-1.$

Subcase(c) n is odd and $n \equiv 0 (\text{mod } 3)$.

Let $g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1(\text{mod } 3), i \le n-3, \\ \mu_2, & \text{if } i \equiv 2(\text{mod } 3), i \le n-3 \text{ and } i = n, \\ \mu_3, & \text{if } i \equiv 0(\text{mod } 3), i \le n-3 \text{ and } i = n-1. \end{cases}$

Now define $f(u_iu_{i+1}) = i, 1 \le i \le n-2,\ f(u_{n-2}u_{n-1}) = n,\ f(u_{n-1}u_n) = n-2, f(u_nu_1) = n-2.$

This completes the proof of the theorem.

Theorem 2.4. *There does not exist a non-constant conjugate S_3-magic labeling* *for the star graph, $K_{1,n}, n \geq 2$.*

Proof. Suppose that $K_{1,n}$ is conjugate S_3-magic with magic constant a. Since there are n pendant vertices in $K_{1,n}$, all pendant edges should be mapped to a under g. So g must be a constant map. We observe that, the star graph $K_{1,n}$ is S_3-magic if and only if either n is odd or $n \equiv 1(\text{mod } 3)$ [4]. Thus we have $K_{1,n}$ is S_3-magic if and only if it is conjugate S_3-magic. So $K_{1,n}$ is conjugate S_3-magic if and only if n is odd or $n \equiv 1(\text{mod } 3)$.

Theorem 2.5. *The bistar graph B_n is conjugate S_3-magic except when n is odd and $n \equiv 1(mod\ 3)$.*

Proof. Let the end vertices of the bridge be k_1 and k_2. Label the pendant vertices of first star by $u_1, u_2, \ldots, u_n$ and the pendant vertices of the second star by $v_1, v_2, \ldots, v_n$.

Case(i) n is even.

Let $f : E(B_n) \to N_{2n+1}$ be any bijective map. Define $g : E(G) \to S_3 \setminus \rho_0$ as $g(e) = \mu_1, \forall e \in E(B_n)$. Then clearly $\ell^*(u) = \mu_1, \forall u \in V(B_n)$.

Case(ii) n is odd and $n \equiv 0(\text{mod } 3)$.

In this case, let f as above and define g as $g(e) = \rho_1, \forall e \in E(B_n)$. Then $\ell^*(u) = \rho_1, \forall u \in V(G)$.

Case(iii) n is odd and $n \equiv 2(\text{mod } 3)$.

In this case also let f as above and define $g(e) = \begin{cases} \rho_2, & \text{if } e = k_1k_2, \\ \rho_1, & \text{otherwise.} \end{cases}$ Clearly $\ell^*(u) = \rho_1, \forall u \in V(B_n)$.

Case(iv) n is odd and $n \equiv 1(\text{mod } 3)$.

Suppose that, B_n is conjugate S_3-magic with magic constant 'a', $a \in S_3$. So each pendant edge should be mapped to a under the map g. Now let $g(k_1k_2) = b, b \in S_3 \setminus \{\rho_0\}$, then there are n possible values for $\ell^*(k_1)$. But in all the cases $\ell^*(k_1) = a$ implies $b = \rho_0$. Which is a contradiction. So B_n is not conjugate S_3-magic when n is odd and $n \equiv 1(\text{mod } 3)$.

This completes the proof of the theorem.

Theorem 2.6. *The cycle graph C_n with a pendant edge is not conjugate S_3-magic.*

Proof. Let G be the graph C_n with a pendant edge e. Denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Without loss of generality, let the one end vertex of the pendant edge e is at u_1 and let the other end vertex of e be denoted by u_{n+1}.

Suppose to the contrary that, the graph G is conjugate S_3-magic with magic constant 'a', where $a \in S_3$. Clearly $a \neq \rho_0$. Let $g(u_iu_{i+1}) = a_i$, where $a_i \in S_3 \setminus \{\rho_0\}$. Suppose that the conjugate magic constant is ρ_1. Then $g(u_1u_{n+1}) = \rho_1$. But $\ell^*(u_1) = \rho_1$ implies that $g(u_1u_2) * g(u_nu_{n+1}) = \rho_0$ also $a_i \in \{\rho_1, \rho_2\}$. Without loss of generality, let $g(u_1u_2) = \rho_1$ and $g(u_nu_1) = \rho_2$. But then $\ell^*(u_2) = \rho_1$ implies $g(u_2u_3) = \rho_0$, which is a contradiction. Similarly, we can prove that the magic constant cannot be ρ_2. Now, suppose that $a = \mu_1$, then $g(u_1u_{n+1}) = \mu_1$. There are 6 possible product for $\ell^*(u_1)$. i.e., $a_1 * a_n * \mu_1 = \mu_1$, $a_n * a_1 * \mu_1 = \mu_1$, $\mu_1 * a_1 * a_n = \mu_1$, $\mu_1 * a_n * a_1 = \mu_1$, $a_1 * \mu_1 * a_n = \mu_1$ or $a_n * \mu_1 * a_1 = \mu_1$. But all the six products leads to a contradiction as above. Hence the proof.

A wheel graph W_n of order $n + 1$, is a graph that contains a cycle of order n and for which every vertex in the cycle is connected to one other vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. Equivalently, $W_n = K_1 + C_n$.

Theorem 2.7. *If $n \geq 3$, the wheel W_n is S_3-magic.*

Proof. Let G be the wheel W_n and let the vertices of C_n be $v_1, v_2, \ldots, v_n$ and the vertex of K_1 be k. Consider the following four cases, for all the following cases let f be any bijection from $E(W_n)$ to N_{2n}:

Case(i) n is odd.

In this case, define $g(e) = \mu_1, \forall e \in E(W_n)$. Clearly W_n is conjugate S_3-magic with magic constant μ_1.

Case(ii) n is even and $n \equiv 0 (\text{mod } 3)$.

Here we define $g : E(W_n) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(e) = \rho_1, \forall e \in E(W_n)$. Then W_n becomes conjugate S_3-magic with constant ρ_0.

Case(iii) n is even and $n \equiv 1 (\text{mod } 3)$.

Here we define f as above and let $g : E(W_n) \to S_3 \setminus \{\rho_0\}$ be defined as : for $1 \leq i \leq n$, $g(v_iv_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$ and $g(kv_i) = \rho_1, i + 1$ is taken modulo n.

Clearly, $\ell^*(u) = \rho_0, \forall u \in V(W_n)$. Hence the theorem is valid in this case.

Case(iv) n is even and $n \equiv 2 (\text{mod } 3)$.

We define g as : $g(e) = \begin{cases} \rho_1, & \text{if } e = kv_i, 1 \leq i \leq n, \\ \rho_2, & \text{if } e = v_iv_{i+1}, 1 \leq i \leq n, n + 1 = 1. \end{cases}$ In this case, W_n is conjugate S_3-magic with magic constant ρ_2.

This completes the proof of the theorem.

The helm H_n is a graph obtained from a wheel W_n by attaching a pendant edge at each vertex of the n cycle.

Theorem 2.8. *The Helm graph $H_n, n \geq 3$ is conjugate S_3-magic if and only if $n \not\equiv 0(mod\ 3)$.*

Proof. Let H_n be the Helm graph of order $2n+1$. Denote the vertices of C_n by $u_1, u_2, \ldots, u_n$, vertex of C_1 be k and denote the other vertices by $v_1, v_2, \ldots, v_n$ such that u_iv_i is a pendant edge. Suppose that $n \not\equiv 0(\text{mod } 3)$. Now consider the following cases:

Case(i) $n \equiv 1(\text{mod } 3)$.

Let f be any bijective map from $E(H_n)$ to N_{3n} and let g be the constant map $g(e) = \rho_1, \forall e \in E(H_n)$. Clearly H_n becomes conjugate S_3-magic with constant ρ_1.

Case(ii) $n \equiv 2(\text{mod } 3)$.

Here, we define f as above and let g be the map

$$g(e) = \begin{cases} \rho_1, & \text{if } e \text{ is a pendant edge,} \\ \rho_2, & \text{otherwise.} \end{cases}$$

Then the above f and g determine a conjugate S_3-magic labeling of H_n with constant ρ_1.

Suppose to the contrary that H_n is conjugate S_3-magic when $n \equiv 0(\text{mod } 3)$. Let the magic constant be a, $a \in S_3$. Observe that $a \neq \rho_0$. Now if possible, let $a \in \{\rho_1, \rho_2\}$ then $g(e) \in \{\rho_1, \rho_2\}$ and f can be any bijection. Also $g(u_iv_i) = a, 1 \leq i \leq n$. Now

$$\ell^*(u_i) = g(u_iv_i) * g(u_iu_{i+1}) * g(u_{i-1}u_i) * g(u_ik). \tag{2.2}$$

Since $\ell^*(u_i) = a$ the equation 2.2 implies $g(u_iu_{i+1}) = g(u_{i-1}u_i) = g(u_ik) = b$, where $b \in \{\rho_1, \rho_2\}$. Thus $\ell^*(k) = \underbrace{b * b * \cdots * b}_{n \text{ times}} = \rho_0$. Which is a contradiction. Hence $a \notin \{\rho_1, \rho_2\}$. Now suppose that $a \in \{\mu_1, \mu_2, \mu_3\}$. Without loss of generality, let $a = \mu_1$ then $g(u_iv_i) = \mu_1, \forall 1 \leq i \leq n$. Let $g(u_1u_2) = p,\ g(u_1u_n) = q,\ g(u_1k) = r,$ where $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$ then

$$\ell^*(u_1) = \prod_{e \in N^*(u_1)} (f(e), g(e)) = \mu_1. \tag{2.3}$$

Suppose $f(u_1u_2) < f(u_1u_n) < f(u_1k)$ then $\ell^*(u_1) = p * q * r * \mu_1$ or $\mu_1 * p * q * r$ or $p * \mu_1 * q * r$ or $p * q * \mu_1 * r$. If $p * q * r * \mu_1 = \mu_1$ or $\mu_1 * p * q * r* = \mu_1$ then

$p * q * r = \rho_0$, which is not possible for any value of $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$. Similarly, we cannot find $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$ satisfying equation 2.3 for any possible values of $f(u_1u_2), f(u_1u_n)$ and $f(u_1k)$. Hence $a \notin \{\mu_1, \mu_2, \mu_3\}$. This completes the proof of the theorem.

A gear graph is a graph G_n obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n cycle.

Theorem 2.9. *The gear graph G_n is conjugate S_3-magic if and only if n is even.*
Proof. Let $G = G_n$. Denote the central vertex of G by k and the vertices of W_n by $u_1, u_2, \ldots, u_n$ and let $v_1, v_2, \ldots, v_n$ be the vertices such that v_i is adjacent to u_i and u_{i+1}. Suppose that n is even. Define f be any bijective map from $E(G)$ to N_{3n}. For $1 \leq i \leq n$, define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as $g(ku_i) = g(u_iv_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$

and $g(v_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \rho_1, & \text{if } i \text{ is odd.} \end{cases}$

Clearly G is conjugate S_3-magic with magic constant ρ_0.

Conversely, suppose that n is odd and G is conjugate S_3-magic with magic constant a. Note that $a \notin \{\mu_1, \mu_2, \mu_3\}$. We have $\ell^*(v_1) = a$ and $g(u_1v_1)$ is conjugate to $g(v_1u_2)$. Observe that the product of two elements from a conjugacy class in S_3 always belongs to $\{\rho_0, \rho_1, \rho_2\}$. Suppose that $a = \rho_0$ and g is a map from $E(G)$ to $\{\rho_1, \rho_2\}$. Without loss of generality, let $g(u_1v_1) = \rho_1$ and $g(v_1u_2) = \rho_2$. Then $\ell^*(u_2) = \rho_0$ implies $g(ku_2) = g(u_2v_2) = \rho_2$. Then $g(u_2v_2) = \rho_2$ implies $g(v_2u_3) = \rho_1$ and $g(ku_3) = \rho_1$. Proceeding like this, we obtain $g(ku_n) = g(u_nv_n) = \rho_1$ and $g(v_nu_1) = \rho_2$. Then $\ell^*(u_1) = \rho_0$ implies $g(ku_1) = \rho_0$. Which is a contradiction. So $g(e) \notin \{\rho_1, \rho_2\}$. Now suppose g is a map from $E(G)$ to $\{\mu_1, \mu_2, \mu_3\}$. Then $a = \rho_0$ implies $g(u_1v_1) = g(v_1u_2)$. Without loss of generality, let $g(u_1v_1) = g(v_1u_2) = \mu_1$. Now $\ell^*(u_2) = \rho_0$ implies $g(ku_2) * \mu_1 * g(u_2v_2) = \rho_0$ or $g(ku_2) * g(u_2v_2) * \mu_1 = \rho_0$. or $g(u_2v_2) * \mu_1 * g(ku_2) = \rho_0$ or $g(u_2v_2) * g(ku_2) * \mu_1 = \rho_0$ or $\mu_1 * g(ku_2) * g(u_2v_2) = \rho_0$ or $\mu_1 * g(u_2v_2) * g(ku_2) = \rho_0$. But we cannot find $g(u_2v_2), g(ku_2) \in \{\mu_1, \mu_2, \mu_3\}$ satisfying any of the above 6 equations.

Now, suppose that $a = \rho_1$ and $g(e) \in \{\rho_1, \rho_2\}$. So $\ell^*(v_1) = \rho_1$ implies $g(u_1v_1) = g(v_1u_2) = \rho_2$. Also $\ell^*(u_2) = \rho_1$ implies $g(u_2v_2) = g(ku_1) = \rho_1$ but then $\ell^*(v_2) = g(u_2v_2) * g(v_2u_3) = \rho_1$ implies $g(v_2u_3) = \rho_0$, which is a contradiction.

Suppose that $a = \rho_1$ with $g(e) \in \{\mu_1, \mu_2, \mu_3\}$. Without loss of generality, let $f(u_1v_1) < f(v_1u_2)$ and $g(u_1v_1) = \mu_1$ and $g(v_1u_2) = \mu_2$. There are 6 possible products for $\ell^*(u_2)$ as above depending on the function f. But all the 6 product leads to a contradiction as in the above cases. So $a \neq \rho_1$. Similarly, we can prove that $a \neq \rho_2$. Hence n can not be an odd number. This completes the proof of the theorem.

A shell $S_{n,n-3}, n \geq 4$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices. The vertex at which all chords are concurrent is called *apex*. The two vertices adjacent to the *apex* have degree 2, the apex has degree $n-1$ and all other vertices have degree 3.

Theorem 2.10. *The shell graph $S_{n,n-3}$ is conjugate S_3-magic for all $n > 4$ except when $n \neq 6$.*

Proof. Let $S_{n,n-3}$ be the shell graph and denote the vertices of $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$. Without loss of generality, let the *apex* be u_1. Consider the following four cases also for all the following cases let f be any bijection from $E(S_{n,n-3})$ to N_{2n-3} and let $u_{n+1} = u_1$. Let $1 \leq i \leq n$ and $3 \leq j \leq n-1$.

Case(i) $n \equiv 1 (\text{mod } 3), n \neq 4$.

In this case, define $g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1, 2, n-1, n, \\ \rho_2, & \text{otherwise.} \end{cases}$ and

$$g(u_1u_j) = \begin{cases} \rho_2, & \text{if } j = 3, n-1, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Then clearly $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

Case(ii) $n \equiv 2 (\text{mod } 3)$.

Here we define $g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1, n, \\ \rho_2, & \text{if } 2 \leq i \leq n-1. \end{cases}$ and

$$g(u_1v_j) = \rho_2, \ 3 \leq j \leq n-1.$$

Hence the above f and g will determine a conjugate S_3-magic labeling of $S_{n,n-3}$ with magic constant ρ_0.

Case(iii) $n \equiv 0 (\text{mod } 3)$ and n odd. In this case, define

$g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 4 \leq i \leq n-1, i = 1, 2, n-1, n, \\ \rho_2, & \text{if } i \text{ is odd and } 3 \leq i \leq n-2. \end{cases}$ and

$g(u_1u_j) = \rho_2$. Hence $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

Case(iv) $n \equiv 0 (\text{mod } 3)$, n is even and $n \neq 6$.

Here we define

$$g(u_1u_2) = g(u_2u_3) = g(u_1u_n) = g(u_{n-1}u_n) = \rho_1,$$
$$g(u_3u_4) = g(u_4u_5) = g(u_5u_6) = g(u_{n-2}u_{n-1}) = g(u_{n-3}u_{n-2}) = \rho_2,$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 6 \leq i \leq n-4, \\ \rho_2, & \text{if } i \text{ is odd and } 7 \leq i \leq n-4. \end{cases} \quad \text{and}$$
$$g(u_1u_j) = \begin{cases} \rho_1, & \text{if } i = 4, 5, n-2, \\ \rho_2, & \text{otherwise.} \end{cases}$$

Thus $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

This completes the proof of the theorem.

Theorem 2.11. *The shell graphs $S_{4,1}$ and $S_{6,3}$ are not conjugate S_3-magic.*
Proof. Let the vertices of $S_{4,1}$ be u_1, u_2, u_3 and u_4. Let the apex be u_1. Suppose that $S_{4,1}$ is conjugate S_3-magic with magic constant a, $a \in S_3$. Since u_2 and u_4 have degree 2, $a \in \{\rho_0, \rho_1, \rho_2\}$ (Theorem 2.1). Consider the following cases.

Case (i) $a = \rho_0$ and $g(e) \in \{\rho_1, \rho_2\}$.
Without loss of generality, let $g(u_1u_2) = \rho_1$. $\ell^*(u_2) = \rho_0$ implies $g(u_2u_3) = \rho_2$. Similarly, $\ell^*(u_3) = \rho_0$ implies $g(u_1u_3) = g(u_3u_4) = \rho_2$. Then $\ell^*(u_4) = \rho_0$ implies $g(u_4u_1) = \rho_1$. Hence $\ell^*(u_1) = g(u_1u_2)*g(u_1u_3)*g(u_4u_1) = \rho_1*\rho_2*\rho_1 = \rho_1 \neq \rho_0$. Which is a contradiction.

Case(ii) $a = \rho_0$ and $g(e) \in \{\mu_1, \mu_2, \mu_3\}$.
In this case, without loss of generality, let $g(u_1u_2) = \mu_1$ then $\ell^*(u_2) = \rho_0$ implies $g(u_2u_3) = \mu_1$. Now $\ell^*(u_3) = \prod_{e \in N^*(u_3)}(f(e), g(e))$. We have $g(u_2u_3), g(u_1u_3) \in \{\mu_1, \mu_2, \mu_3\}$ but the product of any three elements in $\{\mu_1, \mu_2, \mu_3\}$(need not be distinct) does not yield the value ρ_0. Hence $\ell^*(u_3) \neq \rho_0$, which is a contradiction.

The above 2 cases show that $a \neq \rho_0$.

Case(iii) $a = \rho_1$ and $g(e) \in \{\rho_1, \rho_2\}$.
Since $a = \rho_1$, $\ell^*(u_2) = \rho_1$. So $g(u_1u_2) = g(u_2u_3) = \rho_2$. Similarly, $\ell^*(u_3) = \rho_1$ implies $g(u_1u_3) = g(u_3u_4) = \rho_1$. But $\ell^*(u_4) = \rho_1$ implies $g(u_4u_1) = \rho_0$, a contradiction.

Case(iv) $a = \rho_1$ and $g(e) \in \{\mu_1, \mu_2, \mu_3\}$.
In the graph $S_{4,1}$ the vertices u_1 and u_3 are of degree 3. Observe that

$g(u_2u_3), g(u_1u_3), g(u_1u_4) \in \{\mu_1, \mu_2, \mu_3\}$ but the product of any three elements in $\{\mu_1, \mu_2, \mu_3\}$ (need not be distinct) does not yield the values ρ_0, ρ_1 and ρ_2. Hence the above case does not exist.

Case(v) $a = \rho_2$.
We can prove that, the magic constant a cannot be ρ_2 when $g(e) \in \{\rho_1, \rho_2\}$ or $g(e) \in \{\mu_1, \mu_2, \mu_3\}$. The proof is similar to the above cases (iii) and (iv). Hence there does not exist a conjugate S_3 magic labeling for $S_{4,1}$ with magic constant ρ_2.

All the above cases show that there does not exist a conjugate S_3-magic labeling for the shell graph $S_{4,1}$. Similarly, we can prove that $S_{6,1}$ is not conjugate S_3-magic.

A fan graph, denoted by F_n, is defined as $P_n + K_1$, where P_n is a path on n vertices.

Theorem 2.12. *The fan graph F_n is conjugate S_3-magic whenever $n \neq 3, 5$.*
Proof. We have $F_n = P_n + K_1$. Let $V(F_n) = \{k, u_1, u_2, \ldots, v_n\}$, where $u_1, u_2, \ldots, u_n$ be the vertices corresponding to P_n and k be the vertex corresponding to K_1. Now consider the following four cases. For all the cases take f be any bijection from $E(F_n)$ to N_{2n-1}.

Case(i) $n \equiv 0 (\text{mod } 3)$ and $n > 3$.
In this case, define $g : E(F_n) \to \{\rho_1, \rho_2\}$ as:

$$\text{for } 1 \leq i \leq n-1, g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1 \text{ and } i = n-1, \\ \rho_2, & \text{otherwise.} \end{cases}$$

$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 2, i = n-1, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Clearly, f and g will determine a conjugate S_3-magic labeling for F_n with magic constant ρ_2.

Case(ii) $n \equiv 1 (\text{mod } 3)$.
In this case, let g be defined by

$$g(u_iu_{i+1}) = \rho_1, \text{ if } i = 1 \leq i \leq n-1 \text{ and}$$

$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, n, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Then the maps f and g will define a conjugate S_3-magic labeling for F_n with the magic constant ρ_0.

Case(iii) $n \equiv 2 (\text{mod } 3)$ and n is even.

Here we define g as

$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_1, & \text{if } i \text{ is even.} \end{cases} \quad \text{and } g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, i = n, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Clearly $\ell^*(u) = \rho_1, \forall u \in V(F_n)$.

Case(iv) $n \equiv 2(\text{ mod } 3)$, n is odd and $n \neq 5$.

In this case also we define f as above and for $1 \leq i \leq n$ define g as follows:

$$g(u_1u_2) = g(u_{n-1}u_n) = g(u_{n-3}u_{n-2}) = \rho_2,$$
$$g(u_2u_3) = g(u_3u_4) = g(u_{n-4}u_{n-3}) = g(u_{n-2}u_{n-1}) = \rho_1,$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 4 \leq i \leq n-4, \\ \rho_2, & \text{if } i \text{ is odd and } 4 \leq i \leq n-4. \end{cases} \quad \text{and}$$
$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, n, 3, 4, n-4, \\ \rho_1, & \text{otherwise.} \end{cases}$$

By defining f and g as above we get a conjugate S_3 -magic labeling for F_n with magic constant ρ_1.

Theorem 2.13. *The fan graphs F_3 and F_5 are not conjugate S_3-magic.*

Proof. Consider the fan graph F_3. Let the vertices of P_3 be denoted by u_1, u_2 and u_3 and the vertex of K_1 be denoted by k. Suppose on the contrary that F_3 is conjugate S_3-magic with magic constant a, where $a \in \{\rho_0, \rho_1, \rho_2\}$. Suppose that $a = \rho_0$. Without loss of generality, let $g(u_1u_2) = \rho_1$ then $g(u_1k) = \rho_2$ and $g(u_2k) = g(u_2u_3) = \rho_1$. Then $g(u_2u_3) = \rho_1$ implies $g(u_3k) = \rho_2$. But then $\ell^*(k) = \rho_2 * \rho_1 * \rho_2 = \rho_2$, which is a contradiction. Hence $a \neq \rho_0$. Suppose that $a = \rho_1$, then $\ell^*(u_1) = \ell^*(u_3) = \rho_1$ implies $g(u_1u_2) = \rho_2 = g(ku_1) = g(ku_3) = g(u_2u_3)$. But then $\ell^*(u_2) = g(u_1u_2) * g(u_2u_3) * g(u_2k) = \rho_1$ implies $g(ku_2) = \rho_1$, which is a contradiction. Hence $a \neq \rho_1$. Similarly, we can prove that $a \neq \rho_2$. Hence F_3 is not a conjugate S_3-magic graph. In a similar manner we can prove that the fan graph F_5 is not conjugate S_3-magic.

Theorem 2.14. *The complete bipartite graph $K_{m,n}$ is conjugate S_3-magic for $m, n > 1$.*

Proof. Let U and V be the two partite sets of $V(K_{m,n})$. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices in U and V respectively. If m and n are of same parity then the constant map $g(e) = \mu_1$ together with any bijection $f : E(K_{m,n}) \to N_{mn}$ will give a conjugate S_3-magic labeling. Now, without loss of generality, assume that m is an even number and n is an odd number. Now consider the following cases: For all the following cases let f be any bijection from $E(K_{m,n})$ to N_{mn}.

Case(i) m is even and $n \equiv 0 (\text{mod } 3)$.

In this case define $g : E(K_{mn}) \to S_3 \setminus \rho_0$ as follows: For $1 \leq i \leq m$ and $1 \leq j \leq n$ define $g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$

Case(ii) m is even and $n \equiv 1 (\text{mod } 3)$.

For $1 \leq i \leq m$ and $1 \leq j \leq n-3$, define

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } j \text{ is odd },i \text{ is even and } j \text{ is even,} \\ \rho_2, & \text{if } \text{ if } i \text{ is odd and } j \text{ is even },i \text{ is even and } j \text{ is odd} \end{cases} \quad \text{and}$$

$$g(u_i v_{n-2}) = g(u_i v_{n-1}) = g(u_i v_n) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$$

Case(iii) m is even and $n \equiv 2 (\text{mod } 3)$.

For $1 \leq i \leq m$, let

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } 1 \leq j \leq n-1; i \text{ is even and } j = n, \\ \rho_2, & \text{if } i \text{ is even and } 1 \leq j \leq n-1; i \text{ is odd and } j = n. \end{cases}$$

In all the above cases, we can prove that f and g will determine a conjugate S_3-magic labeling for $K_{m,n}$ with magic constant ρ_0. Hence the proof.

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