South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 3 (2023), pp. 301-318

DOI: 10.56827/SEAJMMS.2023.1903.24 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A STUDY ON GRAPHS DEFINED ON L-SLICES

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(**Received:** *Jul. 09, 2023* **Accepted:** *Dec. 14, 2023* **Published:** *Dec. 30, 2023)*

Abstract: Let L be a locale with top element 1_L and J be a join semilattice with bottom element 0_J. The L-slice (σ, J) is the action of the locale on join semilattice satisfying certain properties. The concept of L-slices were modelled in tune with the modules in algebra. The benefit of studying L-slices is that we can approach the structure algebraically as well as topologically.

This paper deals with the graph theoretic approach to L-slices. The idea of relating graphs with algebraic structures was started by the work of Beck in [3]. The algebraic properties of L-slices prompted us to consider the possibility of various graphs that could be associated with L-slices. The article introduces two different graphs on L-slices. The total graph $\Gamma((T(\sigma, J))$ is defined. We derive a characterisation for such graphs to be nonempty. The structural properties of $\Gamma((T(\sigma, J))$ is studied. The weak Zariski Topology on (σ, J) gives us the graph $G_T(\omega^*)$. The conditions under which the graph is nonempty is examined. Also some of the structural properties of $G_T(\omega^*)$ is obtained.

Keywords and Phrases: Locale, L-slices, total graph.

2020 Mathematics Subject Classification: 06D22, 05C25.

1. Introduction and Preliminaries

Marshall Stone made a great impact on the concept of topological spaces through his famous representation theorem.The theorem led to consider the concept of point free topology. Many topologists of the time like Isabel made huge contributions to the field.The complete lattices with meet distributing over arbitrary joins were named as frames/loclaes and was started to be viewed as generalised topological spaces.Most of the topological concepts have been studied in the localic background. The connection between Boolean algebra and Boolean rings instigated the idea of taking locale theory to a more wider area of abstract algebra. The concept of modules have seen a wide range of development in the field of abstract algebra. The thought of connecting locales with modules shaped the construction of L-slices. The idea of relating graphs with algebraic structures was started by the work of Beck in [3]. The algebraic properties of L-slices prompted us to consider the possibility of various graphs that could be associated L-slices.The connection between graph theory and L-slices could be used to solve problems in one theory using tools from the other.The complexity in studying L-slices can be simplified using the techniques in graph theory. Further, we may be able to look into the applications of L-slices in areas like networking. The chapter introduces two different graphs on L-slices. The total graph $\Gamma((T(\sigma, J))$ is defined. We derive a characterisation for such graphs to be nonempty. The structural properties of $\Gamma((T(\sigma, J))$ is studied. The weak Zariski Topology on (σ, J) gives us the graph $G_T(\omega^*)$. The conditions under which the graph will be nonempty is examined. Also, some of the structural properties of $G_T(\omega^*)$ is obtained. Here we consider only finite L-slices and consequently the graphs under consideration would be the finite ones.

1.1. Frames and Locales

Definition 1.1. [12] *A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup\{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.*

Example 1.2. [12] i.The lattice of open subsets of topological space.
ii.The Boolean algebra B of all regularly open subsets of Real line R.

Remark. *The category of frames is denoted by **Frm** and the dual category is the category **Loc** of locales.*

Definition 1.3. [12] *A subset F of locale L is said to be a filter if*
i. F is a sub-meet-semilattice of L; that is $1_L \in F$ and $a \in F$, $b \in F$ imply $a \sqcap b \in F$.
ii. F is an upper set; that is $a \in F$ and $a \sqsubseteq b$ imply $b \in F$.

Definition 1.4. [12] *A filter F is proper if $F \neq L$, that is if $0_L \notin F$.*
A proper filter F in a locale L is prime if $a_1 \sqcup a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.

Definition 1.5. [12] *A proper filter F in a locale L is a completely prime filter if for any indexing set J and $a_i \in L$, $i \in J$, $\bigsqcup a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$. Completely prime filters are denoted by c.p filters.*

1.2. L-slice and its Properties

This section discusses the concept of L-slice and some of its properties. Given a locale L and a join semilattice J with bottom element 0_J, we have introduced a new concept of an action σ of locale L on join semilattice J together with a set of conditions. The pair (σ, J) is called L-slice. L-slice, though algebraic in nature adopts properties of L through the action σ.

Definition 1.6. [13] *Let L be a locale and J be join semilattice with bottom element 0_J. By the "action of L on J" we mean a function $\sigma : L \times J \to J$ such that the following conditions are satisfied.*
i. $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ *for all* $a \in L, x_1, x_2 \in J$.
ii. $\sigma(a, 0_J) = 0_J$ *for all* $a \in L$.
iii. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ *for all* $a, b \in L, x \in J$.
iv. $\sigma(1_L, x) = x$ *and* $\sigma(0_L, x) = 0_J$ *for all* $x \in J$.
v. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ *for* $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice.

Definition 1.7. [13] *1. Let L be a locale and I be any ideal of L. Consider each $x \in I$ as constant map $\boldsymbol{x} : L \to L$. Then (σ, I) is an L-slice. In particular (σ, L) is an L-slice.*
2. Let the locale L be a chain with Top and Bottom elements and J be any join semilattice with bottom element. Define $\sigma : L \times J \to J$ by $\sigma(a, j) = j \ \ \forall a \neq 0$ and $\sigma(0_L, j) = 0_J$. Then σ is an action of L on J and (σ, J) is an L-slice.

Definition 1.8. [13] *Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if*
i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ *for all* $x_1, x_2 \in J$.
ii. $f(\sigma(a, x)) = \mu(a, f(x))$ *for all* $a \in L$ *and all* $x \in (\sigma, J)$.

Definition 1.9. [13] *Let (σ, J) be an L-slice of a locale L. A subjoin semilattice J' of J is said to be L-subslice of J if J' is closed under action by elements of L.*

Example 1.10. [13] 1. Let L be a locale and $O(L)$ denotes the collection of all order preserving maps on L. Then $(\sigma, O(L))$ is an L-slice, where $\sigma : L \times O(L) \to O(L)$ is defined by $\sigma(a, f) = f_a$, where $f_a : L \to L$ is defined by $f_a(x) = f(x) \sqcap a$. Let $K = \{f \in O(L) : f(x) \sqsubseteq x, \forall x \in L\}$. Then (σ, K) is an L-subslice of the L-slice $(\sigma, O(L))$.

2. Let (σ, J) be an L-slice and let $x \in (\sigma, J)$. Define $\langle x \rangle = \{\sigma(a, x); a \in L\}$. Then $(\sigma, \langle x \rangle)$ is an L-subslice of (σ, J) and it is the smallest L-subslice of (σ, J) containing x.

Definition 1.11. [13] *Let (σ, J) be an L-slice of a locale L. For each $a \in L$, the map $\sigma_a : (\sigma, J) \to (\sigma, J)$ defined by $\sigma_a(x) = \sigma(a, x)$ is an L-slice homomorphism.*

Definition 1.12. [13] *A subslice (σ, I) of an L-slice (σ, J) is said to be ideal of (σ, J) if $x \in (\sigma, I)$ and $y \in (\sigma, J)$ are such that $y \leq x$, then $y \in (\sigma, I)$.*

Definition 1.13. *An ideal (σ, I) of an L-slice (σ, J) is a prime ideal if it has the following properties:*
i. If a and b are any two elements of L such that $\sigma(a \sqcap b, x) \in (\sigma, I)$, then either $\sigma(a, x) \in (\sigma, I)$ or $\sigma(b, x) \in (\sigma, I)$.
ii. (σ, I) is not equal to the whole slice (σ, J).

Definition 1.14. [13] *Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if*
i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ for all $x_1, x_2 \in (\sigma, J)$.
ii. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Example 1.15. [13] i. Let (σ, J) be an L-slice and (σ, J') be an L-subslice of (σ, J). Then the inclusion map $i : (\sigma, J') \to (\sigma, J)$ is an L-slice homomorphism.
ii. Let $I = \downarrow(a)$, $J = \downarrow(b)$ be principal ideals of the locale L. Then (σ, I), (σ, J) are L-slices. Then the map $f : (\sigma, I) \to (\sigma, J)$ defined by $f(x) = x \sqcap b$ is an L-slice homomorphism.

Proposition 1.16. [13] *If $f : (\sigma, J) \to (\mu, K)$ is a L-slice homomorphism, then $f(0_J) = 0_K$.*

1.3. L-Prime Elements and $Spec(\sigma, C)$

Definition 1.17. [10] *An element $p \neq 1_C$ of (σ, C) is said to be L-prime element if for every $r \in L$ and $n \in (\sigma, C), \sigma(r, n) \leq p$ implies that either $r \in [1_C \to p]_L$ or $n \leq p$.*

Example 1.18. [10] If we consider the L-slice $(\sqcap, L)$ then the L-prime elements are precisely the meet irreducible elements of L.

We now discuss some of the properties of L-prime elements.

Theorem 1.19. [10] *If p be a L-prime element and $x \in (\sigma, C)$ then $[x \to p]_L$ is a prime ideal of L.*

Corollary 1.20. *If p is a L-prime element then $[1_C \to p]_L$ is a prime ideal of a locale L.*

Definition 1.21. [10] *The set of all L -prime elements of (σ, C) is called the spectrum of (σ, C) and is denoted by $Spec(\sigma, C)$.*

Definition 1.22. [10] *For $n \in (\sigma, C)$ we define $C(n) = \{p \in Spec(\sigma, C) : n \leq p\}$.*

Proposition 1.23. [10] *For $n \in (\sigma, C)$ and $p \in Spec(\sigma, C)$ we have the following :*
i) $C(0_C) = Spec(\sigma, C)$
ii) $C(1_C) = \phi$
iii) $\bigcap_{i \in I} C(n_i) = C(\bigvee_{i \in I} n_i)$,for some indexed set I
iv) $C(n) \cup C(l) \subseteq C(n \wedge l)$.

The above proposition leads us to the following theorem.

Theorem 1.24. [10] *On $Spec(\sigma, C), \Lambda = \{C(n) : n \in (\sigma, C)\}$ forms a basis for some topology Ω.*

Definition 1.25. [10] *$Spec_\wedge(\sigma, C)$ is the set of all $p \in (\sigma, C)$ such that p is meet irreducible as well as an L-prime element of (σ, C).*

Proposition 1.26. [10] *On $Spec_\wedge(\sigma, C)$, $C(n) \cup C(l) = C(n \wedge l)$.*
Proof. We have $C(n) \cup C(l) \subseteq C(n \wedge l)$. If $p \in C(n \wedge l)$ then $n \wedge l \leq p$. The L-prime element p being meet irreducible, either $n \leq p$ or $l \leq p$. That is, either $p \in C(n)$ or $p \in C(l)$. Hence $p \in C(n) \cup C(l)$ and $C(n) \cup C(l) = C(n \wedge l)$.

Proposition 1.27. [10] *The collection $\nu = \{C(n) : n \in (\sigma, C)\}$ defined on $Spec_\wedge(\sigma, C)$ forms a family of closed sets for some topology on $Spec_\wedge(\sigma, C)$.*

Definition 1.28. [10] *The topology Ψ generated by the family of closed sets ν is called the Zariski topology on $Spec_\wedge(\sigma, C)$.*

2. Main Results

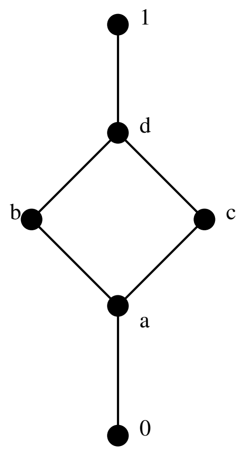
This paper deals with the graph theoretic approach to L-slices. The idea of relating graphs with algebraic structures was started by the work of Beck in [3]. The algebraic properties of L-slices prompted us to consider the possibility of various graphs that could be associated with it. The paper introduces two different graphs on L-slices. The total graph $\Gamma((T(\sigma, J))$ is defined. We derive a characterisation for such graphs to be nonempty. The structural properties of $\Gamma((T(\sigma, J))$ is studied. The weak Zariski Topology on (σ, J) gives us the graph $G_T(\omega^*)$. The conditions under which the graph is nonempty is examined. Also some of the structural properties of $G_T(\omega^*)$ is obtained. Here we consider only finite L-slices and consequently the graphs under consideration would be the finite ones.

2.1. Total Graph of L-Slice

Definition 2.1. *Let (σ, J) be an L-slice and $L^* = L \setminus \{0_L\}$. We define the torsion elements of a L-slice $T(\sigma, J)$ as $T(\sigma, J) = \{x \in (\sigma, J) : \sigma(a, x) = 0_J$ for some $a \in L^*\}$. It is evident that $T(\sigma, J)$ is always nonempty.*

Theorem 2.2. *$T(\sigma, J)$ is an ideal of L-slice (σ, J).*
Proof. If $x, y \in T(\sigma, J)$ then there exists $a, b \in L^*$ such that $\sigma(a, x) = 0_J$ and $\sigma(b, y) = 0_J$. Also, $\sigma(a \sqcap b, x \vee y) = \sigma(a, \sigma(b, x \vee y)) = \sigma(a, \sigma(b, x) \vee \sigma(b, y)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, 0_J) = 0_J$. Therefore $x \vee y \in T(\sigma, J)$.
If $z \leq x$ then $\sigma(a, z) \leq \sigma(a, x)$ implies $\sigma(a, z) = 0_J$. Therefore $z \in T(\sigma, J)$. Consider $\sigma(b, x) \in (\sigma, J)$, then $\sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, 0_J) = 0_J$. Hence $T(\sigma, J)$ is an ideal of (σ, J).

Example 2.3. i) Let L be a locale and let $J =\downarrow x$ for some $x \in L$. Then $(\sqcap, J)$ is a slice and $T(\sqcap, J) = \{y \in J : \sqcap(a, y) = 0_J\} = \{y \in J : a \sqcap y = 0_J\}$.
ii) Consider the locale represented by the following Hasse diagram



Let $J =\downarrow b = \{0, a, b\{\}\}$ then $T(\sqcap, J) = \{0_J\}$.

Note that for any L-slice (σ, J) the annihilator $Ann(J) = \{x \in (\sigma, J) : \sigma(a, x) = 0_J \;\; \forall x \in (\sigma, J)\} \subseteq T(\sigma, J)$. We now define the total graph of the L-slice (σ, J).

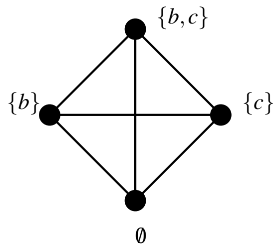
Definition 2.4. *The vertex set V_T of $\Gamma(T(\sigma, J))$ is the set of all elements of the L-slice and the edge set E_T of $\Gamma(T(\sigma, J)) = \{(x, y) : x \vee y \in T(\sigma, J)\}$.*

Theorem 2.5. *The total graph $\Gamma(T(\sigma, J))$ is complete if and only if $T(\sigma, J) = (\sigma, J)$.*

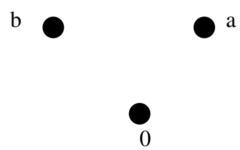
Proof. Suppose $\Gamma(T(\sigma, J))$ is complete then there exists an edge between every $x, y \in V_T$. That is, $x \vee y \in T(\sigma, J)$. In particular, every vertex is adjacent to 0_J. Hence $x \vee 0_J = x \in T(\sigma, J)\ \ \forall x \in (\sigma, J)$. Thus $T(\sigma, J) = (\sigma, J)$. Conversely suppose that $T(\sigma, J) = (\sigma, J)$. Since J is a join semilattice, for any two vertices $u, v \in V_T$ implies $u \vee v \in J = T(\sigma, J)$. Thus $\Gamma(T(\sigma, J))$ is complete.

Corollary 2.6. *The above theorem is necessarily satisfied if* $Ann(J) \neq \{0_J\}$.
Proof. Suppose $a \in Ann(J)$. The definition of $Ann(J)$ shows that $\sigma(a, x) = 0_J,\ \ \forall x \in (\sigma, J)$. Evidently, $T(\sigma, J) = (\sigma, J)$ and $\Gamma(T(\sigma, J))$ is complete.

Example 2.7. i) Let $X = \{a, b, c\}$.Then $\mathfrak{P}(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{b, c\}$. $\downarrow A = \{C \in \mathfrak{P}(X) : C \subseteq A\}$ implies $\downarrow A = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. $\downarrow A$ is a join semilattice under the partial ordering $\subseteq$. Also, $(\mathfrak{P}(X), \subseteq)$ is a locale. Define the action $\sqcap$ on $\downarrow A$ as $\sqcap(B, A_1) = B \cap A_1$ where $A_1 \in \downarrow A$. The annihilator $Ann(\downarrow A) = \{\phi, \{a\}\}$ and $T(\sqcap, \downarrow A) = \{\phi, \{b\}, \{c\}, \{b, c\}\} = \downarrow A$. Therefore $\Gamma(T(\sqcap, \downarrow A))$ is complete. Also, $\Gamma(T(\sqcap, \downarrow A))$ is the complete graph K_4.



ii) In Example 2.3 *ii*), we observed that $T(\sqcap, \downarrow b) = \{0_J\}$. Then the total graph $\Gamma(T(\sqcap, \downarrow b))$ is totally disconnected.



Now we can generalise the above as follows.

Proposition 2.8. $\Gamma(T(\sigma, J))$ *is totally disconnected if and only if* $T(\sigma, J) = \{0_J\}$.
Proof. Since $\Gamma(T(\sigma, J))$ is totally disconnected we have that 0_J is not connected with any other vertices. Hence $x \vee 0_J = x \notin T(\sigma, J)$ implying that $T(\sigma, J) = \{0_J\}$. Conversely, let $T(\sigma, J) = \{0_J\}$. Any two vertices x, y of $\Gamma(T(\sigma, J))$ is connected if and only if $x \vee y = 0_J$ and that is possible if and only if $x = 0_J, y = 0_J$.Hence $\Gamma(T(\sigma, J))$ is totally disconnected.

We have already shown that $T(\sigma, J)$ is an ideal of (σ, J). Now we propose the next two theorems which is a consequence of the structure of $T(\sigma, J)$.

Theorem 2.9. *The subgraph induced by the set $T(\sigma, J)$ is always complete. In particular, if $\mid T(\sigma, J) \mid = n$ then the subgraph induced will be the complete graph K_n.*

Proof. Since $T(\sigma, J)$ is an ideal of (σ, J), if $x, y \in T(\sigma, J)$ then $x \vee y \in T(\sigma, J)$. Therefore the subgraph induced by $T(\sigma, J)$ is always complete.

Corollary 2.10. *The clique number $\omega(\Gamma(T(\sigma, J)))$ is $\mid T(\sigma, J) \mid$.*

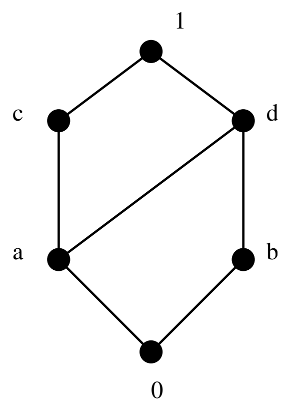
Corollary 2.11. *The subgraph induced by $(\sigma, J) \setminus T(\sigma, J)$ is totally disconnected and the independence number $\beta(\Gamma(T(\sigma, J))) = \mid (\sigma, J) \setminus T(\sigma, J) \mid$.*

Theorem 2.12. *If $T(\sigma, J)$ is a proper ideal of (σ, J) then $\Gamma(T(\sigma, J))$ is always disconnected.*

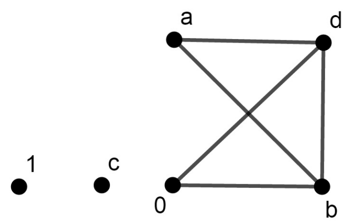
Proof. Let $x, y \in (\sigma, J)$ such that $x \in T(\sigma, J)$ and $y \in (\sigma, J) \setminus T(\sigma, J)$. The subgraph induced by $T(\sigma, J)$ and $(\sigma, J) \setminus T(\sigma, J)$ are disjoint. Suppose they are connected then $x \vee y \in T(\sigma, J)$. But $T(\sigma, J)$ is an ideal would imply that $y \in T(\sigma, J)$, which is a contradiction. Thus the subgraphs induced by $T(\sigma, J)$ and $(\sigma, J) \setminus T(\sigma, J)$ will always be disjoint. Thus $\Gamma(T(\sigma, J))$ is always disconnected.

Let us consider some L-slices and examine the properties of total graph associated with them.

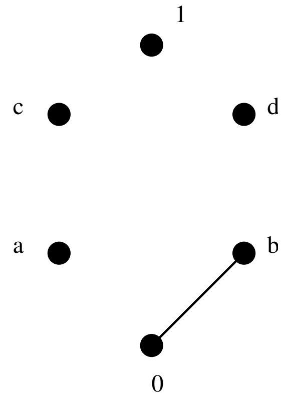
Example 2.13. Let J be given by the Hasse diagram



And let $L = \{0 \leq c \leq 1\}$. Define the action on J as $\sqcap(a, x) = a \sqcap x$ for $a \in L$ and $x \in J$. The ideal $T(\sigma, J) = \{0, b, d\}$ and it is a proper ideal of $(\sqcap, J)$. $\Gamma(T(\sqcap, J))$ is disconnected and the graph is



Thus the total graph of the L-slice is the union of two K_3 graphs and two K_1 graphs.
If we consider $L = \{0 \leq a \leq 1\}$ then $T(\sqcap, J) = \{0, b\}$. Then the total graph of L-slice is



The total graph of the L-slice is the union of one K_2 graph and four K_1 graphs.
If we let $L = \{0 \leq d \leq 1\}$ then $T(\sqcap, J) = \{0\}$ and $\Gamma(T(\sqcap, J))$ is disconnected.

Remark. *A L-slice is said to be a σ-domain over L if there exists no torsion elements for the L-slice. In other words, there exists no $a \in L^*$ such that $\sigma(a, x) = 0_J$.*

Example 2.14. i) If L is a chain then the L-slice $(\sqcap, L)$ is a σ-domain over L.
Let $T^*(\sigma, J) = \{x \neq 0_J : \exists a \in L^* \textit{ such that } \sigma(a, x) = 0_J\}$ then the corresponding total graph is denoted by $\Gamma(T^*(\sigma, J))$.
In this case, if $T^*(\sigma, J)$ is nonempty then $T^*(\sigma, J)$ is a subslice. Also, if an L-slice is a σ-domain over L then the corresponding $\Gamma(T^*(\sigma, J))$ is an empty graph.
ii) If L is a chain then $\Gamma(T^*(\sqcap, L))$ is an empty graph.

Definition 2.15. *A locale L is said to have zero divisors if for $a \in L^*$ there exists $b \in L^*$ such that $a \sqcap b = 0_L$.*

Lemma 2.16. *Let a be a zero divisor of the locale L. If $x \in (\sigma, J)$ then $\sigma(a, x) \in$*

$T(\sigma, J)$.

Proof. If a is a zero divisor of L then there exists $b \in L$ such that $a \sqcap b = 0_L$. Therefore $\sigma(b, \sigma(a, x)) = \sigma(a \sqcap b, x) = \sigma(0_L, x) = 0_J$ implies that $\sigma(a, x) \in T(\sigma, J)$.

Lemma 2.17. *If the top element 1_L of the locale L is the join of n zero divisors of L then every element of the L-slice (σ, J) is the join of n torsion elements.*

Proof. Let $1_L = z_1 \sqcup z_2 \sqcup \ldots\ldots \sqcup z_n$, where each z_i is a zero divisor of L.

For any $x \in (\sigma, J)$,
$$\begin{aligned} \sigma(1_L, x) &= \sigma(z_1 \sqcup z_2 \sqcup \ldots \sqcup z_n, x) \\ &= \sigma(z_1, x) \vee \sigma(z_2, x) \ldots \sigma(z_n, x) \\ \Rightarrow x &= \sigma(z_1, x) \vee \sigma(z_2, x) \ldots \sigma(z_n, x) \end{aligned}$$

The above lemma states that each $\sigma(z_i, x) \in T(\sigma, J)$. Hence the result.

A characterisation of the total graph of an L-slice based on the zero divisors of the locale L.

Theorem 2.18. *If L has a finite basis of zero divisors then the total graph of the L-slice (σ, J) is complete.*

Proof. If $\{z_1, z_2, \ldots. z_n\}$ be the finite basis of zero divisors then from the above lemma $x = \sigma(z_1, x) \vee \sigma(z_2, x) \ldots \sigma(z_n, x)$, where each $\sigma(z_i, x) \in T(\sigma, J)$. And the fact that $T(\sigma, J)$ is an ideal will give us the theorem.

Proposition 2.19. *The chromatic number $\chi\Gamma(T(\sigma, J))$ of $\Gamma(T(\sigma, J))$ is such that either always $\chi\Gamma(T(\sigma, J)) = 1$ or $\chi\Gamma(T(\sigma, J)) = n + 1$, where $n = \mid T(\sigma, J) \mid$.*

Proof. If $T(\sigma, J) = \{0_J\}$ then graph $\Gamma(T(\sigma, J))$ is totally disconnected and $\chi\Gamma(T(\sigma, J))$ is one. We know that the subgraph induced by $T(\sigma, J)$ is the complete graph K_n. Theorem 2.12 shows that if $T(\sigma, J)$ is a proper ideal then $\Gamma(T(\sigma, J))$ is always disconnected. Thus if $T(\sigma, J) \neq \{0_J\}$ then $\chi\Gamma(T(\sigma, J)) = n + 1$.

Remark. *Theorem 2.8 ($t = \{0\}$) and Theorem 2.12 ($t \neq \{0\}$) shows that $\Gamma(T(\sigma, J))$ is never a critical graph.*

Property 1. *The diameter of the graph $diam(\Gamma(T(\sigma, J))) \in \{1, \infty\}$.*

Proof. The theorem 2.5 shows that $diam(\Gamma(T(\sigma, J))) = 1$. If $T(\sigma, J) = \{0_J\}$ then the graph is totally disconnected and $diam()\Gamma(T(\sigma, J))) = \infty$.

Property 2. *The radius $r(\Gamma(T(\sigma, J))) \in \{0, 1\}$.*

Proof. The graph $\Gamma(T(\sigma, J))$ is either complete or disconnected. Hence the radius of the total graph of the L-slice will be either 0 or 1.

Let us denote the subgraph induced by $T(\sigma, J)$ as $\Gamma_n(T(\sigma, J))$, where n denotes the cardinality of the set $T(\sigma, J)$.

Property 3. *If $T(\sigma, J)$ is proper ideal then the diameter and radius of $\Gamma_n(T(\sigma, J))$*

will be the same and equal to 1.
Proof. The property is obtained through the completeness of the subgraph.

Property 4. *Whenever* $|T(\sigma, J)|= n \geq 3$*, then the girth of* $\Gamma_n(T(\sigma, J))$ *denoted as* $gr(\Gamma_n(T(\sigma, J))) = 3$ *and the circumference of* $\Gamma_n(T(\sigma, J))$*,* $c(\Gamma_n(T(\sigma, J))) = n$*.*
Proof. Follows from the completeness of $\Gamma_n(T(\sigma, J))$.

The above property can also be restated as.

Property 5. *If* $|T(\sigma, J)|= n \geq 3$ *if and only if* $gr(\Gamma_n(T(\sigma, J))) = 3$*.*

In this section we have defined and studied various properties of the total graph. We have observed that $\Gamma(T(\sigma, J))$ is complete if $|Ann(J)|\geq 2$. If $|T(\sigma, J)|\geq 2$ then $\Gamma(T(\sigma, J))$ disconnected. Thus $\Gamma(T(\sigma, J))$ is either complete or disconnected.

2.2. Graphs associated with weak Zariski topology ω^* on $Spec(\sigma, C)$

We have shown in [10] that the sets $C(n) = \{p \in Spec(\sigma, C) : n \leq p\}$ forms basis for a topology on prime spectrum $Spec(\sigma, C)$. Also, if $C(n) \cup C(l) = C(n \wedge l)$ the collection $\nu = \{C(n) : n \in (\sigma, C)\}$ will then be the collection of closed sets on $Spec(\sigma, C)$ and the topology so formed may be called weak Zariski topology ω^* on $Spec(\sigma, C)$.
This section deals with graphs associated with this weak Zariski topology ω^*. For a subset T of $Spec(\sigma, C)$ we introduce a graph $G_T(\omega^*)$. We study some of its properties and show that it has a bipartite subgraph.

Definition 2.20. *Let* T *be a nonempty subset of* $Spec(\sigma, C)$*. The graph* $G_T(\omega^*)$ *has as vertex set* $V(G_T(\omega^*)) = \{n \in (\sigma, C) : \exists\ l \in (\sigma, C)$ *such that* $C(n) \cup C(l) = T\}$*. Also, two vertices* n *and* k *are adjacent if and only if* $C(n) \cup C(k) = T$*. In other words, the graph* $G_T(\omega^*)$ *has* n *as vertex if and only if there exists a* $l \in (\sigma, C)$ *such that* $C(n \wedge l) = T$*.*

Remark. *We study the properties of graphs associated with the weak Zariski topology. The definition itself gives us two conditions for such a graph to exist. We state them as our next two propositions.*

Proposition 2.21. $G_T(\omega^*) \neq \phi$ *if and only if* T *is closed and is not an irreducible subset of* $Spec(\sigma, C)$*.*
Proof. Follows directly from the definition of $G_T(\omega^*)$.

The above proposition can be rephrased as follows.

Proposition 2.22. $G_T(\omega^*) \neq \phi$ *if and only if* $T = C(\bigwedge T)$ *and* T *is not an irreducible subset of* $Spec(\sigma, C)$*.*
Proof. Suppose $G_T(\omega^*) \neq \phi$. The above proposition shows that T is closed.So it remains to show that $T = C(\bigwedge T)$. We know that $T \subseteq C(\bigwedge T)$. Let $C(n)$

be any closed subset of $Spec(\sigma, C)$ containing T. Then $n \leq p\ \forall p \in T$ implies that $n \leq \bigwedge_{p \in T} p = q$. Therefore for evey $l \in C(q)$ implies $l \in C(n)$. That is, $C(n) \supseteq C(q)$. Hence $C(q)$ is the smallest closed set containing T. Thus $T = C(q) = C(\bigwedge_{p \in T} p) = C(\bigwedge T)$.

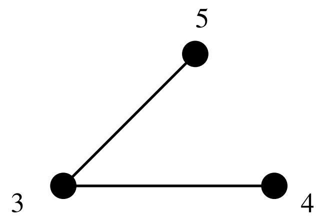
Theorem 2.23. *The weak Zariski topology graph $G_T(\omega^*)$ is connected and the diameter of the graph, $diam(G_T(\omega^*)) \leq 2$.*

Proof. If n and k are not adjacent then $C(n) \cup C(k) \neq T$. Now there exists vertices l and k such that $C(n) \cup C(l) = C(n \wedge l) = T$ and $C(m) \cup C(k) = C(m \wedge k) = T$. If $l = m$ then $n - l - k$ is a path of length two. If $l \neq m$ then $n - (l \wedge m) - k$ is a path of length two. Hence $G_T(\omega^*)$ is connected and $diam(G_T(\omega^*)) \leq 2$.

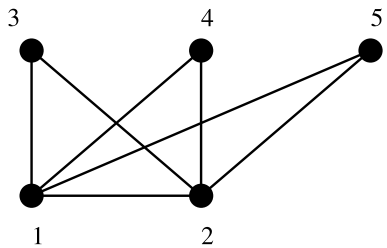
Corollary 2.24. *If $G_T(\omega^*)$ contains a cycle then the girth $g(G_T(\omega^*)) \leq 3$.*

Proof. Suppose $g(G_T(\omega^*)) = k > 3$. Let $n_1 - n_2 - n_3 - \cdots - n_{k-1} - n_k - n_1$ be a cycle with length k. Then clearly $n_1 - (n_2 \wedge n_{k-1}) - n_k - n_1$ is a cycle of length 3. Hence a contradiction. Therefore $g(G_T(\omega^*)) \leq 3$.

Example 2.25. i) Let $C = \{1, 2, 3, 4, 5\}$ and $(C, \leq)$ be complete lattice with $\leq$ as the usual ordering 'less than or equal to'. Let the locale be $(L = \{1, 2, 5\}, \leq)$. The action σ defined as $\sigma(a, x) = a \sqcap x$ will make C an L-component $(\sqcap, C)$.
In this case $Spec(\sigma, C) = \{2, 3, 4\}$ and $C(1) = Spec(\sigma, C)$, $C(2) = \{2, 3, 4\}$, $C(3) = \{3, 4\}$, $C(4) = \{4\}$, $C(5) = \phi$. Also, $C(n) \cup C(m) = C(n \wedge m)$ for every $n, m \in (\sqcap, C)$. If $T = \{3, 4\}$ then $V(G_T(\omega^*)) = \{3, 4, 5\}$. The graph $G_T(\omega^*)$ is $K_{1,2}$.

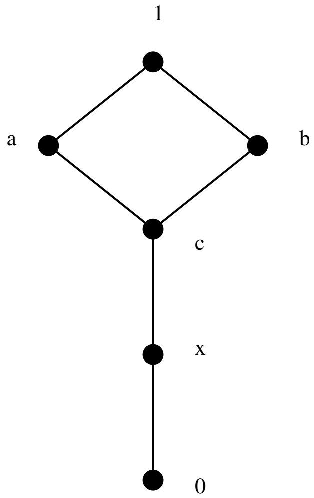


If $T = \{2, 3, 4\}$ then $V(G_T(\omega^*)) = \{1, 2, 3, 4, 5\}$ and the graph $G_T(\omega^*)$ is $K_{2,3}$.

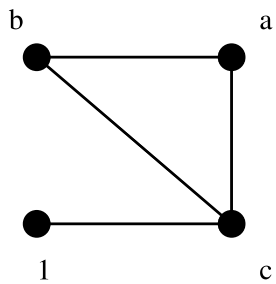


Also if $T = \{2, 4\}$ then $G_T(\omega^* = \phi)$

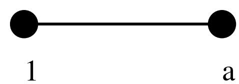
ii) Consider the complete lattice



Let $L = \{0, a, 1\}$. The spectrum $Spec(\sigma, C) = \{x, a, b\}$. $C(0) = Spec(\sigma, C)$, $C(1) = \phi$, $C(x) = \{x, a, b\}$, $C(c) = \{a, b\}$, $C(a) = \{a\}$, $C(b) = \{b\}$. It can be easily verified that $C(n) \cup C(m) = C(n \wedge m)$ for every pair $n, m \in (\sigma, C)$. For $T = \{x, a\}$ then $G_T(\omega^*) = \phi$. If $T = \{a, b\}$ then $V(G_T(\omega^*)) = \{1, c, a, b\}$ and the graph is



If $T = \{a\}$ then $V(G_T(\omega^*)) = \{1, a\}$ and the graph is



Remark. *Since $C(1_C) = \emptyset$ the top element 1_C will always belong to the vertex set and deg$(1_C) \geq 1$. Also deg(1_C) is the cardinality of the set $\{n \in (\sigma, C) : C(n) = T\}$.*

Proposition 2.26. *For any finite set T and $G_T(\omega^*) \neq \phi$ we have that*

$T \cap V(G_T(\omega^*) \neq \phi$.

Proof. Let $p \in T$ then we have $C(p) \cup C(\bigwedge_{q \in T, q \neq p} q) = T$.Therefore, $p \in V(G_T(\omega^*))$.

2.3. The subgraph $G_T'(\omega^*)$

Definition 2.27. *The subgraph $G_T'(\omega^*)$ of $G_T(\omega^*)$ has vertex set $V(G_T'(\omega^*))$ defined as $\{n \in (\sigma, C)$: there exist $l \in (\sigma, C)$ such that $C(n) \cup C(l) = T$, $C(n), C(l) \neq T, C(n) \cap C(l) = \phi\}$, where $(u, v) \in E(G_T'(\omega^*))$ if and only if $C(u) \cup C(v) = T, C(u) \cap C(v) = \phi$.*

Note that the degree of u is the number of vertices k with $C(v) = C(k)$.

Proposition 2.28. *$G_T'(\omega^*) \neq \phi$ if and only if $T = C(\bigwedge_{q \in T} q)$ and is disconnected.*

Proof. We have already shown $G_T(\omega^*) \neq \phi$ then $T = C(\bigwedge_{q \in T} q)$. Let $n, l \in V(G_T'(\omega^*))$, then $C(n) \cup C(l) = T, C(n) \cap C(l) = \phi$. Thus, T is disconnected. The converse follows easily from the definition.

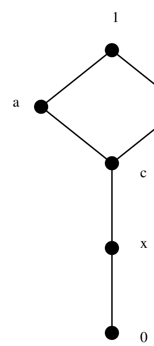
Theorem 2.29. *$G_T'(\omega^*)$ is a bipartite graph.*

Proof. A graph is bipartite if and only if it does not contain an odd cycle [4]. We will show that $G_T'(\omega^*)$ does not have an odd cycle. Suppose $g(G_T'(\omega^*)) = k > 4$. Consider the cycle $n_1 - n_2 - n_3 - \cdots - n_{k-1} - n_k - n_1$ of length k. It is evident that $C(n_{k-1}) = C(n_1)$. The cycle $n_1 - n_2 - n_3 - \cdots - n_{k-2} - n_1$ is of length $k - 1$. Thus $g(G_T'(\omega^*)) \leq 4$. We show that $g(G_T'(\omega^*)) \neq 3$. Suppose $n_1 - n_2 - n_3 - n_1$ is 3-cycle. Then $\phi = (C(n_1) \cap C(n_2)) \cup (C(n_3) \cap C(n_1)) = C(n_1) \cap (C(n_2) \cup C(n_3)) = C(n_1) \cap T = C(n_1)$.Thus we arrive at a contradiction. Hence the graph does not contain an odd cycle.

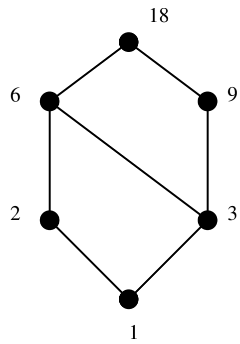
Corollary 2.30. *If $G_T'(\omega^*)$ contains a cycle then $gr(G_T'(\omega^*)) = 4$.*

Remark. *$G_T'(\omega^*)$ is a complete bipartite graph if and only if $C(n) = C(l)$ for every vertices l, n belonging to same vertex set.*

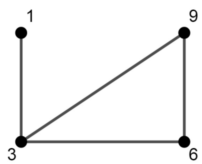
Example 2.31. i) Consider the complete lattice C to be



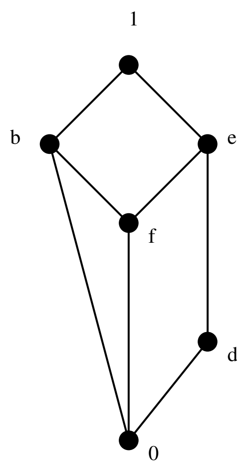
For $L = \{0, a, 1\}, Spec(\sigma, C) = \{x, a, b\}$.If $T = \{a, b\}$, then $V(G'_T(\omega^*)) = \{a, b\}$. Hence $G'_T(\omega^*)$ is $K_{1,1}$.Also if $T = \{a\}$, then $V(G'_T(\omega^*)) = \phi$.
ii) Consider the complete lattice C to be



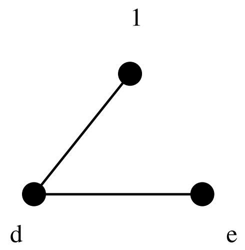
For $L = \{1, 2, 18\}, Spec(\sigma, C) = \{2, 6, 9\}$.If $T = \{6, 9\}$,
then $V(G_T(\omega^*)) = \{3, 6, 9, 18\}$ and the corresponding graph is



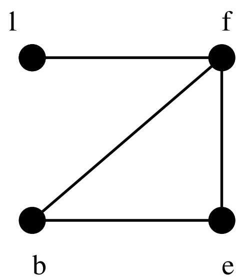
and $V(G'_T(\omega^*)) = \{6, 9\}$ and the graph is $K_{1,1}$.
iii) Consider the complete lattice C to be



$L = \{0, d, e, 1\}, Spec(\sigma, C) = \{b, d, e\}$. If $T = \{d, e\}$, then $V(G_T(\omega^*)) = \{d, e, 1\}$ and the corresponding graph is



and $V(G'_T(\omega^*)) = \phi$. If $T = \{b, e\}$, then $V(G_T(\omega^*)) = \{f, b, e, 1\}$ and the corresponding graph is



$V(G'_T(\omega^*)) = \{b, e\}$ and $G'_T(\omega^*)$ is $K_{1,1}$.

3. Conclusion

This paper introduces the concepts of total graphs and that of graphs associated with the weak Zariski topology. The introduction of concepts of algebraic graph theory into L-slices is initiated through this article. Different types of graphs can be studied in the background of L-slices. The topological properties of L-slices can be used to study the graphs associated with them.Graph theoretic development of L-slices led us to the total graph $\Gamma(T(\sigma, J))$ and $G_T(\omega^*)$ on L-slices. We have shown that if $T(\sigma, J)$ is a proper ideal of (σ, J) then $\Gamma(T(\sigma, J))$ is disconnected. Also we showed that $\Gamma(T(\sigma, J))$ is complete if and only if the L-slice (σ, J) is not faithful. We were also able to prove that the weak Zariski topology graph $G_T(\omega^*)$ is connected and $diam(G_T(\omega^*)) \leq 2$.

References

[1] Balakrishnan R., Ranganathan K., A Text Book of Graph Theory, Springer, 1991.

[2] Birkhoff G., Lattice Theory, American Mathematical Society.

[3] Beck I., Colouring of commutative rings, Journal of Algebra, 116 (1988), 208-226.

[4] Frank H., Graph Theory, Narosa Publishing House, 1988.

[5] George G., General lattice theory, Birkhauser, 2003.

[6] John C., Derek Allan H., A First Look at Graph Theory, World Scientific, 1991.

[7] Johnstone P. T., The point of pointless topology, Bulletin of American Mathematical society, 1983.

[8] Johnstone P. T., Stone Spaces, Cambridge University Press, 1982.

[9] Leonard G., Jerrison M., Rings of Continuous functions, Van Nostrand, 1960.

[10] Mary Elizabeth Antony, Sabna K. S., Mangalambal N. R., The Topology $\Omega^* \quad on\ Spec(\sigma, C)$ of L-component (σ, C), Malaya Journal of Matematik, University Press, Special Issue, No. 1 (2019), 441-444.

[11] Narsingh D., Graph theory with applications to engineering and computer science, PHI Learning Private Limited, 2011.

[12] Picado J., Pultr, Frames and locales: Topology without points, Front. Math., Springer, Basel, 2012.

[13] Sabna K.S., A Study on Properties of Locales, Action of Locales and their Applications, Ph.D. thesis, University of Calicut, 2018.

[14] Shariefuddin P., Rameez Raja, On graphs associated with modules over commutative rings, J. Korean Math. Soc., 5 (2016), 1167-1182.

[15] Vickers S., Topology via Logic, in: Cambridge Tracts in Theoretical Computer Science, Vol. 5, Cambridge University Press, Cambridge, 1985.

[16] Zoran P., The Total graph of a module, Matematicki vesnik, 63(4) (2011), 305-312.

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