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IDEALS AND ALMOST PRINCIPAL IDEALS IN EUCLIDEAN $\Gamma-$SEMIRINGS

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Abstract: The generalization of the ring of ordinary integers and their properties into an Euclidean ring is well known. Every ideal in an Euclidean ring is a principal ideal, as is also widely known. That is, the Euclidean ring is a principal ideal ring. This paper aims to generalize the $\Gamma-$semiring of non-negative integers and their properties by defining Euclidean $\Gamma-$semiring. A Euclidean $\Gamma-$semiring is one of the many special classes of $\Gamma-$semirings. Finally, the special class of $\Gamma-$semirings discussed in this paper is the class of almost principal ideal $\Gamma-$semirings.

Keywords and Phrases: Euclidean $\Gamma-$semiring, Almost Principal ideals and Almost Principal ideal $\Gamma-$semirings.
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1. Introduction

Semirings were first considered explicitly by Vandiver in (1934) [15] in connection with the axiomatization of the arithmetic of the natural numbers. Many scholars have investigated semirings, either independently or as part of an effort to branch out from ring theory or semigroup theory, or in connection with applications. Semirings never gained widespread acceptance, and although interest in them never fully waned among algebraists, it did gradually. Redei [8] and Almeida Costa [1] are the only authors to attempt to explain how the algebraic theory of semirings fits into modern algebra. The theory of rings and semigroups substantially impacted the development of the theory of semirings. Since ideals play an important role in advanced studies, generalizing ideals in algebraic structures is necessary for further study of algebraic structure.

As a generalization of the ring, the notion of a $\Gamma-$ ring was introduced by Nobusawa in (1964) [5]. The notion of $\Gamma-$ semigroup was introduced by Sen in (1981) [9] as a generalization of $\Gamma-$ groups. Murali Krishna Rao(1995) [6] introduced the notion of $\Gamma-$semiring as a generalization of $\Gamma-$ ring, ring, ternary semiring and semiring. The important reason for developing $\Gamma-$ semiring is a generalization of the results of rings, $\Gamma-$ rings, semirings, semigroup, and ternary semirings. Later on, much has been developed on this concept by different researchers.

The motivation for this paper is [2, 3, 4], in which Euclidean semiring is defined and the structure of ideals, as well as almost principal ideals, are given. This paper aims to generalize/characterize the $\Gamma-$ semiring of non-negative integers and their properties by defining Euclidean $\Gamma-$ semirings. Since the set of non-negative integers is not a principal ideal $\Gamma-$ semiring, a Euclidean $\Gamma-$ semiring is not expected to be a principal ideal $\Gamma-$ semiring. The structure of ideals in an Euclidean $\Gamma-$ semiring R is closely related to the function f with R. We will show that if B is a basis for an ideal in R, then f is restricted to B is bounded. Further, we prove an analogue of some of the well-known results for Euclidean $\Gamma-$ semirings by showing that a Euclidean $\Gamma-$ semiring is a principal $\Gamma-$ semiring and almost principal ideal $\Gamma-$ semiring. The final special class of $\Gamma-$ semirings is the class of almost principal ideal $\Gamma-$ semirings(**APIS**). Recall that an ideal is a principal ideal if a single element generates it.

2. Preliminaries

First, we recall some definitions of the basic concepts of $\Gamma-$semirings with examples and their ideals that we need in the sequel. For this, we follow [6, 7, 10, 11, 12].

Definition 2.1. *Let R and Γ be two additive commutative semigroups. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying the following conditions:*

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$.

(ii) $x\alpha(y + z) = x\alpha y + x\alpha z$.

(iii) $x(\alpha + \beta)z = x\alpha z + x\beta z$.

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$ *for all* $x, y, z \in R$ *and* $\alpha, \beta \in \Gamma$.

Example 2.2. Let A and B be semirings and let $R = Hom(A,B)$ and $\Gamma = Hom(B,A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings.

Example 2.3. Let M be a $\Gamma-$ ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring.

Example 2.4. Let $R = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\alpha, \beta\}$. We define operations with the following tables.

$+$	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_3	x_4	x_1
x_3	x_3	x_4	x_1	x_2
x_4	x_4	x_1	x_2	x_3

$+$	α	β
α	α	β
β	β	α

One can easily see that R and Γ are commutative semigroups.
Further,

α	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_1	x_1
x_3	x_1	x_1	x_1	x_1
x_4	x_1	x_1	x_1	x_1

β	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_3	x_4
x_3	x_1	x_3	x_1	x_3
x_4	x_1	x_4	x_3	x_2

Then R is a $\Gamma-$semiring.

The following definitions are from [6, 10, 12, 13, 14].

Definition 2.5. *A* $\Gamma-$*semiring* R *is said to have a zero element if* $0\gamma x = 0 = x\gamma 0$ *and* $x + 0 = x = 0 + x$ *for all* $x \in R$ *and* $\gamma \in \Gamma$.

Definition 2.6. *A* $\Gamma-$*semiring* R *is said to have an identity element* e *if for all* $x \in R$ *there exists* $\alpha \in \Gamma$ *such that* $e\alpha x = x = x\alpha e$.

Definition 2.7. *A* $\Gamma-$*semiring* R *is said to have a strong identity element* e, *if for all* $x \in R$, $e\alpha x = x = x\alpha e$, *for all* $\alpha \in \Gamma$.

Definition 2.8. *A* $\Gamma-$*semiring* R *is commutative if* $x\gamma y = y\gamma x$ *for all* $x, y \in R$ *and for all* $\gamma \in \Gamma$.

Definition 2.9. *A non empty subset* S *of a* $\Gamma-$*semiring* R *is said to be a sub* $\Gamma-$*semiring of* R *if* $(S, +)$ *is a sub semi group of* $(R, +)$ *and* $x\gamma y \in S$ *for all* $x, y \in S$ *and* $\gamma \in \Gamma$.

Definition 2.10. *A non-empty subset I of R is said to be the left (right) ideal of R if I is sub semigroup of $(R,+)$ and $x\alpha y \in I(y\alpha x \in I)$ for all $y \in I, x \in R$ and $\alpha \in \Gamma$.*

If R is a $\Gamma-$semiring with zero elements, then it is easy to verify that every ideal of R has zero elements.

Definition 2.11. *If I is the left and right ideal of R, then I is known to be an ideal of R.*

Definition 2.12. *A proper ideal M of a $\Gamma-$ semiring R is said to be a maximal ideal if there does not exist any other proper ideal of R containing M properly.*

Definition 2.13. *A non-zero element x in a $\Gamma-$ semiring R is a left zero divisor if and only if there exists a non-zero element $y \in R$ and $\alpha \in \Gamma$ satisfying $x\alpha y = 0$. It is a right zero divisor if and only if non-zero $y \in R$ exists and $\alpha \in \Gamma$ satisfying $y\alpha x = 0$. It is a zero divisor if and only if it is both a left and right zero divisor.*

Definition 2.14. *let R be a $\Gamma-$semiring. Then $x, y \in R$ are associates if there exists an element $u \in U(\Gamma R)$ and $\alpha \in \Gamma$ such that $x = u\alpha y$.*

Note that if R is a $\Gamma-$semiring with strong identity then $y = u^{-1}\beta x$ for some $\beta \in \Gamma$ and $R\Gamma x = R\Gamma y$.

Definition 2.15. *A function $f : R \to S$, where R and S are $\Gamma-$semirings is said to be a $\Gamma-$morphism of $\Gamma-$semirings if*

(i) $f(x + y) = f(x) + f(y)$,

(ii) $f(x\alpha y) = f(x)\alpha f(y)$ *for all* $x, y \in R$ *and* $\alpha \in \Gamma$.

Throughout this paper, we will consider R as an Euclidean $\Gamma-$semiring, with strong identity e.

3. Ideals in Euclidean $\Gamma-$semirings

The problem of generalizing the non-negative integers and their characteristics is interesting. In an ordinary Euclidean ring, the function defined on the ring satisfies specific properties for the division algorithm and the product of two elements. However, there is no connection between the sum of the two components. It is essential to set a condition on the function related to the sum of two elements to study the ideals in Euclidean semirings. In case of non-negative integers, we have $|x + y| = |x| + |y| \geq |x|$. Dale and Hanson [2] have defined Euclidean semiring by adding this property to the definition of an Euclidean ring. Analogously, we define Euclidean $\Gamma-$semiring in definition 3.6.

The following definition of antisimple(principal) $\Gamma-$semiring is similar to the definition defined for principal semirings in [2].

Definition 3.1. *Let R be a commutative $\Gamma-$semiring with strong identity e. Define the Principal part of R as $P(R) = \{r \in R \mid$ there exists $x \in R$ such that $r = x + e\} \cup \{0\}$. If $R = P(R)$ then R is called an antisimple(principal) $\Gamma-$semiring.*

It is noted that the identity e may or may not be in $P(R)$. However, $0 \in P(R)$.

Example 3.2. [2] Let $R = \mathbb{N}$ be a $\Gamma-$semiring , $\mathbb{N}$ be a set of non negative integers. Let $x \in R$. Since $0 \in P(R)$ and $1 = 0 + 1 \in P(R)$, so take $x \neq 0, x \neq 1$. As x is not an identity, so, to Peano's postulates, there exists $y \in R$ such that $x = y + 1$. Hence, R is a principal $\Gamma-$semiring.

Example 3.3. [2] Let $a, b \in \mathbb{N}, \mathbb{N}$ be a set of non negative integers such that $b > a$ and $R = [a, b]$. Let us define $x + y = max(x, y)$ and $x\alpha y = min(x, y)$ in R. Then, R is closed, commutative, and associative under the operations of addition and multiplication. Also for all $x \in R, a + x = max(a, x) = x$ and $x\alpha b = min(x, b) = x$. Therefore, a and b are additive and multiplicative identities of R. Hence R is a commutative $\Gamma-$semiring with identities a and b. Now let $x \neq b \in R$ and let there exists $y \in R$ such that $x = y + b = max(y, b) = b$, which is a contradiction. Hence $P(R) = b \cup a = \{b, a\}$ and R is not a principal $\Gamma-$semiring.

The above examples indicate that the principal part of a $\Gamma-$semiring may be trivial or non-trivial. In this paper, we consider the case when $R = P(R)$ is principal $\Gamma-$semirings.

Theorem 3.4. *Let R be a commutative $\Gamma-$semiring with a strong identity e, then $P(R)$ is a sub-$\Gamma-$semiring of R.*

Free of zero divisors property of a Euclidean $\Gamma-$semiring R is directly follows from the definition of zero divisors as follows:
Let $x\alpha y = 0$ with $y \neq 0$ for all $x, y \in R$, and for all $\alpha \in \Gamma$. Then $f(x)g(\alpha)f(y) = f(x\alpha y) = f(0) = 0$. Since $y \neq 0$ so $f(y) \neq 0$. Therefore, $f(x) = 0$ implies that $x = 0$.

Definition 3.5. *Let R be a Euclidean $\Gamma-$semiring and $0 \neq x \in R$. Then an element $y \in R$ is divisor of x, if there exists $r \in R, \alpha \in \Gamma$ such that $x = r\alpha y$.*

The following definition is analogous to the definition given in [2, 4].

Definition 3.6. *Let R be a principal $\Gamma-$semiring and free of zero divisors with a function $f : R \to \mathbb{N}$, $\mathbb{N}$ the set of non negative integers and $g : \Gamma \to \{1\}$ then R is called a Euclidean $\Gamma-$semiring if it satisfies the following properties:*

(i) for $x \in R$, $f(x) = 0$ if and only if $x = 0$,

(ii) *for all* $x, y \in R$, *if* $x + y \neq 0$ *then* $f(x + y) \geq f(x)$.

(iii) *for all* $x, y \in R$ *and* $\alpha \in \Gamma$, $f(x\alpha y) = f(x)g(\alpha)f(y))$,

(iv) *for all* $x, y \neq 0 \in R$, *there exists* $q, r \in R, \alpha \in \Gamma$ *such that* $x = q\alpha y + r$ *where either* $r = 0$ *or* $f(r) < f(y)$.

Example 3.7. The set of non-negative integers $\mathbb{N}$ with $\Gamma = \{1\}$, is a Euclidean $\Gamma-$semiring. For, let $f(n) = n$ for all $n \in \mathbb{N}$. The four properties of Euclidean $\Gamma-$semirings are satisfied.

We now characterize the ideals in a Euclidean $\Gamma-$semiring.
Let R be a Euclidean $\Gamma-$semiring and $a \in R$, then we denote the set $T_a = \{0\} \cup \{x \in R \mid f(x) \geq f(a)\}$.

Theorem 3.8. *Let* R *be a Euclidean* $\Gamma-$*semiring and* $a \in R$, *then* T_a *is an ideal of* R.

Remark 3.9. *Let* R *be an Euclidean* $\Gamma-$*semiring with strong identity* e. *Since* $f(a) = 0$ *if and only if* $a = 0$ *and* $f(e) = 1$, *it is clear that* $T_0 = T_e = R$. *Similarly, we give some properties of ideals of the form* T_a *in the following theorem, which is proved in* [2] *and holds for* $\Gamma-$*semirings.*

Theorem 3.10. *Let* R *be a Euclidean* $\Gamma-$*semiring and* $a, b \in R$. *Then*

(i) $T_a \subseteq T_b$ *if and only if* $f(a) \geq f(b)$,

(ii) $T_a \cup T_b = T_c$ *where* $f(c) = min\{f(a), f(b)\}$.

(iii) $T_a \cap T_b = T_c$, *where* $f(c) = max\{f(a), f(b)\}$,

(iv) *If* $\{a_i\}$ *is a sequence of elements in* R *such that* $f(a_i) < f(a_{i+1})$, *then* $\cap T_{a_i} = \phi$

Proof. The proofs of $(i) - (iii)$ are simple and straightforward.
(iv) Let $x \in T_{a_i}$ and $f(x) = n$. Since $\{f(a_i)\}$ is an increasing sequence of positive integers, therefore there is an a_j such that $f(a_j) > n = f(x)$. This implies that $x \notin \cap T_a$, and so $\cap T_{a_i} = \phi$.

Theorem 3.11. *Let* R *be an Euclidean* $\Gamma-$*semiring with strong identity* e. *Let* T_a *be an ideal of* R *and* $B_a = \{x \in T_a \mid x = a + y, y \in R$ *and* $f(y) < f(a)\}$. *For* $a, b \in R$, *let* $R[a, b) = \{x \in R \,|f(a) \leq f(x) \leq f(b)\}$. *Then* $B_a = R[a, 2a)$.
Proof. First, we will show that $f(B_a)$ is bounded. Let $x = a + y \in B_a$. Then $f(y) < f(a)$ and for all $a, y \in R$ there exist $q, r \in R, \alpha \in \Gamma$ such that $a = q\alpha y + r$, where either $r = 0$ or $f(r) < f(y)$. Since R is a principal $\Gamma-$semiring, $q = q' + e$

for some $q' \in R$, so $f(2a) = f(a+a) = f(a+q\alpha y+r) = f(a+(q'+e)\alpha y+r) = f(a+y+q'\alpha y+r) \geq f(a+y) = f(x)$. Thus, $f(B_a)$ is bounded by $f(2a)$. But by definition of B_a, $x \neq 2a$ and it follows that $B_a \subseteq R[a, 2a)$. Now let $t \in R[a, 2a)$. Then $f(a) \leq f(t) \leq f(2a)$. Therefore, for all $a, t \in R$ there exist $p, r \in R, \beta \in \Gamma$ such that $t = p\beta a+r$, where either $r = 0$ or $f(r) < f(a)$. If $p \neq e$, then there exists p' such that $p = p'+e$. Therefore, $t = p\beta a+r$ gives that $t = (p'+e)\beta a+r = p'\beta a+a+r$ If $p' = e$, then $f(t) = f(a+a+r) \geq f(a+a) = f(2a)$, which is a contradiction. If $p' \neq e$, then $p' = p''+e$ for some $p'' \in R$. Then, $t = p'\beta a+a+r = (p''+e)\beta a+a+r = p''\beta a+a+a+r$ and $f(t) = f(p''\beta a+a+a+r) \geq f(a+a) = f(2a)$,which is again a contradiction. Therefore, $p = e$ and $t = a+r \in B_a$. Thus, $R[a, 2a) \subseteq B_a$ and it follows that $B_a = R[a, 2a)$.

Theorem 3.12. *Let R be an Euclidean $\Gamma-$semiring with strong identity e. Then $R[a, 2a)$ is a basis for T_a.*

Proof. Let $x \in T_a$. Then for all $a, x \in R$ there exists $q, r \in R, \alpha \in \Gamma$ such that $x = q\alpha a+r$ where either $r = 0$ or $f(r) \leq f(a)$. If $r = 0$ or $q = e$, then the result is obvious. Again, let $r \neq 0$ and $q \neq e$. Since R is a principal $\Gamma-$semiring, therefore $q = q'+e$ for some $q' \in R$. Then $x = q\alpha a+r = (q'+e)\alpha a+r = q'\alpha a+(a+r)$, where $a+r \in R[a, 2a)$. Hence, $R[a, 2a)$ is a basis for T_a.

Let I be an ideal of R and $R[a, 2a) \subseteq I$ for some $a \in I$, then it is clear that $T_a \subseteq I$. We will find some more conditions which will guarantee that $T_a \subseteq I$ for some $a \in I$. Some of such conditions are given in the following three results:

Theorem 3.13. *Let R be an Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R. If $a \in I$ and $R[k\alpha a, (k+e)\alpha a] \subseteq I$ for some $k \in I, \alpha \in \Gamma$, then $T_{k\alpha a} \subseteq I$.*

Proof. Let $x \in R[k\alpha a, 2(k\alpha a)] - R[k\alpha a, (k+e)\alpha a]$. Then $f(k\alpha a+a) < f(x) < f(2(k\alpha a))$. Also, for all $a, x \in R$ there exist $p, r \in R, \alpha, \beta \in \Gamma$ such that $x = p\beta(k+e)\alpha a+r$, where either $r = 0$ or $f(r) \leq f(k\alpha a+a)$. Again, for all $a, r \in R$ there exist $s, t \in R, \gamma, \beta \in \Gamma$ such that $r = s\gamma a+t$ where either $t = 0$ or $f(t) < f(a)$. Since R is a principal $\Gamma-$semiring, so there exists $q \in R$ such that $p = q+e$. Now, $x = p\beta(k+e)\alpha a+r = (q+e)\beta(k\alpha a+a)+(s\gamma a+t) = q\beta(k\alpha a+a)+(k\alpha a+a)+(s\gamma a+t) = q\beta(k\alpha a+a)+(s\gamma a+a)+(k\alpha a+t) = q\beta(k\alpha a+a)+(s+e)\gamma a+(k\alpha a+t)$. Now, $q\beta(k\alpha a+a) \in I$, $(s+e)\gamma a \in I$ and $k\alpha a+t \in I$, since $f(k\alpha a) \leq f(k\alpha a+t) < f(k\alpha a+a) = f((k+e)\alpha a)$ This implies that $x \in I$ and $R[k\alpha a, 2(k\alpha a)) \subseteq I$. Hence, by Theorem 3.12, $T_{k\alpha a} \subseteq I$.

Theorem 3.14. *Let R be a commutative Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R. If there exists $a \in I$ such that $a+e \in I$ then $T_{a\alpha a} \subseteq I$.*

Proof. Let $a, a+e \in I$ and $x \in R(a\alpha a, a\alpha a + a)$. Then $x = a\alpha a + z$ where $f(z) < f(a)$. Now, for all $a, z \in R$ there exists $p, r \in R, \beta \in \Gamma$ such that $a = p\beta z + r$, where either $r = 0$ or $f(r) < f(z)$. Since R is a principal $\Gamma-$semiring, so $p = q + e$ for some $q \in R$. Thus, $x = a\alpha a + z = (p\beta z + r)\alpha a + z = p\beta z\alpha a + r\alpha a + z = z\beta(p\alpha a + e) + r\alpha a = z\beta[(q+e)\alpha a + e] + r\alpha a = z\beta[q\alpha a + (a+e)] + r\alpha a \in I$. Thus, $R[a\alpha a, a\alpha a + a) \subseteq I$, so by theorem 3.13, $T_{a\alpha a} \subseteq I$.

The following definition is analogous to the definition in [2].

Definition 3.15. *Let R be a Euclidean $\Gamma-$semiring and $x \neq 0, y \in R$. An element $d \in R$ will be called the greatest common divisor of x and y if:*

(i) d is a common divisor of both x and y.

(ii) If c is another common divisor of both x and y, then $f(c) \leq f(d)$.

Definition 3.16. *Let R be an Euclidean $\Gamma-$semiring, then for any $ab \in R$. We say a and b are relatively prime if their greatest common divisor is identity e.*

In Euclidean rings, for any two integers a and b, we know that there exists a greatest common divisor d, for which there are integers s and t such that $d = sa + tb$. Moreover, for Euclidean semirings, we will assume that if $a, b \in \mathbb{N}$, a set of non-negative integers, then either $sa = tb + d$ or $tb = sa + d$. We now extend this property to Euclidean $\Gamma-$semirings.

It is easy to check that in a Euclidean $\Gamma-$ semiring, the greatest common divisor of two elements a and b can be written in the form $d = s\alpha a + t\beta b$, where $s, t \in R$, and some $\alpha, \beta \in \Gamma$. Consequently for Euclidean $\Gamma-$semrings, we will assume that either $s\alpha a = t\beta b + d$ or $t\beta b = s\alpha a + d$. Studying several forms of the division algorithm in Euclidean semirings is necessary. It will be seen that this is true also for the study of Euclidean $\Gamma-$semirings.

Theorem 3.17. *Let R be an Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R. If $a, b \in I$, such that a and b are relatively prime, then there exists $c \in I$ such that $T_c \subseteq I$.*

Proof. Let a and b be relatively prime, then there exist $x, y \in R$ and $\alpha, \beta \in \Gamma$, such that either $x\alpha a = y\beta b + e$ or $y\beta b = x\alpha a + e$. Suppose $x\alpha a = y\beta b + e$. Now, $y\beta b \in I$ and $y\beta b + e = x\alpha a \in I$. So by theorem 3.13, $T_{y\beta b} \subseteq I$. Similarly, if $y\beta b = x\alpha a + e$ then it follows that $T_{x\alpha a} \subseteq I$. Therefore, in either case, there exists c such that $T_c \subseteq I$.

If $f(x) \neq f(y)$, then it is clear that $f(T_x)$ and $f(T_y)$ can differ by only a finite number of non-negative integers. Therefore, if I is an ideal containing T_a, then $R \supseteq I \supseteq T_a$. Thus, $f(R)$ and $f(I)$ can differ by only finitely many numbers of non-negative integers.

Theorem 3.18. *Let R be an Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R such that $T_a \subseteq I$ for some $a \in I$. Then there exists $x \in I$ such that T_x is maximal in I and $I = L \cup T_x$, where $L = \{y \in I \mid 0 < f(y) < f(x)\}$.*
Proof. The result follows the principle of well-ordering and Theorem 3.10.

It is easy to show that if T_a is an ideal of $R, b \in R$, then $b\Gamma T_a$ is an ideal of R. From the above theorem, it is clear that any basis of I in $L \cup R[x, 2x)$. If I is an ideal of R, then it may not contain any ideal of the form T_a. For this, if $a \in I$ with $f(a) > 1$, then for any x, there is no ideal of the form T_x in principal ideal (a). This is obvious as $(a) = \{a\alpha x \mid x \in R\}$ and $f(a\alpha x) = f(a)g(\alpha)f(x)$. Therefore, $f((a))$ has elements only, multiples of $f(a)$. Thus, $(a) = a\Gamma T_e$. By considering the ideal of the form $b\Gamma T_a$, we want to generalize this case. It is easy to show that if T_a is an ideal of $R, b \in R$, then $b\Gamma T_a$ is an ideal of R. By theorem 3.10, it follows that $b\Gamma T_a \subseteq b\Gamma T_c$ if and only if $f(a) \geq f(b)$.

Now, we want to show that if I is an ideal of R such that I contains no ideal of the form T_a, then I has an ideal of the form $b\Gamma T_a$. For this, we need some properties for this type of ideal.

Theorem 3.19. *Let R be an Euclidean $\Gamma-$semiring with strong identity e. Then $b\Gamma R[a, 2a)$ is a basis for $b\Gamma T_a$.*
Proof. This proof directly follows from Theorem 3.12.

It is clear now that for any $a \in I$, the ideal $(a) = a\Gamma T_e \subseteq I$. But we want to find ideals of the form $b\Gamma T_a$ such that $a \neq e$, and also find conditions that will ensure that an ideal I will contain $b\Gamma T_a$. As $R[a, 2a) = B_a = \{y \mid y = a + r$ where $f(r) < f(a)\}$, therefore for any $\alpha \in \Gamma$, we have $d\alpha R[a, 2a) = \{y \mid y = d\alpha a + r$ where $f(r) < f(d\alpha a)$ and d divides $r\}$.

Theorem 3.20. *Let R be an Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R. If $a \in I$ and $d\Gamma R[k\alpha a, (k + e)\alpha a] \subseteq I$, then $d\Gamma T_{k\alpha a} \subseteq I$.*

Theorem 3.21. *Let R be a commutative Euclidean $\Gamma-$semiring with strong identity e and I be an ideal of R. If d is a divisor of a and $a + d \in I$ then $d\Gamma T_a \subseteq I$.*
Proof. If d is divisor of a, then there exists $k \in R$ and $\alpha \in \Gamma$ such that $a = d\alpha k$. Let $x \in d\Gamma R[a, a + d]$. Then $d\Gamma R[a, a + d] = d\Gamma R[d\alpha k, (k + e)\alpha d]$. This implies that for some $\beta \in \Gamma$, we have $x = d\beta(d\alpha k) + z$, where $f(z) < f(d\alpha k)$ and d divisor of z. Therefore, there exists $y \in R$ and $\delta \in \Gamma$ such that $z = y\delta d$. Further, for all $d, z \in R$, there exist $p, r \in R$, $\gamma \in \Gamma$ such that $d = p\gamma z + r$, where either $r = 0$ or $f(r) < f(z)$. Since R is a principal $\Gamma-$semiring, we have $d = t + e$ and $p = q + e$. From all this, we have, $x = d\beta(d\alpha k) + z = (d\alpha k)\beta d + z = (d\alpha k)\beta(p\gamma z + r) + z = (a\beta p\gamma z + z) + a\beta r = y\delta(a\beta p\gamma d + d) + a\beta r = y\delta(a\beta(q + e)\gamma(t + e) + d) + a\beta r = y\delta(a\beta(q\gamma t + q + t + e) + d) + a\beta r = y\delta(a\beta(q\gamma t + q + t) + (a + d)) + a\beta r$ But $a, a + d \in I$,

therefore it follows that $x \in I$. Thus, $d\Gamma R[a, a + d] \subseteq I$ and from Lemma 3.20 it follows that $d\Gamma T_a \subseteq I$.

Corollary 3.22. *Let R be a commutative Euclidean $\Gamma-$semiring and I be an ideal of R. Let $a, b \in R$ and d be a greatest common divisor of a, b, then $d\Gamma T_c \subseteq I$ for some $c \in I$.*

The proof of the following theorem is simple.

Theorem 3.23. *Let R be a commutative Euclidean $\Gamma-$semiring and $p, q, c, d \in R$. Then*

(i) If p is divisor of q, then $d\Gamma T_q \subseteq d\Gamma T_p$.

(ii) If c is divisor of d, then $d\Gamma T_p \subseteq c\Gamma T_p$.

Finally, we have the structure theorem whose semiring version is in [2].

Structure Theorem

Let I be an ideal of R such that it does not contain any ideal of the form T_x for any $x \in I$. Let $a, b \in R$ with d as the greatest common divisor. Now, $d \neq e$, otherwise from theorem 3.17, we get $T_c \subseteq I$ for some $c \in I$, which is a contradiction. By Corollary 3.22 it follows that $d\Gamma T_p \subseteq I$ for some $p \in I$. Now, let $U = \{d \mid d$ is the greatest common divisor of some $a, b \in I\}$ and let $k \in U$ such that $f(k)$ is minimum. Let $V = \{p \mid d\Gamma T_p \subseteq I\}$ and $q \in V$ is such that $f(q)$ is minimum. As no two elements in I are co-prime, $k \neq e$ and k divide d for all $d \in U$. Then, by applying theorem 3.23 and theorem 3.10, we get $k\Gamma T_a$ is maximal in I. Now if we set $L = \{t \in I | f(t) < f(k\alpha q)\}$ for all $\alpha \in \Gamma$ we get $I = L \cup k\Gamma T_q$, where $L \cap k\Gamma T_q = \{0\}$. Now $k\Gamma R[q, 2q)$ is a basis for $k\Gamma T_a$. Therefore, $L \cup k\Gamma R[q, 2q)$ is a basis for I and moreover this basis is bounded by $\underset{\alpha \in \Gamma}{Min}\{f(2(k\alpha q)\}$.

Now, the theorem 3.18 proves the following structure theorem for ideals in a Euclidean $\Gamma-$semiring.

Theorem 3.24. *Let R be an Euclidean $\Gamma-$semiring and I be an ideal of R. Let $L = \{t \in I \mid f(t) < f(d\alpha p)\}$, for all $\alpha \in \Gamma$, then $I = L \cup d\Gamma T_p$, where $d\Gamma T_p$ is maximal in I and $L \cap d\Gamma T_p = \{0\}$. Moreover, $L \cup d\Gamma R[p, 2p)$ is a basis for I whose images are bounded by $\underset{\alpha \in \Gamma}{Min}\{f(2(d\alpha p)\}$.*

4. Almost Principal Ideals in Euclidean $\Gamma-$ Semirings

A Euclidean $\Gamma-$semiring is one of the many special classes of $\Gamma-$semirings. The classes of $\Gamma-$semirings discussed in this section are analogous to some of the semirings discussed in [3]. In $\Gamma-$semiring R, an element u is called a unit if there exists $a \in R, \alpha \in \Gamma$ such that $u\alpha a = e$. We denote the set of all units in R by $U(\Gamma R)$.

The set $U(\Gamma R)$ is multiplicatively closed. Let $R^* = R - \{0\}$. A $\Gamma-$semiring R is called $\Gamma-$ division semiring if $R^* = U(\Gamma R)$.

Definition 4.1. *Let R be a $\Gamma-$semiring. An ideal I of R is an almost principal ideal of R if a finite set F exists such that $I \cup F$ is a principal ideal. A $\Gamma-$semiring R is called an almost principal ideal $\Gamma-$semiring(APIS) if every ideal in R is an almost principal ideal.*

Example 4.2. Every principal ideal domain is an almost principal ideal $\Gamma-$semiring. The set of non-negative integers $\mathbb{N}$ is an almost principal ideal $\Gamma-$semiring.

Theorem 4.3. *Let R be a Euclidean $\Gamma-$semiring and $u \in R$. Then u is a unit in R if and only if $f(u) = 1$.*
Proof. Let $u \in R$ be a unit. Then there exists $u' \in R, \alpha \in \Gamma$ such that $u\alpha u' = e$. Therefore $f(u)g(\alpha)f(u') = f(u\alpha u') = f(e) = 1$. Since $f(u)$ and $f(u')$ are non-negative integers. Therefore it follows that $f(u) = 1$. Conversely, suppose that $f(u) = 1$. Then by applying the division algorithm, there exists $q, r \in R, \beta \in \Gamma$ such that $e = q\beta u + r$ where $r = 0$ or $f(u) > f(r)$. But $1 = f(u) > f(r)$ implies that $f(r) = 0$ and thus $r = 0$. Therefore, $e = q\beta u$, and hence u is a unit.

Corollary 4.4. *Let R be a Euclidean $\Gamma-$semiring. If u is a unit and $a \in R$ then u divides a.*

Theorem 4.5. *Let R be a Euclidean $\Gamma-$semiring. Then R is a $\Gamma-$ division semiring if and only if f is bounded on R.*
Proof. If R be a $\Gamma-$ division semiring and $a \in R$. Then, either $a = 0$ or a is a unit. Therefore $f(a) = 0$ or $f(a) = 1$ and f is bounded on R. Conversely, let f be bounded on R. Then there exists an integer m such that $f(a) \leq m$ for all $a \in R$. Let $y \in R$ such that $y \neq 0$. If $f(y) = k > 1$, then there exists an integer, say n such that $f[(y\alpha)^{n-1}y] = [f(y)]^{n-1}\alpha f(y) > m$, which contradicts the assumption that f is bounded on R. Therefore, $f(y) = 1$. So y is a unit. Hence, R is a $\Gamma-$division semiring.

We introduce the cancellation property relative to the function f defined on $\Gamma-$semiring.

Definition 4.6. *A Euclidean Γ-semiring R is said to have f-cancellation if $a, b, c, d \in R, a + b = c + d$ and $f(a) = f(c)$, then $f(b) = f(d)$.*

Example 4.7. Clearly, the Euclidean $\Gamma-$ semiring $\mathbb{N}$, the set of non negative integer has the $f-$ cancellation property.

Theorem 4.8. *Let R be a Euclidean $\Gamma-$semiring with $f-$cancellation and strong identity e. Then $f(a) = f(b)$ if and only if a and b are associates.*

Proof. Let a and b be associates, then there exists $\alpha \in \Gamma$ and a unit u such that $a = b\alpha u$. Now, $f(a) = f(b\alpha u) = f(b)g(\alpha)f(u) = f(b)$. Conversely, let $f(a) = f(b)$. Then by division algorithm we have $a = b\beta q + r$ for some $q \in R, \beta \in \Gamma$ where $r = 0$ or $f(r) < f(b)$. Therefore, $f(a) = f(b\beta q + r) \geq f(b\beta q) = f(b)g(\beta)f(q)$ Consequently, $f(q) = 1$. This implies that q is a unit. Therefore $f(a) = f(b\beta q)$. Now as $a + 0 = b\beta q + r$, therefore $f-$cancellation property gives that $f(r) = 0$. Thus $r = 0$. Hence, a and b are associates.

Remark 4.9. *Let Γ be finite and consider two classes of $\Gamma-$semirings, the class with finite elements of units and the class in which every non-zero element is a unit (the class of $\Gamma-$division semirings, in which the only ideals are the trivial ideals. Hence these are principal ideal $\Gamma-$semirings and therefore almost principal ideal $\Gamma-$semirings.*

Now we restrict our attention to the former class of $\Gamma-$semirings and show that every Euclidean $\Gamma-$semiring, with finite Γ and $U(\Gamma R)$ is an almost principal ideal $\Gamma-$semiring.

Theorem 4.10. *Let R be a Euclidean $\Gamma-$semiring with $f-$ cancellation. If Γ and $U(\Gamma R)$ are finite, then for all $a, d \in R$, $d\Gamma T_a$ is an almost principal ideal.*
Proof. We first show that $R - T_a$ is finite. If $x \in R - T_a$, then $f(x) < f(a)$. Let us suppose that $R - T_a$ is not finite. Then $R - T_a$ must have a infinite subset with distinct elements, say $K = \{x_0, x_1, \ldots, x_n, \ldots\}$ such that $f(x_i) = f(x_j) = m < f(a)$ for all non-negative integers i and j. Since $U(\Gamma R)$ is finite, so let $U(\Gamma R) = \{u_1, u_2, \ldots, u_t\}$. Again, $f(x_0) = f(x_1)$, it follows from theorem 4.8 that $x_0 = u_{i_1}\alpha_1 x_1$ where $\alpha_1 \in \Gamma$ and $1 \leq i_1 \leq t$. Continuing this way, we get $x_0 = u_{i_k}\alpha_k x_k$ for each positive k and $\alpha_k \in \Gamma$. Let $s > t$, then $x_0 = u_{i_s}\alpha_s x_s$ where $\alpha_s \in \Gamma$ and $1 \leq i_s \leq t$. But $U(\Gamma R)$ has only t elements, therefore $u_{i_s} = u_{i_r}$ for some i_r with $1 \leq i_r \leq t$. Hence $u_{i_s}\alpha_s x_s = x_0 = u_{i_r}\alpha_r x_r = u_{i_s}\alpha_r x_r$ and since u_{i_s} is a unit, this implies that $x_r = x_s$. But this contradicts the hypothesis that K has distinct elements. Therefore, $R - T_a$ is finite. Consequently, $d\Gamma[R - T_a]$ is finite. Let $(d) - d\Gamma T_a = d\Gamma R - d\Gamma T_a = d\Gamma[R - T_a]$. Hence $(d) - d\Gamma T_a$ is finite and $d\Gamma T_a \cup [(d) - T_a] = (d)$ is a principal ideal. Hence, $d\Gamma T_a$ is an almost principal ideal.

Theorem 4.11. *Let R be a Euclidean $\Gamma-$semiring with $f-$cancellation and $U(\Gamma R)$ and Γ are finite, then R is an almost principal ideal $\Gamma-$semiring.*
Proof. Let I be an ideal in R such that $I \neq \{0\}$. Then By Theorem 3.24 $I = L \cup d\Gamma T_a$, where $d\Gamma T_a$ is maximal in I, $L = \{t \in I | f(t) < f(d)g(\alpha)f(a) = f(d\alpha a)\}$ and $L \cap d\Gamma T_a = \{0\}$. Let $y \in L$. Therefore $f(y) < f(d\alpha a)$, for all $\alpha \in \Gamma$. Let g be the greatest common divisor of y and $d\alpha a$. Then, by Corollary 3.22, it follows

that there exists $c \in I$ such that $g\Gamma T_c \subseteq I$. Now as $d\Gamma T_a$ is maximal in I, therefore $g\Gamma T_c \subseteq d\Gamma T_a$ and this implies that d divides g. But g divides y, therefore d is a divisor of y. Therefore we have $y = d\beta x$ for some $x \in R, \beta \in \Gamma$. Hence $y \in (d)$. Moreover $f(d)g(\beta)f(x) = f(d\beta x) = f(y) < f(d)g(\beta)f(a)$. Therefore it follows that $f(x) < f(a)$ and $y \in (d) - d\Gamma T_a$. Then $L \subseteq (d) - d\Gamma T_a$. Also $(d) - T_a$ is finite, therefore L is finite. So by theorem 4.10 $d\Gamma T_a$ is an almost principal ideal. Now $d\Gamma T_a \subseteq I = L \cup d\Gamma T_a \subseteq (d)$ implies that I is an almost principal ideal.

Hence, Euclidean $\Gamma-$semirings, with finite Γ and have finite units, as well as those Euclidean $\Gamma-$semirings in which every non-zero element is a unit, are almost principal ideal $\Gamma-$semirings.

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