South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 3 (2023), pp. 253-260

DOI: 10.56827/SEAJMMS.2023.1903.20 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

LIE ALGEBRA BUNDLES OF FINITE TYPE AND NUMERABLE BUNDLES

K. Kamalakshi and R. Rajendra

Department of Mathematics
Field Marshal K. M. Cariappa College,
Madikeri, Kodagu - 571201, Karnataka, INDIA

E-mail : kamalakshiaithal@yahoo.com, rrajendrar@gmail.com

(**Received:** *Jun. 19, 2023* **Accepted:** *Dec. 19, 2023* **Published:** *Dec. 30, 2023)*

Abstract: In this paper, we study the relation between numerable Lie algebra bundle and Lie algebra bundle of finite type. The effect of finite type on shrinkable maps and homotopy equivalence are examined. Further, we obtain some results on Section Extension Property, Covering Homotopy Property and Weak Covering Homotopy Property for Lie algebra bundles of finite type.

Keywords and Phrases: Lie algebra bundles of finite type, universal bundle, shrinkable maps, homotopy, numerable covering, numerable bundle.

2020 Mathematics Subject Classification: 17BXX, 18AXX, 55PXX, 55P10, 55RXX, 57R22.

1. Introduction

Following the procedure of Huebsch [5] and Hurewicz [6], Dold got the results on Covering Homotopy property(CHP) as a consequence of section extension theorem [3]. The necessary and sufficient condition for Covering Homotopy Property(CHP) was examined and CHP for induced spaces has been studied in [3]. To study fibre homotopy equivalence, Dold considered Weak Covering Homotopy Property(WCHP). The results were customised for spaces with numerable covering. The notions of numerable covering and numerable bundles introduced by Dold in [3] pave the way for some results on bundles of finite type, that we examine in this paper. Here all underlying vector spaces are real and finite dimensional.

Let ξ and ξ' be vector bundles over a space X. If P is a property of continuous maps, then we say that $p : \xi \to X$ (resp. $f : \xi \to \xi'$) has the property P over $Y \subset X$ if p_Y (resp. f_Y) has the property P. For example, p is trivial over Y, f is a fibre homotopy equivalence over Y, etc. The map f has the property P locally if every $x \in X$ has a neighborhood U such that f has the property P over U.

We recall the following definitions:

Definition 1.1. *A Lie algebra bundle is a vector bundle $\xi = (\xi, p, X)$ together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces a Lie algebra structure on each fibre ξ_x.*

Definition 1.2. *A locally trivial Lie algebra bundle is a s vector bundle ξ in which each fibre is a Lie algebra and for each x in X, there is an open set U in X containing x, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each x in U, $\phi_x : \{x\} \times L \to p^{-1}(x)$ is a Lie algebra isomorphism.*

Definition 1.3. *A section of a Lie algebra bundle (ξ, p, X) is a map $s : X \to \xi$ such that $p \circ s = id_X$. $\Gamma(\xi)$ denote the set of all sections of ξ.*

Definition 1.4. *Let $\xi = (\xi, p, X)$ and $\eta = (\eta, q, X)$ be two Lie algebra bundles over the same base space X. A Lie algebra bundle morphism $f : \xi \to \eta$ is continuous map such that $p = qf$ and for each x in X, $f_x : \xi_x \to \eta_x$ is a Lie algebra homomorphism. We say that a morphism is an isomorphism if f is bijective and f^{-1} is a continuous map.*

Definition 1.5. *A Lie algebra bundle ξ over an arbitrary space X is of finite type if there is a finite partition S of unity on X (that is a finite set S of non-negative continuous functions on X whose sum is 1) such that the restriction of the bundle to the set $\{x \in X \mid f(x) \neq 0\}$ is a trivial Lie algebra bundle for each f in S.*

Definition 1.6. *A covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of a space X is called numerable if it admits a refinement by a locally finite partition of unity. That is, if there exists a locally finite partition of unity $\{f_i : X \to [0, 1]\}_{i \in S}$ such that every set $f_i^{-1}(0, 1]$ is contained in some U_λ.*

Remark 1.7. *If ξ is a Lie algebra bundle of finite type over X and $\{U_i\}$ is an open cover corresponding to the partition of unity $\{f_i\}$ (i.e., $U_i = f_i^{-1}(0, 1]$), then $\{U_i\}$ is numerable.*

Definition 1.8. *A Lie algebra bundle $\xi = (\xi, p, X)$ is said to be numerable if X has a numerable covering $\{V_\lambda\}_{\lambda \in \Lambda}$ such that $\xi|_{V_\lambda}$ is trivial for each $\lambda \in \Lambda$.*

Remark 1.9. *Any Lie algebra bundle over compact space is of finite type and any Lie algebra bundle over paracompact space is numerable. A Lie algebra bundle of* *finite type is always numerable. Converse is not true. For, consider the paracompact space $\mathbb{R}_l$ ($\mathbb{R}$), with the lower limit topology. Any Lie algebra bundle over $\mathbb{R}_l$ is numerable as $\mathbb{R}_l$ is paracompact. But it is not of finite type, since finitely many open sets cannot cover $\mathbb{R}_l$.*

For each space X, let $k_G(X)$ denote the set of isomorphism classes of numerable principal G-bundles over X. Let $\{\xi\}$ denote the isomrphism class of principal G-bundles ξ over X. For a homotopy class $[f] : B \to C$, we define a function $k_G([f]) : k_G(C) \to k_G(B)$ by the relation $k_G([f])\{\xi\} = \{f^*(\xi)\}$.

Let **H** denote the category of all spaces and homotopy classes of maps. We observe that the collection of functions k_G: $\mathbf{H} \to \mathbf{ens}$ is a cofunctor (**ens** is the category of sets and functions [7, Theorem 10.1]). Let $\xi = (\xi, p, X)$ be a fixed numerable principal G-bundle. For each space B we define a function $\phi_\xi(B) : [B, X] \to k_G(B)$ by the relation $\phi_\xi(X)[u] = \{u^*(\xi)\}$.

Definition 1.10. *A principal G-bundle $\xi = (\xi, p, X)$ is said to be universal if ξ is numerable and $\phi_\xi : [-, X] \to k_G$ is an isomorphism. The space X is called a classifying space of G.*

Under ordinary composition, the maps over X form a category, which is denoted by $\mathcal{C}_X$.

Definition 1.11. *A homotopy $\Theta : \xi \times I \to \xi'$ is called a homotopy over X or vertical homotopy if $\Theta_t : \xi \to \xi'$, $\Theta_t(e) = \Theta(e, t)$ is a map over X for every $t \in I$. Two maps $f_0, f_1 : \xi \to \xi'$ are vertically homotiopic, $f_0 \simeq_X f_1$, if there exists a vertical homotopy Θ with $\Theta_0 = f_0$, $\Theta_1 = f_1$. The relation $\simeq_X$ is an equivalence relation between maps over X which is compatible with composition. By identifying equivalent maps, we get a new category $\bar{\mathcal{C}_X}$ whose elements are those of $\mathcal{C}_X$ and whose morphisms are vertical homotopy classes of maps over X.*

Definition 1.12. *A halo around $Y \subset X$ is a subset V of X such that there exists a continuous function $f : X \to [0, 1]$ with $Y \subset f^{-1}(1), CV \subset f^{-1}(0)$, ($CV$ is the complement of V).*

Definition 1.13. *Let ξ and ξ' be two Lie algebra bundles over X. We say $p : \xi \to X$ is dominated by $p' : \xi' \to X$ (or p' dominates p) if there exist maps $f : \xi \to \xi'$, $g : \xi' \to \xi$ over X such that $gf \simeq_X id_\xi$(i.e., p is a retract of p' in the category $\bar{\mathcal{C}_B}$).*

We observe that the following properties of a Lie algebra bundle $p : \xi \to X$ are equivalent:

(a) p is a fibre homotopy equivalence(treated as a map over X into id_X)

(b) p is dominated by id_X

(c) there exists a section s and a vertical homotopy $\Theta : sp \cong_X id_\xi$.

If any one of the above properties holds, then p is called **shrinkable**.

2. Section Extension Property

In this section, we discuss the section extension property for Lie algebra bundles.

Definition 2.1. *A Lie algebra bundle $p : \xi \to X$ has the section extension property (SEP) if the following holds:*
For every $Y \subset X$ and every section s over Y which admits an extension to halo V around Y, there exists an extension S over X, i.e., a section $S : X \to \xi$ with $S|_Y = s$.

Theorem 2.2. (Section Extension Theorem) *Let $p : \xi \to X$ be a Lie algebra bundle of finite type and let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity, $U_{\alpha_i} = \{x \in X \mid f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial such that p has SEP over each U_{α_i}. Then p has the SEP.*

Proof. We have $f_{\alpha_i} : X \to [0, 1]$ and $U_{\alpha_i} = f_{\alpha_i}^{-1}(0, 1]$. Take a section s over $Y \subset X$ which admits an extension to a halo V. Let $f : X \to [0, 1]$ be a haloing function and denote the extension s by the same letter, $s : V \to E$. We have to find a section $S : X \to \xi$ such that $S|_Y = s|_Y$. We observe that $V \cap U_{\alpha_i}$ is a halo around $Y \cap U_{\alpha_i}$ and $s : V \cap U_{\alpha_i} \to \xi$ is an extension of the section s on $Y \cap U_{\alpha_i}$. Since p has SEP over each U_{α_i}, there exists a section $S_{\alpha_i} : U_{\alpha_i} \to \xi$ such that $S_{\alpha_i}|_{Y \cap U_{\alpha_i}} = s|_{Y \cap U_{\alpha_i}}$. Now define a section S on X by, $S(x) = S_{\alpha_k}(x)$, where k is the maximum such that $x \in U_{\alpha_k}$. We have, $S|_Y = s|_Y$.

Corollary 2.3. *Let $\xi \to X$ be a Lie algebra bundle of finite type. Take a section s over $Y \subset X$ which admits an extension to a halo V around Y. Let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity, $U_{\alpha_i} = \{x \in X \mid f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial such that $p_{U_{\alpha_i}}$ is shrinkable for each α_i (i.e., fibre homotopy equivalent to a trivial space $U_{\alpha_i} \times W$ with contractible W). Then there exists a section $S : X \to \xi$ with $S|_Y = s$.*

Proof. Proof follows from above theorem and Corollary 2.8 in [3], as $\{U_{\alpha_i}\}$ is a numerable covering.

Theorem 2.4. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Let $\{f_{\alpha_i}\}_{i \in S}$ be a finite partition of unity, $U_{\alpha_i} = \{x \in X \mid f_{\alpha_i}(x) \neq 0\}$ with $\xi|_{U_{\alpha_i}}$ is trivial. If p is shrinkable over each U_{α_i}, then p is shrinkable.*

Proof. Proof follows from Corollary 3.2 in [3], as $\{U_{\alpha_i}\}$ is a numerable covering.

3. Covering Homotopy Property

In this section, we discuss the covering homotopy property for Lie algebra bundles.

Definition 3.1. *Let $p : \xi \to X$ be a Lie algebra bundle over X and $\bar{H} : A \times I \to X$ be a homotopy. We say p has the covering homotopy property (CHP) for $\bar{H}$ if the following holds:*
Given $h : A \to \xi$ with $ph(a) = \bar{H}(a, 0)$, given further $f : A \to I$ and $H' : f^{-1}(0, 1] \times I \to \xi$ with $pH'(a, t) = \bar{H}(a, t)$, $H'(a, 0) = h(a), a \in f^{-1}(0, 1], t \in I$, there exists $H : A \times I \to \xi$ with $pH = \bar{H}, H|_{f^{-1}(1)} = H'|_{f^{-1}(1)}, H(a, 0) = h(a), a \in A$.
We use analogous terminology if I is replaced by an arbitrary interval $[b, c], b < c$.

We say that p has CHP for A, if it has the CHP for all homotopies $\bar{H}$ with range $A \times I$. If it has the CHP for all spaces, then we say it has the CHP.

Definition 3.2. *Let ξ^I denote the space of all the paths in ξ. For every $\bar{H} : A \times I \to X$, $h : A \to \xi$ with $ph(a) = \bar{H}(a, 0)$, we define a fibre bundle $q : \eta \to A$ over A as follows:*

$$\eta = \{(a, w) \in A \times \xi^I \mid h(a) = w(0) \text{ and } pw(t) = \bar{H}(a, t)\}, \quad q(a, w) = a.$$

A covering homotopy gives a section S for q by $S(a) = (a, H_a)$, where $H_a(t) = H(a, t)$.

We recall the following two lemmas from [3]:

Lemma 3.3. [3, Lemma 4.5] *The map p has the CHP for $\bar{H}$ if and only if $q = q_h$ has the SEP for all $h : A \to \xi$ with $ph(a) = \bar{H}(a, 0)$.*

Lemma 3.4. [3, Lemma 4.6] *Let $b < c < d$ be real numbers and $\bar{H} : A \times [b, d] \to X$. If p has the CHP for $\bar{H}|_{A \times [b,c]}$ and $\bar{H}|_{A \times [c,d]}$, then for $\bar{H}$ itself.*

Theorem 3.5. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type and $\bar{H} : A \times I \to X$ be a homotopy. Suppose $\{f_{\alpha_i}\}_{i=1}^n$ is a partition of unity on A, with $U_{\alpha_i} = f_{\alpha_i}^{-1}(0, 1]$ and $\{U_{\alpha_i}\}$ cover A. If for every α_i, real numbers $0 = t_0^{\alpha_i} < t_1^{\alpha_i} < \cdots < t_{r_{\alpha_i}}^{\alpha_i} = 1$ such that p has the CHP for $\bar{H}|_{U_{\alpha_i} \times [t_k^{\alpha_i}, t_{k+1}^{\alpha_i}]}$, then p has the CHP for $\bar{H}$.*

Proof. By Lemma 3.4, p has the CHP for $\bar{H}|_{U_{\alpha_i} \times I}$ for all α_i. Then $q : \eta \to A$ has the SEP over each U_{α_i}, by Lemma 3.3. Hence q itself has the SEP (Theorem 2.2). Thus, again by Lemma 3.3, p has the CHP for $\bar{H}$.

Theorem 3.6. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Suppose $\{f_\alpha\}_{\alpha \in S}$ is a partition of unity, with $U_\alpha = f_\alpha^{-1}(0, 1]$ and $\xi|_{U_\alpha}$ is trivial. If p has the CHP over every set U_α, then p has the CHP for all spaces A.*

Proof. Let $\bar{H} : A \times I \to X$ be a homotopy. We need to show that p has the CHP for $\bar{H}$. Let $a \in A$. Then $\bar{H}(a, t) \in X = \bigcup_{\alpha \in S} U_\alpha, \forall t \in I$. Let $\bar{H}(a, t) \in \bigcup_{i=1}^{k} U_{\alpha_i}$. Then

$f_{\alpha_i}\bar{H}(a,t) \neq 0, \forall i = 1, \ldots, k$. Define

$$g_{\alpha_1 \ldots \alpha_k}(a) = \prod_{i=1}^{k} Min\{f_{\alpha_i}\bar{H}(a,t) \mid t \in [i-1/k, i/k]\}.$$

Note that $g_{\alpha_1 \ldots \alpha_k}(a) \neq 0$ if and only if $\bar{H}(a \times [i-1/k, i/k]) \subset U_{\alpha_i}, \forall i$. Then the functions in $\{g_{\alpha_1 \ldots \alpha_k}\}$ are well defined and $\{V_{\alpha_1 \ldots \alpha_k} = g^{-1}_{\alpha_1 \ldots \alpha_k}(0,1]\}$ covers A. We observe that p has the CHP for all $\bar{H}|_{V_{\alpha_1 \ldots \alpha_k} \times [i-1/k, i/k]}$ and hence the result follows from Theorem 3.5 (If needed, divide each function in $\{g_{\alpha_1 \ldots \alpha_k}\}$ by sum of all the functions to get the partition of unity).

4. Weak Covering Homotopy Property

In this section, we discuss the weak covering homotopy property for Lie algebra bundles.

Definition 4.1. *Let $p : \xi \to X$ be a Lie algebra bundle over X and $\bar{H} : A \times [0,1] \to X$ be a homotopy. We say that p has the weak covering homotopy property (WCHP) for $\bar{H}$, if it has the ordinary CHP for the following:*

$$\hat{H} : A \times [-1,1] \to X, \;\; \hat{H}(a \times [-1,0]) = \bar{H}(a,0), \;\; \hat{H}|_{A \times [0,1]} = \bar{H}.$$

Every map $H : A \times [-1,1] \to \xi$ with $pH = \hat{H}$ will be called a weak covering homotopy of $\bar{H}$.

We use analogous terminology if $[0,1], [-1,1]$ is replaced by $[c,d]$, $[b,d]$ ($b < c < d$). We say p has the WCHP for X if it has the WCHP for all $\bar{H}$ with range $A \times I$.

Theorem 4.2. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Suppose $\{f_{\alpha_i}\}_{i=1}^{n}$ is a partition of unity, with $U_{\alpha_i} = f^{-1}_{\alpha_i}(0,1]$ and $\xi|_{U_{\alpha_i}}$ is trivial. If p has the WCHP over every set U_{α_i}, then p has the WCHP for all spaces A.*

Proof. Proof follows from [3, Theorem 5.12], since $\{U_{\alpha_i}\}$ is a numerable covering.

Theorem 4.3. *Let $p : \xi \to X$ be a Lie algebra bundle of finite type. Suppose $\{f_{\alpha_i}\}_{i=1}^{n}$ is a partition of unity, with $U_{\alpha_i} = f^{-1}_{\alpha_i}(0,1]$, $\xi|_{U_{\alpha_i}}$ is trivial and the inclusion map $U_{\alpha_i} \to X$ is null-homotopic for every α_i. Then $p : \xi \to X$ has the WCHP if and only if p is fibre homotopy equivalent over each U_{α_i} to a trivial space.*

Proof. We have $\{U_{\alpha_i}\}$ is a numerable covering. Therefore, proof follows from [3, Theorem 6.4].

Theorem 4.4. (Classification theorem) *A principal G-bundle $\eta = (\eta, p, X)$ of finite type is universal if and only if its bundle space X is contractible.*

Proof. Since every bundle of finite type is numerable, the result follows from [3, Theorem 7.5].

Theorem 4.5. (Covering Homotopy theorem for Bundle maps) *Let ξ and η be principal G-bundles of finite type, $\phi : \xi \to \eta$ be a bundle map, and $D : X_\xi \times [0,1] \to X_\eta$ a deformation of X_ϕ (i.e., $D(x,0) = X_\phi(x)$). Then there exists a bundle map $\Phi : \xi \times [0,1] \to \eta$ such that $X_\Phi = D$ and $\Phi(z,0) = \phi(z), z \in \xi$.*
Proof. Proof follows from [3, Theorem 7.8], as any Lie algebra bundle of finite type is numerable.

Corollary 4.6. *If η is a Lie algebra bundle of finite type over X and $f_0, f_1 : B \to X_\eta$ are homotopic maps, then the induced bundles $f_0^{-1}(\eta), f_1^{-1}(\eta)$ are equivalent.*
Proof. Since every Lie algebra bundle of finite type is numerable, result follows from [3, Theorem 7.10].

Acknowledgements

We thank Dr. H. M. Prasad, Department of Mathematics, Srinivas Institute of Engineering and Technology, Mangalore and Prof. P. S. K. Reddy, Department of Mathematics, JSS Science and Technology University, Mysuru, for their valuable comments and suggestions.

References

[1] Atiyah, M. F., K-Theory, W. A. Benjamin, Inc., New York, 1967.

[2] Cartan H. and Eilenberg S., Homological Algebra, Princeton Mathematical Series 19, Princeton, New Jersey, 1956.

[3] Dold A., Partitions of unity in the theory of fibrations, Annals of Mathematics, 78(2) (1963), 223-255.

[4] Douady A. and Lazard M., Espaces fibrès en algebrès de Lie et en groupes, Inventiones mathematicae, 1 (1966), 133-151.

[5] Huebsch, W., Covering Homotopy, Duke Mathematical Journal, 23(2) (1956), 281-291.

[6] Hurewicz, W., On the concept of fibre space, Proc. Natl. Acad. Sci. USA, 41(11) (1955), 956-961.

[7] Husemoller, D., Fibre Bundles, Springer-Verlag, New York, Heidelberg, Berlin, 1994.

[8] James, I. M. and Whitehead, J. H. C., Note on fibre spaces, Proceedings of the London Mathematical Society, s3-4(1) (1954), 129-137.

[9] Kiranagi, B. S., Lie algebra bundles, Bull. Sci. Math., 2e series, 102 (1978), 57-62.

[10] Kiranagi, B. S., Lie algebra bundles and Lie rings, Proc. Nat. Acad. Sci. India, 54(A) I (1984), 38-44.

[11] Kumar, R., Prema, G. and Kiranagi, B. S., Lie algebra bundles of finite type, Acta Universitatis Apulensis, 39 (2014), 151-160.

[12] Michael, E., Local properties of topological Spaces, Duke Mathematical Journal, 21 (1954), 163-171.

[13] Mackenzie, K. C. H., General Theory of Lie Groupoids and Lie Algebroids, Cambridge University Press, Cambridge, 2005.

[14] Steenrod, N. E., The Topology of Fibre Bundles, Princeton University Press, Princeton, 1951.

[15] Swan, R. G., Vector bundles and projective modules, Trans. Amer. Math. Soc., 105 (1962), 264-277.

[16] Vaserstein, L. N., Vector bundles and projective modules, Trans. Amer. Math. Soc., 294(2) (1986), 749–755.