South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 3 (2023), pp. 233-242

DOI: 10.56827/SEAJMMS.2023.1903.18 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

DECOMPOSITIONS OF CONTINUITY VIA SIMPLY-OPEN SETS

Talal Ali Al-Hawary and Jalal Hattem Hussein*

Mathematics Department,
Yarmouk University, Irbid, JORDAN

E-mail : talalhawary@yahoo.com

*Mathematics Department,
College of Science for Woman,
University of Baghdad, Baghdad, IRAQ

E-mail : jalalintuch@yahoo.com

(**Received:** *May 18, 2023* **Accepted:** *Dec. 27, 2023* **Published:** *Dec. 30, 2023)*

Abstract: In [7, 9, 13, 14], the class of simply –open sets was introduced and explored. In this paper, we introduce what we call SM$-$ continuity and $SMM-$continuity and we give several characterizations and two decompositions of $SM-$continuity. Finally, new decompositions of continuity are provided.

Keywords and Phrases: Simply-open, M-continuity, continuity.

2020 Mathematics Subject Classification: 54C08, 54C05, 54C10.

1. Introduction

Let $(X, \mathfrak{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl_{\mathfrak{T}}(A)$ and $Int_{\mathfrak{T}}(A)$, respectively. If no ambiguity appears, we use $\overline{A}$ and A^o, respectively. By X, Y and Z we mean topological spaces with no separation axioms imposed. $\mathfrak{T}_{standard}$, $\mathfrak{T}_{indiscrete}$, $\mathfrak{T}_{leftray}$ and $\mathfrak{T}_{cocountable}$ will stand for the standard, indiscrete, left ray and the cocountable topologies, respectively. A space $(X, \mathfrak{T})$ isanti locally countable if all non-empty open subsets are uncountable.

In [7, 9, 13], a subset A of a space $(X, \mathfrak{T})$ is called simply –open if $A = O \cup N$, where O is open and N is nwd subset of X. The class of all simply–open sets in X

will be denoted by $SMO(X,\mathfrak{T})$ or simply $SMO(X)$. The simply-interior of a set A is the union of all simply-open subsets of A and is denoted by $Int_{SI}(A)$. Clearly A is simply-open if and only if $A = Int_{SI}(A)$.

Several decompositions of continuity and waek continuity were established: In [11], it was proved that a map f is A–continuous if and only if it is semicontinuous and LC–continuous, f is continuous if and only if it is α–continuous and LC–continuous if and only if it is precontinuous and LC–continuous if and only if it is precontinuous and A–continuous and f is α–continuous if and only if is precontinuous and semicontinuous.

Analogous to [1, 2, 3, 4, 5, 8, 10, 11, 16, 17], in Section 2 we introduce the relatively new notion of SM-continuity: a map $f : X \to Y$ is $SM-$continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an $SM-$open subset U in X containing x such that $f(U) \subseteq V$ and f is $SM-$continuous if it is $SM-$continuous at every $x \in X$, which is closely related to continuity. Moreover, we show that SM-continuity preserves Lindelof property and a space $(X, \mathfrak{T})$ is Lindelof if and only if $(X, \mathfrak{T}_{SM})$ is Lindelof, where $\mathfrak{T}_{SM}$ is the collection of all simply open subsets of X. Sections 3 is devoted for studying four weaker notions of SM-continuity by which we provide two decompositions of SM-continuity. Finally, in Section 4 we give several decompositions of continuity which seem to be new.

2. SM-continuous Mappings

We begin this section by introducing the notion of SM-open set.

Definition 2.1. *A subset A of a space $(X, \mathfrak{T})$ is called SM-open if for every $x \in A$, there exists an open subset $U_x \subseteq X$ containing x such that $U_x \backslash Int_{SI}(A)$ is countable. The complement of an SM-open subset is called SM-closed.*

Since every open set is simply open, every open set is SM-open. But the converses need not be true.

Example 2.2. Let $X = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, X, \{a\}\}$. Set $A = \{b\}$. Then A is SM-open but not open.

Corollary 2.3. *If $(X, \mathfrak{T})$ is anti locally countable and A is SM-closed, then $Int_{\mathfrak{T}}(A) = Int_{SI}(A)$ of $SM-$continuous mappings. Several characterizations of this class of mappings are also provided.*

Now, using the notion of SM-open set, we introduce SM-continuity.

Definition 2.4. *A map $f : X \to Y$ is $SM-$continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an $SM-$open subset U in X containing x such that $f(U) \subseteq V$. f is $SM-$continuous if it is $SM-$continuous at every $x \in X$.*

As every open set is $SM-$open, every continuous map is $SM-$continuous. The converse need not be true.

Example 2.5. Let $X = \{a, b\}$, $\mathfrak{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathfrak{T}_2 = \{\varnothing, X, \{b\}\}$. Then the identity map $id : (X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is $SM-$continuous but not continuous.

The proofs of the following three results are immediate and are thus omitted.

Lemma 2.6. *Let X, Y and Z be spaces. Then*

(1) *If $f : X \to Y$ is $SM-$continuous surjection and $g : Y \to Z$ is continuous surjection, then $g \circ f$ is $SM-$continuous.*

(2) *If $f : X \to Y$ is $SM-$continuous surjection and $A \subseteq X$, then $f|_A$ is $SM-$continuous.*

(3) *If $f : X \to Y$ is a map such that $X = X_1 \cup X_2$ where X_1 and X_2 are closed and both $f|_{X_1}$ and $f|_{X_2}$ are $SM-$continuous, then f is $SM-$continuous.*

(4) *If $f_1 : X \to X_1$ and $f_2 : X \to X_2$ are maps and $g : X \to X_1 \times X_2$ is the map defined by $g(x) = (f_1(x), f_2(x))$ for all $x \in X$, then g is $SM-$continuous if and only if f_1 and f_2 are $SM-$continuous.*

Lemma 2.7. *For a map $f : X \to Y$, the following are equivalent:*

(1) *f is $SM-$continuous.*

(2) *The inverse image of every open subset of Y is $SM-$open in X.*

(3) *The inverse image of every closed subset of Y is $SM-$closed in X.*

(4) *The inverse image of every basic open subset of Y is $SM-$open in X.*

(5) *The inverse image of every subbasic open subset of Y is $SM-$open in X.*

Lemma 2.8. *A space $(X, \mathfrak{T}_X)$ is Lindelof if and only if $(X, SMO(X))$ is Lindelof.*

Next we show that being Lindelof is preserved under $SM-$continuity.

Theorem 2.9. *If $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is $SM-$continuous and X is Lindelof, then Y is Lindelof.*

Proof. Let $\mathfrak{B} = \{V_\alpha : \alpha \in \nabla\}$ be an open cover of Y. Since f is $SM-$continuous, $\mathfrak{A} = \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a cover of X by $SM-$open subsets and as X is Lindelof, by Lemma 2.8, $\mathfrak{A}$ has a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$. Now $Y = f(X) = f(\cup\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N})\} \subseteq \cup\{V_{\alpha_n} : n \in \mathbb{N}\}$. Therefore Y is Lindelof.

If X is a countable space, then every subset of X is $SM-$open and hence every map $f : X \to Y$ is $SM-$continuous. Next, we show that if X is uncountable such that every $SM-$continuous map $f : X \to Y$ is a constant map, then X has to be connected.

Theorem 2.10. *If X is uncountable space such that every $SM-$continuous map $f : X \to Y$ is a constant map, then X is connected.*

Proof. If X is disconnected, then there exists a non-empty proper subset A of X which is both open and closed. Let $Y = \{a, b\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{b\}\}$ and $f : X \to Y$ defined by $f(A) = \{a\}$ and $f(X \backslash A) = \{b\}$. Then f is a non-constant $SM-$continuous map.

The converse of the preceding result need not be true even when X is uncountable.

Example 2.11. The identity map $id : (\mathbb{R}, \mathfrak{T}_{leftray}) \to (\mathbb{R}, \mathfrak{T}_{indiscrete})$ is a non-constant $SM-$continuous.

3. Decompositions of $SM-$continuity

We begin by recalling the following well-known two definitions:

Definition 3.1. *A map $f : X \to Y$ is weakly continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is weakly continuous if it is weakly continuous at every $x \in X$.*

Definition 3.2. *A map $f : X \to Y$ is W*-continuous if for every open subset V in Y, $f^{-1}(Fr(V))$ is closed in X, where $Fr(V) = \overline{V} \backslash \overset{o}{V}$.*

Weakly continuity and W*-continuity are independent notions that are weaker than continuity and the two together characterize continuity (see for example [15]). Next we give two relatively new such definitions.

Definition 3.3. *A map $f : X \to Y$ is weakly $SM-$ continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an $SM-$open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is weakly $SM-$continuous if it is weakly $SM-$continuous at every $x \in X$.*

Clearly, every $SM-$continuous and every weakly continuous map is weakly $SM-$continuous. Non of the converses need be true as shown next.

Example 3.4. Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = a$ for all $x \in \mathbb{R}$. Then f is weakly $SM-$continuous but not $SM-$continuous.

Example 3.5. Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map

$f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly continuous and hence weakly $SM-$continuous but not $SM-$continuous.

Definition 3.6. *A map $f : X \to Y$ is coweakly $SM-$ continuous if for every open subset V in Y, $f^{-1}(Fr(V))$ is $SM-$closed in X, where $Fr(V) = \overline{V}\backslash\overset{o}{V}$.*

Clearly, every $SM-$continuous is coweakly $SM-$continuous. The converse need not be true.

Example 3.7. Let $X = Y = \{a, b\}$, $\mathfrak{T}_X = \{\varnothing, X\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{a\}, \{b\}\}$Then the identity map $id : X \to Y$ is coweakly $SM-$continuous but not $SM-$continuous.

Our first characterization of $SM-$continuity in terms of the preceding two notions of continuity is given next.

Theorem 3.8. *The following are equivalent for a map $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$:*

(1) *f is $SM-$continuous.*

(2) *$f : (X, SMO(X)) \to (Y, \mathfrak{T}_Y)$ is continuous.*

(3) *$f : (X, SMO(X)) \to (Y, \mathfrak{T}_Y)$ is weakly continuous and W*-continuous.*

Proof. (1) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (3) : Follows from Lemma 2.7.

(3) $\Rightarrow$ (1) : Since $f : (X, SMO-) \to (Y, \mathfrak{T}_Y)$ is W*-continuous, it is coweakly $SM-$continuous and as it is weakly-continuous, it is weakly $SM-$continuous. Thus by Lemma 2.7, $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is $SM-$continuous.

We show that weakly $SM-$continuity and coweakly $SM-$continuity are independent notions, but together they characterize $SM-$continuity.

Example 3.9. The map *id* in Example 3.7 is coweakly $SM-$continuous but not weakly $SM-$continuous.

Example 3.10. Let $Y = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}\}$. Then the map $f :$ $(\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defnined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly $SM-$continuous but not coweakly $SM-$continuous.

Theorem 3.11. *A map $f : X \to Y$ is $SM-$continuous if and only if f is both weakly and coweakly $SM-$continuous.*

Proof. $SM-$continuity implies weakly and coweakly $SM-$continuity is obvious. Conversely, suppose $f : X \to Y$ is both weakly and coweakly $SM-$continuous and let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Then as

f is weakly $SM-$continuous, there exists an $SM-$open subset U of X containing x such that $f(U) \subseteq \overline{V}$. Now $Fr(V) = \overline{V}\backslash V$ and hence $f(x) \notin Fr(V)$. So $x \in U\backslash f^{-1}(Fr(V))$ which is $SM-$open in X since f is coweakly $SM-$continuous. For every $y \in f(U\backslash f^{-1}(Fr(V)))$, $y = f(a)$ for some $a \in U\backslash f^{-1}(Fr(V))$ and hence $f(a) = y \in f(U) \subseteq \overline{V}$ and $y \notin Fr(V)$. Thus $f(a) = y \notin Fr(V)$ and thus $f(a) \in V$. Therefore, $f(U\backslash f^{-1}(Fr(V))) \subseteq V$ and hence f is $SM-$continuous.

Next, we define a new class of open sets that is independent of ω-open class, but together they characterize $SM-$open.

Definition 3.12. *For a space $(X, \mathfrak{T})$, let $SMM(X) =: \{A \subseteq X : Int_{SMO-}(A) = Int_{\mathfrak{T}_\omega}(A)\}$. A is SMM-set if $A \in SMM(X)$.*

Clearly every $SM-$open set is SMM-set, but the converse need not be true.

Example 3.13. Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{standard}$. Then $\mathbb{Q}$ is an SMM-set which is neither $SM-$open nor ω-open.

Even an ω-open subset need not be an SMM-set.

Example 3.14. Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{standard}$. Then $\mathbb{R}\backslash\mathbb{Q}$ is an ω-open which is not an SMM-set.

Theorem 3.15. *A subset A of a space X is $SM-$open if and only if A is ω-open and an SMM-set.*

Proof. Trivially every $SM-$open is ω-open and an SMM-set. Conversely, let A be an ω-open set that is ω_ω^o-set. Then $A = Int_{\mathfrak{T}_\omega}(A) = Int_{SMO-}(A)$ and therefore A is $SM-$open.

Definition 3.16. *A map $f : X \to Y$ is SMM-continuous if the inverse image of every open subset of Y is an SMM-set.*

Clearly every $SM-$continuous map is SMM-continuous, but the converse need not be true as not every SMM-set is $SM-$open. An immediate consequence of Theorem 3.15 is the following decomposition of $SM-$continuity.

Theorem 3.17. *A map $f : X \to Y$ is $SM-$continuous if and only if f is ω-continuous and SMM-continuous.*

4. Decompositions of Continuity

We begin this section by introducing the notion of an $SMOO$-set. We then introduce the notion of ω_X^o-continuity which gives an immediate decomposition of continuity.

Definition 4.1. *For a space $(X, \mathfrak{T})$, let $SMOO(X) =: \{A \subseteq X : Int_{SMO-}(A) = Int_{\mathfrak{T}}(A)\}$. A is an $SMOO$-set if $A \in SMOO(X)$.*

The proof of the following result follows immediately from Corollary 2.3.

Corollary 4.2. *If $(X, \mathfrak{T})$ is anti locally countable, then $SMOO(X)$ contains all $SM-$closed subsets of X.*

We remark that, in general, an $SM-$closed set need not be an $SMOO$-set as shown in the next example.

Example 4.3. Let $X = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, X, \{a\}\}$. Set $A = \{b\}$. Then A is $SM-$closed but not an $SMOO$-set.

As every open set is $SM-$open, every open set is an $SMOO$-set but the converse need not be true.

Example 4.4. Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{standard}$. Then $\mathbb{Q}$ is an $SMOO$-set which is not open.

Next, we show that the notions of $SMOO$-set and $SM-$open are independent, but together they characterize open sets.

Example 4.5. In Example 4.3, A is $SM-$open but not an $SMOO$-set.

Example 4.6. In Example 4.4, $\mathbb{Q}$ is an $SMOO$-set which is not $SM-$open.

Theorem 4.7. *A subset A of a space X is open if and only if A is $SM-$open and an $SMOO$-set.*

Proof. Trivially every open set is $SM-$open and an ω_X^o-set.. Conversely, let A be an $SM-$open set that is $SMOO$-set. Then $A = Int_{SMO-}(A) = Int_{\mathfrak{T}}(A)$ and therefore A is open.

In a similar manner, for a space $(X, \mathfrak{T})$ let $SMOM =: \{A \subseteq X : Int_{\mathfrak{T}_\omega}(A) = Int_{\mathfrak{T}}(A)\}$ and call a subset A is $SMOM$-set if $A \in \omega_X$. Then we have the following result.

Theorem 4.8. *A subset A of a space X is open if and only if A is ω-open and an $SMOM$-set.*

Definition 4.9. *A map $f : X \to Y$ is $SMOM$-continuous (respectively, $SMOM$-continuous) if the inverse image of every open subset of Y is an $SMOM$-set (respectively, $SMOM$-set).*

Clearly every continuous map is $SMOO$-continuous, but the converse need not be true as not every $SMOO$-set is open. An immediate consequence of Theorems 3.11, 3.17, 4.7 and 4.8 are the following decompositions of continuity, which seem to be new.

Theorem 4.10. *For a map $f : X \to Y$, the following are equivalent:*

(1) *f is continuous.*

(2) *f is $SM-$continuous and $SMOM$-continuous.*

(3) *f is ω-continuous and $SMOM$-continuous.*

(4) *f is both weakly $SM-$continuous, coweakly $SM-$continuous and $SMOM$-continuous.*

(5) *f is ω-continuous, $SMOO$-continuous and $SMOM$-continuous.*

5. Conclusion

Once an author introduced a new class of open sets, one would hunt for a result that gives a sufficient and necessary condition for this open set with another kind of weak open set to be an open. So after the class of simply –open sets was introduced and explored., we introduced what we call SM− continuity and $SMM-$continuity and we gave several characterizations and two decompositions of $SM-$continuity. Finally, new decompositions of continuity were provided.

References

[1] Al-Hawary, T., and Al-Omari, A., Between Open and ω-Open Sets, Q &A in General Topology, 24 (2006), 67-77.

[2] Al-Hawary, T., and Al-Omari, A., Decompositions of Continuity, Turkish J. Math., 30(20) (2006), 187-195.

[3] Al-Hawary, T., Fuzzy ω_0-open sets, Bull. Korian Math. Soc., 45(4) (2008), 749-755.

[4] Al-Hawary, T., Decomposition of Continuity via α-open sets, Acta Universitatis aplulensis, 34 (2013), 137-142.

[5] Al-Hawary, T., $\in$-Closed sets, Thai J. of Math., 16(3) (2018), 675-681.

[6] Al-Sa'ad, W., Study of Some Weak Forms of Continuous Functions, Master's Thesis, Jordan University, 1993.

[7] Biswas, H., On some mappings in topological spaces, Bull.Cal. Math. Soc., 61 (1969), 127-135.

[8] Hussein, J. H. and Al-Hawary, T., On σ-Topological Vector Spaces, AIP Conference Proceedings, 2023.

[9] Kljushin, V. L. and Hussein, V. L., On simply-open sets. Vestnik UDC, 3 (2011), 34-38.

[10] Kaymakci, A. K. and Hatir, A. K., A decomposition of continuity, XVI. ULUSAL MATEMATİK SEMPOZYUMUAt: YÜZÜNCÜ YIL ÜNİVERSİT ESİ (2003).

[11] Ganster, M. and Reilly, I., A Decomposition of Continuity, Acta Math. Hung., 56(3-4) (1990), 299-301.

[12] Munkres, J., Topology a First Course, Prentice_Hall inc., New Jersey, 1975.

[13] Nasef, A. and Mareay, R., More on simply open sets and its applications, South Asian J. Math., 5(3) (2015), 100-108.

[14] Neubrunnov á, A., On transfinite sequences of certain types of functions, Acta Fac. Rer. Natur. Univ. Com. Math., 30 (1975), 121-126.

[15] Noiri, T., On Weakly Continuous Mappings, Proc. Amer. Math. Soc., 46 (1074), 120-124.

[16] Tong, J., A Decomposition of Continuity, Acta Math. Hung., 48(1-2) (1986), 11-15.

[17] Tong, J., On Decompositions of Continuity in Topological Spaces, Acta Math. Hung., 54(1-2) (1989), 51-55.

This page intentionally left blank.