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On Certain New Bailey Pairs and Their Applications

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Abstract: In this paper, we have established certain new Bailey pairs which have been used to obtain transformation formulae for q-series.
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1. Introduction, Notations and Definitions

In the present paper, we shall adopt the following notations and definitions. The q-rising factorial is defined by, for $|q| < 1$,

$$[a;q]_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad n = 1, 2, 3, ...,$$

$$[a;q]_0 = 1,$$

$$[a;q]_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

and

$$[a_1, a_2, ..., a_r; q]_n = [a_1;q]_n[a_2;q]_n...[a_r;q]_n.$$

A basic hypergeometric series (q-series) is defined by,

$$ {}_r\Phi_s\left[\begin{array}{c} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n q^{\lambda\binom{n}{2}}}{(q, b_1, b_2, ..., b_s; q)_n}, \tag{1.1}$$

where $\binom{n}{2} = n(n-1)/2$. Series (1.1) converges for all values of z if λ is a positive integer. For $\lambda = 0$, it converges for $|z| < 1$.

The sequences $\langle \alpha_n, \beta_n \rangle$ are said to be the Bailey pair relative to the parameter 'a' if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} = \frac{1}{(q,aq;q)_n} \sum_{r=0}^{n} \frac{(-)^r (q^{1+n})^r (q^{-n};q)_r}{q^{r(r+1)/2}(aq^{1+n};q)_r}. \tag{1.2}$$

Bailey [1,2] used the q-Gauss sum

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; c/ab \\ c \end{matrix} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty},$$

to get that if $\langle \alpha_n, \beta_n \rangle$ is a Bailey pair relative to 'a', then

$$\sum_{n=0}^{\infty} (y,z;q)_n \left(\frac{aq}{yz}\right) \beta_n = \frac{\left(\frac{aq}{y}, \frac{aq}{z}; q\right)_\infty}{\left(aq, \frac{aq}{yz}; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(aq/y, aq/z;q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n. \tag{1.3}$$

Following summation formulae are needed in our analysis,

$${}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; -q^{n-\frac{1}{2}} \\ \sqrt{a}, -\sqrt{a}, aq^{n+1} \end{matrix} \right]$$

$$= \frac{1}{2} \frac{(aq, -q^{-1/2}; q)_n}{(\sqrt{aq}; q)_n (-q\sqrt{a}; q)_{n-1}} + \frac{1}{2} \frac{(aq, -q^{-1/2}; q)_n}{(-\sqrt{aq}; q)_n (q\sqrt{a}; q)_{n-1}} \tag{1.4}$$

[Verma and Jain 3; (4.1)]

$${}_3\Phi_2 \left[\begin{matrix} a, q\sqrt{a}, q^{-n}; q; -q^n \\ \sqrt{a}, aq^{n+1} \end{matrix} \right]$$

$$= \frac{1}{2} \frac{(aq, -1; q)_n (1+\sqrt{a})}{(aq; q^2)_n} + \frac{1}{2} \frac{(aq, -1; q)_n (1-\sqrt{a})}{(\sqrt{a}, -q\sqrt{a}; q)_n} \tag{1.5}$$

[Verma and Jain 3; (4.2)]

$${}_2\Phi_1 \left[\begin{matrix} a, q^{-n}; q; -q^{n+\frac{1}{2}} \\ aq^{n+1} \end{matrix} \right]$$

$$= \frac{1}{2} \frac{(aq, -\sqrt{q}; q)_n (1+\sqrt{a})}{(-\sqrt{aq}, q\sqrt{a}; q)_n} + \frac{1}{2} \frac{(aq, -\sqrt{q}; q)_n (1-\sqrt{a})}{(\sqrt{aq}, -q\sqrt{a}; q)_n} \tag{1.6}$$

[Verma and Jain 3; (4.3)]

$$
{}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; -q^{\frac{1}{2}+n} \\ \sqrt{a}, -\sqrt{a}, aq^{n+1} \end{array}\right]
$$

$$
= \frac{1}{2\sqrt{a}} \frac{(aq, -q^{-1/2}; q)_n}{(\sqrt{aq}; q)_n(-q\sqrt{a}; q)_{n-1}} - \frac{1}{2\sqrt{a}} \frac{(aq, -q^{-1/2}; q)_n}{(-\sqrt{aq}; q)_n(q\sqrt{a}; q)_{n-1}} \tag{1.7}
$$

[Verma and Jain 3; (4.5)]

$$
{}_3\Phi_2\left[\begin{array}{l} a, q\sqrt{a}, q^{-m}; q; -q^{1+m} \\ \sqrt{a}, aq^{m+1} \end{array}\right]
$$

$$
= \frac{1}{2\sqrt{a}} \frac{(aq, -1; q)_m(1+\sqrt{a})}{(aq; q^2)_m} - \frac{1}{2\sqrt{a}} \frac{(aq, -1; q)_m(1-\sqrt{a})}{(\sqrt{a}, -q\sqrt{a}; q)_m}. \tag{1.8}
$$

[Verma and Jain 3; (4.6)]

$$
{}_2\Phi_1\left[\begin{array}{l} a, q^{-m}; q; -q^{\frac{1}{2}+m} \\ aq^{1+m} \end{array}\right]
$$

$$
= \frac{1}{2\sqrt{a}} \frac{(aq, -\sqrt{q}; q)_m(1+\sqrt{a})}{(-\sqrt{aq}, q\sqrt{a}; q)_m} - \frac{1}{2\sqrt{a}} \frac{(aq, -\sqrt{q}; q)_m(1-\sqrt{a})}{(\sqrt{aq}, -q\sqrt{a}; q)_m} \tag{1.9}
$$

[Verma and Jain 3; (4.7)]

2. Main Results

In this section we shall establish certain new Bailey pairs.

(a) Choosing $\alpha_r = \dfrac{q^{r(r+1)/2}(a, q\sqrt{a}, -q\sqrt{a}; q)_r q^{-3r/2}}{(q, \sqrt{a}, -\sqrt{a}; q)_r}$ in (1.2) and, using (1.4) we get,

$$
\beta_n = \frac{1}{2}\frac{(-q^{-1/2}; q)_n}{(q; q)_n}\left\{\frac{1+\sqrt{a}q^n}{(\sqrt{aq}, -q\sqrt{a}; q)_n} + \frac{1-\sqrt{a}q^n}{(-\sqrt{aq}, q\sqrt{a}; q)_n}\right\}. \tag{2.1}
$$

Thus $\langle \alpha_n, \beta_n \rangle$ given in (2.1) form a Bailey pair.

(b) Choosing $\alpha_r = \dfrac{q^{r(r+1)/2}(a, q\sqrt{a}; q)_r q^{-r}}{(q, \sqrt{a}; q)_r}$ in (1.2) and, using (1.5) we have,

$$
\beta_n = \frac{(-1; q)_n}{2(q; q)_n}\left\{\frac{1+\sqrt{a}}{(\sqrt{aq}, -\sqrt{aq}; q)_n} + \frac{1-\sqrt{a}}{(\sqrt{a}, -q\sqrt{a}; q)_n}\right\}. \tag{2.2}
$$

So, $\langle \alpha_n, \beta_n \rangle$ given in (2.2) form a Bailey pair.

(c) Putting $\alpha_r = \dfrac{q^{r(r+1)/2}(a;q)_r q^{-r/2}}{(q;q)_r}$ in (1.2) and, making use of (1.6) we get,

$$\beta_n = \frac{1}{2(q;q)_n}\left\{\frac{(-\sqrt{q};q)_n(1+\sqrt{a})}{(-\sqrt{aq},q\sqrt{a};q)_n} + \frac{(-\sqrt{q};q)_n(1-\sqrt{a})}{(\sqrt{aq},-q\sqrt{a};q)_n}\right\}. \tag{2.3}$$

Thus $\langle \alpha_n, \beta_n \rangle$ given in (2.3) form a Bailey pair.

(d) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a,q\sqrt{a},-q\sqrt{a};q)_r q^{-r/2}}{(q,\sqrt{a},-\sqrt{a};q)_r}$ in (1.2) and, making use of (1.7) we find,

$$\beta_n = \frac{(-q^{-1/2};q)_n}{2\sqrt{a}(q;q)_n}\left\{\frac{(1+q^n\sqrt{a})}{(\sqrt{aq},-q\sqrt{a};q)_n} - \frac{(1-q^n\sqrt{a})}{(-\sqrt{aq},q\sqrt{a};q)_n}\right\}. \tag{2.4}$$

So, $\langle \alpha_n, \beta_n \rangle$ given in (2.4) form a Bailey pair.

(e) Choosing $\alpha_r = \dfrac{q^{r(r+1)/2}(a,q\sqrt{a};q)_r}{(q,\sqrt{a};q)_r}$ in (1.2) and, using (1.8) we have,

$$\beta_n = \frac{1}{2\sqrt{a}}\frac{(-1;q)_n}{(q;q)_n}\left\{\frac{1+\sqrt{a}}{(\sqrt{aq},-\sqrt{aq};q)_n} - \frac{1-\sqrt{a}}{(\sqrt{a},-q\sqrt{a};q)_n}\right\}. \tag{2.5}$$

So, $\langle \alpha_n, \beta_n \rangle$ given in (2.5) form a Bailey pair.

(f) Lastly, taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a;q)_r q^{r/2}}{(q;q)_r}$ in (1.2) and, making use of (1.9) we find,

$$\beta_n = \frac{(-\sqrt{q};q)_n}{2\sqrt{a}(q;q)_n}\left\{\frac{(1+\sqrt{a})}{(-\sqrt{aq},q\sqrt{a};q)_n} - \frac{(1-\sqrt{a})}{(\sqrt{aq},-q\sqrt{a};q)_n}\right\}. \tag{2.6}$$

Thus $\langle \alpha_n, \beta_n \rangle$ given in (2.6) form a Bailey pair.

3. Applications of Bailey pairs (2.1-(2.6))

In this section we shall use the Bailey pairs of previous section in order to establish transformation formulae for q-series.

(a) Putting the Bailey pair given in (2.1) in (1.3) we get,

$${}_3\Phi_2\left[\begin{matrix} y,z,-q^{-1/2};q;aq/yz \\ \sqrt{aq},-q\sqrt{a} \end{matrix}\right] + \sqrt{a}\ {}_3\Phi_2\left[\begin{matrix} y,z,-q^{-1/2};q;aq^2/yz \\ \sqrt{aq},-q\sqrt{a} \end{matrix}\right]$$

$$+\ {}_3\Phi_2\left[\begin{matrix} y,z,-q^{-1/2};q;aq/yz \\ -\sqrt{aq},q\sqrt{a} \end{matrix}\right] - \sqrt{a}\ {}_3\Phi_2\left[\begin{matrix} y,z,-q^{-1/2};q;aq^2/yz \\ -\sqrt{aq},q\sqrt{a} \end{matrix}\right]$$

$$= \frac{2(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \, {}_5\Phi_4 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, y, z; q; aq^{1/2}/yz \\ \sqrt{a}, -\sqrt{a}, aq/y, aq/z; q \end{matrix} \right], \tag{3.1}$$

where $|aq/yz| < 1$.
(b) Putting the Bailey pair given in (2.2) in (1.3) we find,

$$\frac{1+\sqrt{a}}{2} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -1; q; aq/yz \\ \sqrt{aq}, -\sqrt{aq} \end{matrix} \right] + \frac{1-\sqrt{a}}{2} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -1; q; aq/yz \\ \sqrt{a}, -q\sqrt{a} \end{matrix} \right]$$

$$= \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \, {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, y, z; q; aq/yz \\ \sqrt{a}, aq/y, aq/z; q \end{matrix} \right], \tag{3.2}$$

where $|aq/yz| < 1$.
(c) Putting the Bailey pair given in (2.3) in (1.3) we obtain,

$$\frac{1+\sqrt{a}}{2} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -\sqrt{q}; q; aq/yz \\ -\sqrt{aq}, q\sqrt{a} \end{matrix} \right] + \frac{1-\sqrt{a}}{2} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -\sqrt{q}; q; aq/yz \\ \sqrt{aq}, -q\sqrt{a} \end{matrix} \right]$$

$$= \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \, {}_3\Phi_2 \left[\begin{matrix} a, y, z; q; aq^{3/2}/yz \\ aq/y, aq/z; q \end{matrix} \right], \tag{3.3}$$

where $|aq/yz| < 1$.
(d) Making use of Bailey pair in (2.4) in (1.3) we find,

$$\frac{1}{\sqrt{a}} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -q^{-1/2}; q; aq/yz \\ \sqrt{aq}, -q\sqrt{a} \end{matrix} \right] + {}_3\Phi_2 \left[\begin{matrix} y, z, -q^{-1/2}; q; aq^2/yz \\ \sqrt{aq}, -q\sqrt{a} \end{matrix} \right]$$

$$-\frac{1}{\sqrt{a}} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -q^{-1/2}; q; aq/yz \\ -\sqrt{aq}, q\sqrt{a} \end{matrix} \right] + {}_3\Phi_2 \left[\begin{matrix} y, z, -q^{-1/2}; q; aq^2/yz \\ -\sqrt{aq}, q\sqrt{a} \end{matrix} \right]$$

$$= \frac{2(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \, {}_5\Phi_4 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, y, z; q; aq^{3/2}/yz \\ \sqrt{a}, -\sqrt{a}, aq/y, aq/z; q \end{matrix} \right], \tag{3.4}$$

where $|aq/yz| < 1$.
(e) Using the Bailey pair of (2.5) in (1.3) we have,

$$\frac{1+\sqrt{a}}{2\sqrt{a}} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -1; q; aq/yz \\ \sqrt{aq}, -\sqrt{aq} \end{matrix} \right] - \frac{1-\sqrt{a}}{2\sqrt{a}} \, {}_3\Phi_2 \left[\begin{matrix} y, z, -1; q; aq/yz \\ \sqrt{a}, -q\sqrt{a} \end{matrix} \right]$$

$$= \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \, {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, y, z; q; aq^2/yz \\ \sqrt{a}, aq/y, aq/z; q \end{matrix} \right], \tag{3.5}$$

where $|aq/yz| < 1$.

(f) Putting the Bailey pair of (2.6) in (1.3) we obtain,

$$\frac{1+\sqrt{a}}{2\sqrt{a}}\ {}_3\Phi_2\left[\begin{matrix} y, z, -\sqrt{q}; q; aq/yz \\ -\sqrt{aq}, q\sqrt{a} \end{matrix}\right] - \frac{1-\sqrt{a}}{2\sqrt{a}}\ {}_3\Phi_2\left[\begin{matrix} y, z, -\sqrt{q}; q; aq/yz \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right]$$

$$= \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} a, y, z; q; aq^{5/2}/yz \\ aq/y, aq/z; q \end{matrix}\right], \tag{3.6}$$

where $|aq/yz| < 1$.

4. Special Cases

In this section we shall deduce certain special cases of the results established in section 3.

(i) Taking a=1 in (3.1) we get,

$${}_3\Phi_2\left[\begin{matrix} y, z, -q^{-1/2}; q; q/yz \\ \sqrt{q}, -q \end{matrix}\right] + {}_3\Phi_2\left[\begin{matrix} y, z, -q^{-1/2}; q; q^2/yz \\ \sqrt{q}, -q \end{matrix}\right]$$

$$+ {}_3\Phi_2\left[\begin{matrix} y, z, -q^{-1/2}; q; q/yz \\ -\sqrt{q}, q \end{matrix}\right] - {}_3\Phi_2\left[\begin{matrix} y, z, -q^{-1/2}; q; q^2/yz \\ -\sqrt{q}, q \end{matrix}\right]$$

$$= \frac{2(q/y, q/z; q)_\infty}{(q, q/yz; q)_\infty}\left\{1 + \sum_{n=1}^{\infty} \frac{(y, z; q)_n (1+q^n) q^{n^2/2}}{(q/y, q/z; q)_n (yz)^n}\right\}. \tag{4.1}$$

(ii) As $y, z \to \infty$ in (3.1) we get,

$$\sum_{n=0}^{\infty} \frac{(-q^{-1/2}; q)_n q^{n^2} a^n}{(q;q)_n(\sqrt{aq}, -q\sqrt{a}; q)_n} + \sum_{n=0}^{\infty} \frac{(-q^{-1/2}; q)_n q^{n^2} a^n}{(q, -\sqrt{aq}, q\sqrt{a}; q)_n}$$

$$+\sqrt{a}\sum_{n=0}^{\infty} \frac{(-q^{-1/2}; q)_n q^{n(n+1)} a^n}{(q, q\sqrt{aq}, -q\sqrt{a}; q)_n} - \sqrt{a}\sum_{n=0}^{\infty} \frac{(-q^{-1/2}; q)_n q^{n(n+1)} a^n}{(q, -\sqrt{aq}, q\sqrt{a}; q)_n}$$

$$= \frac{2}{(aq;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}; q; aq^{1/2} \\ \sqrt{a}, -\sqrt{a}; q^3 \end{matrix}\right]. \tag{4.2}$$

(iii) Taking $y, z \to \infty$ in (3.2) we obtain,

$$\frac{1+\sqrt{a}}{2}\ {}_1\Phi_2\left[\begin{matrix} -1; q; aq \\ \sqrt{aq}, -\sqrt{aq}; q^2 \end{matrix}\right] + \frac{1-\sqrt{a}}{2}\ {}_1\Phi_2\left[\begin{matrix} -1; q; aq \\ \sqrt{a}, -q\sqrt{a}; q^2 \end{matrix}\right]$$

$$= \frac{1}{(aq;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} a, q\sqrt{a}; q; aq \\ \sqrt{a}; q^3 \end{matrix}\right]. \tag{4.3}$$

(iv) Taking $y, z \to \infty$ in (3.3) we find,

$$\frac{1+\sqrt{a}}{2}\,{}_1\Phi_2\left[\begin{array}{l} -q^{1/2}; q; aq \\ -\sqrt{aq}, q\sqrt{a}; q^2 \end{array}\right] + \frac{1-\sqrt{a}}{2}\,{}_1\Phi_2\left[\begin{array}{l} -\sqrt{q}; q; aq \\ \sqrt{aq}, -q\sqrt{a}; q^2 \end{array}\right]$$

$$= \frac{1}{(aq;q)_\infty}\,{}_1\Phi_0\left[\begin{array}{l} a; q; aq^{3/2} \\ -; q^3 \end{array}\right]. \qquad (4.4)$$

Similar other results can also be secured.

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