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ON TOPOLOGICAL J - QUOTIENT MAPS

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Abstract: In this paper, J - Quotient maps, Strongly J - Quotient maps, $[J]$ - Quotient maps and Strongly J - Open maps utilizing J - Closed sets are introduced. The newly defined Quotient maps are analysed with various existing Quotient maps. Interrelations between J - Quotient maps, Strongly J - Quotient maps, $[J]$ - Quotient maps and Strongly J - Open maps are investigated. Here the properties of those functions are presented.

Keywords and Phrases: J - Closed, J - Continuous, JTC - space, J - Open, J - Irresolute.

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1. Introduction

Regular open sets were introduced by Stone [14] and using the concept semi-regularization of a topological space is constructed. In 1968, Velicko [17] proposed a concept namely δ-open sets stronger than open sets. Levine [4] has brought generalized closed sets in 1970. Dunham [2] has established Cl^* using the concept of g - closed sets. In 2016, Annalakshmi [1] has introduced regular*-open sets using Cl^*. In 2019, the authors Meenakshi.PL and Sivakamasundari.K have introduced unification of $regular^*$ - open sets namely η^*-open sets [5] which lies between δ - open sets and open sets. Its basic properties are studied and the concepts of η^* - cluster points, η^* - adherent points and η^* - derived sets are introduced. Using η^* - open sets, the authors have introduced J - closed sets [6] and their features. In the year 2020, the concept of J - continuous functions is initiated by the author [9]. Further the author [10], [11] has initiated J - Irresolute functions, J - Closed Functions, J - open Functions in topological spaces respectively. In this paper, J - Quotient maps, Strongly J - Quotient maps, $[J]$ - Quotient maps and Strongly J - Open maps utilizing J - Closed sets are introduced. The newly defined Quotient maps are analysed with various existing Quotient maps. Interrelations between J - Quotient maps, Strongly J - Quotient maps, $[J]$ - Quotient maps and Strongly J - Open maps are investigated. Here the properties of those functions are presented.

For this paper some basic definitions and results in topological spaces are needed which are given in section 2. Throughout this paper, a topological space is represented by (R, ζ).

2. Preliminaries

Definition 2.1. *Let (R, ζ) be a topological space. If D is a non-empty subset of (R, ζ) then the intersection of all closed sets containing D is called closure of D and is denoted by $Cl(D)$. The union of all open sets contained in D is called interior of D and is denoted by $int(D)$.*

Definition 2.2. *A subset D of a topological space (R, ζ) is called generalized closed (briefly g - closed)* [4] *if $Cl(D) \subseteq M$ whenever $D \subseteq M$ and M is open in (R, ζ).*

Definition 2.3. [2] *If $D \subseteq R$, then*
(i) $Cl^*(D)$ is the intersection of all generalized closed sets in R containing D which is said to be generalized closure of D.
(ii) $int^*(D)$ is the unification of all generalized open sets in R contained in D which is said to be generalized interior of D.

Definition 2.4. *A subset D of (R, ζ) is called a*
1) regular closed set [14] *if $D = Cl(int(D))$*
2) semi-closed set [3] *if $int(Cl(D)) \subseteq D$*

The corresponding complements are their corresponding open sets. The corresponding closures are defined as the intersection of corresponding sets containing D.

Definition 2.5. [17] *A set D is called δ - open when D can be represented as the union of regular open sets. The δ - closure and δ - interior are defined as usual using δ - closed sets and δ - open sets.*

Definition 2.6. *A set D is called regular*-open (or r^*-open)* [1] *if $D = int(Cl^*(D))$. The corresponding complement, closure, interior are defined in the usual manner.*

Definition 2.7. [5] *A subset D is known as η^*-open if it is a union of regular*-*

open sets (r^*-open sets).The corresponding complement of a η^*-open set is called a η^*-closed set.

Definition 2.8. *A subset D in R is known as J-closed set* [6] *if $Cl(D) \subseteq M$ whenever $D \subseteq M$, M is η^*-open in R. We represent the collection of all J-closed sets of (R, ζ) by $JC(R, \zeta)$. The complement of J-closed is said to be J-open* [7] *in (R, ζ).*

Definition 2.9. *A subset D in R is known as $\hat{g}$-closed* [16] *if $cl(D) \subseteq M$ whenever $D \subseteq M$ and M is semi-open in (R, ς). The complement of $\hat{g}$ -closed is said to be $\hat{g}$ -open in (R, ζ).*

Definition 2.10. *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be continuous if $f^{-1}(U)$ is a closed set in (R, ζ) for every closed set U in (Z, σ).*

Definition 2.11. [9] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be J-Continuous if the inverse image of every closed set in (Z, σ) is J-closed in (R, ζ).*

Definition 2.12. [15] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be $\hat{g}$ -continuous if the inverse image of every closed set in (Z, σ) is $\hat{g}$ -closed in (R, ζ).*

Definition 2.13. [8] *A topological space (R, ζ) is called a JTC-space if each J-closed in (R, ζ) is closed in the space (R, ζ).*

Definition 2.14. [10] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be J-Closed function if the image of every closed set in (R, ζ) is J-closed in (Z, σ).*

Definition 2.15. [11] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be J-open function if the image of every open set in (R, ζ) is J-open in (Z, σ).*

Definition 2.16. [14] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be $\hat{g}$ -open function if the image of every open set in (R, ζ) is $\hat{g}$ -open in (Z, σ).*

Definition 2.17. [11] *A function $f : (R, \zeta) \to (Z, \sigma)$ is said to be J-irresolute function if the inverse image of every J-open in (Z, σ) is J-open in (R, ζ).*

Definition 2.18. [12] *A map f from a topological space R onto a topological space Z is called a quotient map if for every subset V of Z, the set $f^{-1}(V)$ is open in R if and only if V is open in Z.*

Definition 2.19. [13] *A map f from a topological space R onto a topological space Z is called a $\hat{g}$-quotient map if for every subset V of Z, the set $f^{-1}(V)$ is open in R if and only if V is $\hat{g}$-open in Z.*

Remark 2.20. *(i) Every $\hat{g}$-open set is a J-open set.*

(ii) Every continuous and g-continuous function is a J-continuous function.

(iii) A J-irresolute function $f : (R, \zeta) \to (Z, \sigma)$ is a J-continuous function.

(iv) Every closed set is a J-closed set. Also each open set is J-open.

Proposition 2.21. [11] *If $f : (R, \zeta) \to (Z, \sigma)$ is any function, $g : (Z, \sigma) \to (P, \mu)$ is an injective function and also a J-irresolute function, their composite function $g \circ f : (R, \zeta) \to (P, \mu)$ is a J-open function then f is a J-open function in (Z, σ).*

Proposition 2.22. [11] *Let $f : (R, \zeta) \to (Z, \sigma)$ be a function where (Z, σ) is a JTC-space. Then f is a J-irresolute function if and only if it is a J-continuous function.*

Note 2.23. *(i) In diagrams, $X \to Z$ represents X implies Z where reverse implication map not hold good.*

(ii) $JO(R, \zeta)$ is the collection of all J-open sets in (R, ζ).

(iii) $\hat{g}O(R, \sigma)$ is the collection of all $\hat{g}$- open sets in (R, σ).

(iv) $JC(R, \zeta)$ is the collection of all J- closed sets in (R, ζ).

(v) $P(R)$ is the power set of R.

3. J-Quotient Maps

Definition 3.1. *A surjective function $e : (R, \zeta) \to (Z, \sigma)$ is said to be a J-Quotient map if e is a J-continuous function and $e^{-1}(U)$ is open in (R, ζ) implies U is J-open in (Z, σ).*

Example 3.2. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Let $\zeta = \{R, \phi, \{i\}, \{i, t\}\}$ and $\sigma = \{Z, \phi, \{i, t\}\}$. Here a surjective function e is a J-continuous function as $JO(R, \zeta) = P(R) - \{\{t, l\}\}$ and $e^{-1}(U) = \{i\} \Rightarrow U = e(\{i\}) = \{i\}, e^{-1}(U) = \{i, t\} \Rightarrow U = e(\{i, t\}) = \{i, t\}$ is a J-open set in (Z, σ) as $JO(Z, \sigma) = P(Z)$. Hence a function e is a J-Quotient map.

Definition 3.3. *A function $e : (R, \zeta) \to (Z, \sigma)$ is said to be strongly J-open function if the image of every J-open set in (R, ζ) is J-open in (Z, σ).*

Example 3.4. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Let $\sigma = \{R, \phi, \{i\}, \{t\}, \{i, t\}\}$ and $\zeta = \{Z, \phi, \{i, t\}\}$. Here $JO(R, \zeta) = \{R, \phi, \{i\}, \{t\}, \{i, t\}\}$ and $JO(Z, \sigma) = P(Z)$. Then e is a strongly J-open function.

Proposition 3.5. *A strongly J-open function $e : (R, \zeta) \to (Z, \sigma)$ is a J-open function but the converse is not true.*

Proof. Consider $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-open function. Let U be an open set in (R, ζ). By Remark 2.20.(iv).,U is a J-open set in (R, ζ). Since e is a strongly J-open function, $e(U)$ is J-open in (Z, σ). Hence e is a J-open function.

Counter Example 3.6. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$

is an identity function. Consider $R = Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{i\}\}$ and $\sigma = \{Z, \phi, \{i, t\}\}$. Here $JO(Z, \sigma) = \{R, \phi, \{i\}, \{t\}, \{i, t\}\}$ and $JO(R, \zeta) = P(R) - \{\{t, l\}\}$. Then e is a J-open function but not a strongly J-open function. Because for a J-open set $\{l\}$ in (R, ζ), the image of $\{l\}$ is not a J-open set in (Z, σ).

Proposition 3.7. *If $e : (R, \zeta) \to (Z, \sigma)$ is a function, $g : (Z, \sigma) \to (P, \mu)$ is an injective function and also a J-irresolute function, their composite function $g \circ e : (R, \zeta) \to (P, \mu)$ is a strongly J-open function then e is a strongly J-open function in (Z, σ).*
Proof. The proof follows from Proposition 3.5. and Proposition 2.21.

Proposition 3.8. *If a surjective function $e : (R, \zeta) \to (Z, \sigma)$ is a J-continuous function and J-open function, then e is a J-quotient map.*
Proof. Consider $e^{-1}(U)$ is open in (R, ζ). Since e is a J-open function and surjective, $e(e^{-1}(U)) = U$ is J-open in (Z, σ) and since e is J-continuous. e is a J-quotient map.

Proposition 3.9. *If a surjective function $e : (R, \zeta) \to (Z, \sigma)$ is a J-continuous function and J-closed function, then e is a J-quotient map.*
Proof. Consider $e^{-1}(U)$ is open in (R, ζ). Then $R - e^{-1}(U)$ is closed in (R, σ). Since e is a J-closed function, $R - e(e^{-1}(U))$ is J-closed in (Z, σ). This implies that $e(e^{-1}(U)) = U$ is J-open in (Z, σ), since e is surjective. Therefore e is a J-quotient map.

Proposition 3.10. *Every $\hat{g}$-quotient map is a J-quotient map but not conversely.*
Proof. By hypothesis, e is surjective. We know that every $\hat{g}$-continuous function is a J-continuous function (by Remark 2.20.(ii).). Let $e^{-1}(U)$ is open in (R, ζ). Since e is a $\hat{g}$ -quotient map, U is $\hat{g}$-open in (Z, σ). By Remark 2.20.(i). as every $\hat{g}$-open is J-open, U is J-open in (Z, σ). Hence e is J-quotient.

Counter Example 3.11. Let $e : (R, \zeta) \to (Z, \sigma)$ be the surjective function defined by $e(i) = i, e(t) = l$ and $e(l) = t$. Consider $R = Z = \{i, t, l\}$ with $\sigma = \{R, \phi, \{i\}, \{i, t\}\}$ and $\zeta = \{Z, \phi, \{i, t\}\}$. Here $e^{-1}(U) = \{i\} \Rightarrow U = e(\{i\}) = \{i\}, e^{-1}(U) = \{i, t\} \Rightarrow U = e(\{i, t\}) = \{i, l\}$. Then e is a J-quotient map as $JO(Z, \sigma) = P(Z)$. But e is not a $\hat{g}$-quotient map, since $\{i, l\}$ is not a $\hat{g}$-open set in (Z, σ) as $\hat{g}O(Z, \sigma) = \{Z, \phi, \{i\}, \{t\}, \{i, l\}\}$.

4. Strong Form of J-Quotient Maps

Definition 4.1. *A surjective function $e : (R, \zeta) \to (Z, \sigma)$ is said to be a strongly J-quotient map provided a subset U of (Z, σ) is open $\Leftrightarrow e^{-1}(U)$ is J-open in (R, ζ).*

Example 4.2. Let $e : (R, \zeta) \to (Z, \sigma)$ be the many one onto function defined by

$e(i) = i = e(l), e(t) = t, e(s) = l$. Consider $R = \{i, t, l, s\}, Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{l\}, \{i, t\}, \{i, t, l\}\}$ and $\sigma = \{Z, \phi, \{i\}, \{t\}, \{i, t\}\}$. Then a surjective function e is a J-continuous function as $JO(R, \zeta) = \{R, \phi, \{i\}, \{t\}, \{l\}, \{i, t\}, \{t, l\}, \{i, l\}, \{i, t, l\}\}$ and $e^{-1}(U) = \{i\} \Rightarrow U = e(\{i\}) = \{i\}$, $e^{-1}(U) = \{t\} \Rightarrow U = e(\{t\}) = \{t\}$, $e^{-1}(U) = \{l\} \Rightarrow U = e(\{l\}) = \{i\}$, $e^{-1}(U) = \{i, t\} \Rightarrow U = e(\{i, t\}) = \{i, t, l\} = Z$, $e^{-1}(U) = \{t, l\} \Rightarrow U = e(\{t, l\}) = \{i, t\}$, $e^{-1}(U) = \{i, l\} \Rightarrow U = e(\{i, l\}) = \{i\}$, $e^{-1}(U) = \{i, t, l\} \Rightarrow U = e(\{i, t, l\}) = \{i, t\}$ are open sets in (Z, σ). Hence a function f is a strongly J-quotient map.

Proposition 4.3. *A strongly J-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a J-open function but the converse is not true.*

Proof. Consider $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-quotient map. Let U be an open set in (R, ζ). By Remark 2.20.(iv).,as every open set is J-open, U is a J-open set in (R, ζ). Since e is a surjective function, $e^{-1}(e(U)) = U$ is a J-open set in (R, ζ). By the given condition, e is Strongly J-quotient map, $e(U)$ is an open set in (Z, σ). By Remark 2.20.(iv)., $e(U)$ is a J-open set in (Z, σ). Hence e is a J-open function.

Counter Example 4.4. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Consider $R = Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{i\}\}$ and $\sigma = \{Z, \phi, \{i, t\}\}$. Here $JO(Z, \sigma) = P(Z)$. Then e is a J-open function but not a strongly J-quotient map. Because $e^{-1}(U) = \{i, l\} \Rightarrow U = e(\{i, l\}) = \{i, l\}$ is not open in (Z, σ).

Proposition 4.5. *A strongly J-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-open function but the converse is not true.*

Proof. Consider $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-quotient map. Let U be a J-open set in (R, ζ). Since e is a surjective function, $e^{-1}(e(U)) = U$ and so it is a J-open set in (R, ζ). By the given condition, e is strongly J-quotient map, $e(U)$ is an open set in (Z, σ). By Remark 2.20.(iv)., $e(U)$ is J-open set in (Z, σ). Hence e is a strongly J-open function.

Counter Example 4.6. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Consider $R = Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{i\}, \{t\}, \{i, t\}\}$ and $\sigma = \{Z, \phi, \{i, t\}\}$. Here $JO(R, \zeta) = \{R, \phi, \{i\}, \{t\}, \{i, t\}\}$ and $JO(Z, \sigma) = P(Z)$. Then e is a strongly J-open function but not a strongly J-quotient map. Because $e^{-1}(U) = \{i\} \Rightarrow U = e(\{i\}) = \{i\}$ is not open in (Z, σ).

Proposition 4.7. *A strongly J-quotient map is a J-continuous function.*

Proof. In Definition 4.1., a subset U of (Z, σ) is open $e^{-1}(U)$ is J-open in (R, ζ) gives that e is a J-continuous function.

Counter Example 4.8. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Consider $R = Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{i, t\}\}$ and $\sigma = \{Z, \phi, \{i\}, \{t\}, \{i, t\}\}$. Here $\sigma^c = \{Z, \phi, \{l\}, \{t, l\}, \{i, l\}\}$ and $\zeta^c = \{R, \phi, \{l\}\}$. Then e is J-continuous as $JC(R, \zeta) = P(R)$ but not a strongly J-quotient map. Because for the J-open set $e^{-1}(U) = \{t, l\}$ in $(R, \zeta), U = \{t, l\}$ is not open in (Z, σ).

Definition 4.9. *A surjective function $e : (R, \zeta) \to (Z, \sigma)$ is said to be a $[J]$-quotient map if e is a J-irresolute function and $e^{-1}(U)$ is J-open in (R, ζ) implies U is open in (Z, σ).*

Example 4.10. Let $e : (R, \zeta) \to (Z, \sigma)$ be the many one onto function defined by $e(i) = i = e(l), e(t) = t, e(s) = l$. Consider $R = \{i, t, l, s\}, Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{l\}, \{i, t\}, \{i, t, l\}\}$ and $\sigma = \{Z, \phi, \{i\}, \{t\}, \{i, t\}\} = JO(Z, \sigma)$. Then a surjective function e is a J-irresolute function as $JO(R, \zeta) = \{R, \phi, \{i\}, \{t\}, \{l\}, \{i, t\}, \{t, l\}, \{i, l\}, \{i, t, l\}\}$ and $e^{-1}(U) = \{i\} \Rightarrow U = e(\{i\}) = \{i\}, e^{-1}(U) = \{t\} \Rightarrow U = e(\{t\}) = \{t\}, e^{-1}(U) = \{l\} \Rightarrow U = e(\{l\}) = \{i\}, e^{-1}(U) = \{i, t\} \Rightarrow U = e(\{i, t\}) = \{i, t, l\} = Z, e^{-1}(U) = \{t, l\} \Rightarrow U = e(\{t, l\}) = \{i, t\}, e^{-1}(U) = \{i, l\} \Rightarrow U = e(\{i, l\}) = \{i\}, e^{-1}(U) = \{i, t, l\} \Rightarrow U = e(\{i, t, l\}) = \{i, t\}$ is a open set in (Z, σ). Hence a function e is a $[J]$-quotient map.

Proposition 4.11. *A $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a J-irresolute function but the converse is not true.*

Proof. The proof follows from the Definition 4.9.

Counter Example 4.12. Consider $R = Z = \{i, t, l\}$ and $e : (R, \zeta) \to (Z, \sigma)$ is an identity function. Consider $R = Z = \{i, t, l\}$ with $\zeta = \{R, \phi, \{i\}\}$ and $\sigma = \{Z, \phi, \{i\}, \{t\}, \{i, t\}\} = JO(Z, \sigma)$. Here $JO(R, \zeta) = P(R) - \{\{t, l\}\}$. Then e is a J-irresolute function but not a $[J]$-quotient map , since $e^{-1}(U) = \{i, l\} \Rightarrow U = e(\{i, l\}) = \{i, l\}$ is not an open set in (Z, σ).

Proposition 4.13. *A $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-open function but the converse is not true.*

Proof. Let U be an open set in (Z, σ). By Remark 2.20.(iv)., U is a J-open set in (Z, σ). Since e is $[J]$-quotient, $e^{-1}(U)$ is a J-open set in (R, ζ). Let $e^{-1}(U)$ is a J-open set in (R, σ). By hypothesis, e is $[J]$-quotient, U is an open set in (Z, σ). Hence e is a strongly J-open function.

Counter Example 4.14. In Counter Example 4.6., e is a strongly J-open function but not a $[J]$-quotient map, since e is not a J-irresolute function. Because for a J-open set $\{l\}$ in (Z, σ),its inverse image $\{l\}$ is not a J-open set in (R, ζ).

Proposition 4.15. *A $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a strongly J-quotient map but the converse is not true.*
Proof. From Remark 2.20.(iii).,every J-irresolute function is a J-continuous function which is required to prove the implication U is an open set in (Z, σ) implies $e^{-1}(U)$ is J-open in (R, ζ). The other implication $e^{-1}(U)$ is J-open in (R, ζ) implies U is open in (Z, σ) follows from the definition of $[J]$-quotient map. Hence the result follows.

Proposition 4.16. *Every strongly J-quotient map is a J-quotient map but not conversely.*
Proof. Let U be an open set in (Z, σ). Since e is strongly J-quotient map, $e^{-1}(U)$ is J-open in (R, ζ) gives that e is a J-continuous function. Let $e^{-1}(U)$ is open in (R, ζ). By Remark 2.20.(iv)., $e^{-1}(U)$ is J-open in (R, ζ). Since e is strongly J-quotient map, U is open in (Z, σ). By Remark 2.20.(iv)., U is J-open in (Z, σ). Hence e is a J-quotient map.

Counter Example 4.17. In Example 3.2., is a J-quotient map but not a strongly J-quotient map. Because $e^{-1}(U) = \{i, l\}$ implies $U = e(\{i, l\}) = \{i, l\}$ is not open in (Z, σ).

Proposition 4.18. *Every quotient map is a J-quotient map but not conversely.*
Proof. By hypothesis, e is surjective. We know that every continuous function is a J -continuous function (by Remark 2.20.(ii).).Since e is a quotient map, $e^{-1}(U)$ is open in (R, ζ) implies U is open in (Z, σ). By Remark 2.20.(iv). as every open is J-open, U is J-open in (Z, σ). Hence e is J-quotient.

Counter Example 4.19. In the above Example 3.2., $e^{-1}(U) = i \Rightarrow U = e(i) = i$ is not an open set in (Z, σ). Hence e is a J-quotient but not a quotient map.

Proposition 4.20. *A $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a J-quotient map but the converse need not be true.*
Proof. By Proposition 4.15. and Proposition 4.16.,we get the proof.

Counter Example 4.21. By Example 3.2., e is a J-quotient map but not a $[J]$-quotient map. Because for a J-open set, $e^{-1}(U) = \{i, l\} \Rightarrow U = e(\{i, l\}) = \{i, l\}$ is not an open set in (Z, σ).

Proposition 4.22. *If an open surjective function $f : (R, \zeta) \to (Z, \sigma)$ is a J-irresolute function and $g : (Z, \sigma) \to (P, \mu)$ is a J-quotient map, then $g \circ e : (R, \zeta) \to (P, \mu)$ is a J-quotient map.*
Proof. Let U be an open subset in (P, μ). Hence $g^{-1}(U)$ is J-open in (Z, σ) as g is a J-quotient map. Since e is J-irresolute, therefore $e^{-1}(g^{-1}(U)) = (g \circ e)^{-1}(U)$

is J-open in (R,ζ). Therefore $g\circ e$ is J-continuous —–(1). Consider $(g\circ e)^{-1}(U) = e^{-1}(g^{-1}(U))$ is open in (R,ζ). Since e is open surjective, $e(e^{-1}(g^{-1}(U))) = g^{-1}(U)$ is open in (Z,σ). By the given condition that g is J-quotient map, U is J-open in (P,μ) —–(2). From (1) and (2), $g\circ e : (R,\zeta) \to (P,\mu)$ is a J-quotient map.

Proposition 4.23. *If a map $h : (R,\zeta) \to (Z,\sigma)$ is a J-quotient map and $g : (R,\zeta) \to (P,\mu)$ is a continuous function that is constant on each set $h^{-1}(R)$, for $R \in Z$, then g induces a J-continuous function $e : (Z,\sigma) \to (P,\mu)$ such that $e \circ h = g$.*

Proof. For the given functions h and g, we have to produce a J-continuous function $e : (Z,\sigma) \to (P,\mu)$, using the fact that g is constant on $h^{-1}(R)$ whenever $R \in Z$, we get $g(h^{-1}(R))$ is a singleton set in (P,μ). Then e is well-defined whenever $e(R)$ denotes this set for every R. Then for each $x \in R$, $e(h(x)) = g(x)$. For if, for an open set U in (P,μ), since g is continuous, we get $g^{-1}(U)$ is an open set in (R,ζ). We have $g^{-1}(U) = h^{-1}(e^{-1}(U))$. Using these facts, we get $e^{-1}(U)$ is a J-open set by using h is being J-quotient. Hence e is a J-continuous function from Z to P.

Proposition 4.24. *If a surjective function $e : (R,\zeta) \to (Z,\sigma)$ is strongly J-open, J-irresolute and $g : (Z,\sigma) \to (P,\mu)$ is a strongly J-quotient map, then $g\circ e : (R,\zeta) \to (P,\mu)$ is a strongly J-quotient map.*

Proof. Given $g : (Z,\sigma) \to (P,\mu)$ is a strongly J-quotient map. Let U be an open subset in (P,μ). Hence $g^{-1}(U)$ is J-open in (Z,σ). Since e is J-irresolute, therefore $e^{-1}(g^{-1}(U)) = (g\circ e)^{-1}(U)$ is J-open in (R,ζ) —–(1). Thus we get $g\circ e$ is J-continuous. Now, take $e^{-1}(g^{-1}(U))$ is J-open in (R,ζ). Since e is strongly J-open, $e(e^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z,σ). By the given condition that g is strongly J-quotient map, U is open in (P,μ) —–(2). From (1) and (2), $g\circ e : (R,\zeta) \to (P,\mu)$ is a strongly J-quotient map.

Proposition 4.25. *If a surjective function $e : (R,\zeta) \to (Z,\sigma)$ is strongly J-open, J-irresolute and $g : (Z,\sigma) \to (P,\mu)$ is a $[J]$-quotient map , then $g\circ e : (R,\zeta) \to (P,\mu)$ is a $[J]$-quotient map.*

Proof. Given $g : (Z,\sigma) \to (P,\mu)$ is a $[J]$-quotient map. Let U be a J-open subset in (P,μ) and using the continuity of g, we get $g^{-1}(U)$ is J-open in (Z,σ). Since e is J-irresolute, $e^{-1}(g^{-1}(U)) = (g\circ e)^{-1}(U)$ is J-open in (R,ζ). Since e is surjective and strongly J-open, $e(e^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z,σ). By the given condition that g is $[J]$-quotient map, U is open in (P,μ). Hence $g\circ e : (R,\zeta) \to (P,\mu)$ is a $[J]$-quotient map.

Proposition 4.26. *If a surjective function $e : (R,\zeta) \to (Z,\sigma)$ is strongly J-open,*

J-irresolute and $g : (Z, \sigma) \to (P, \mu)$ is a J-quotient map, then $g \circ e : (R, \zeta) \to (P, \mu)$ is a J-quotient map.

Proof. Given $g : (Z, \sigma) \to (P, \mu)$ is a J-quotient map. Let U be an open subset in (P, μ). Hence $g^{-1}(U)$ is J-open in (Z, σ). Since e is J-irresolute, therefore $e^{-1}(g^{-1}(U)) = (g \circ e)^{-1}(U)$ is J-open in (R, ζ). Therefore $g \circ e$ is J-continuous —–(1). Conversely, take $e^{-1}(g^{-1}(U))$ is open in (R, ζ). By Remark 2.20.(iv)., $e^{-1}(g^{-1}(U))$ is J-open in (R, ζ). Since e is strongly J-open, $e(e^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z, σ). By the given condition that g is a J-quotient map, U is open in (P, μ) —–(2). From (1) and (2), $g \circ e : (R, \zeta) \to (P, \mu)$ is a J-quotient map.

Proposition 4.27. *If a function $e : (R, \zeta) \to (Z, \sigma)$ is strongly J-quotient, J-irresolute and $g : (Z, \sigma) \to (P, \mu)$ is a $[J]$-quotient map, then $g \circ e : (R, \zeta) \to (P, \mu)$ is a $[J]$-quotient map.*

Proof. Given $g : (Z, \sigma) \to (P, \mu)$ is a $[J]$-quotient map. Let U be an J-open subset in (P, μ). Since g is $[J]$-quotient, using the continuity part, $g^{-1}(U)$ is J-open in (Z, σ) and using the part that e is J-irresolute, $e^{-1}(g^{-1}(U)) = (g \circ e)^{-1}(U)$ is J-open in (R, ζ). By the given condition that e is a strongly J-quotient map, $g^{-1}(U)$ is open in (Z, σ) which will give that $g^{-1}(U)$ is J-open in (Z, σ) by Remark 2.20.(iv). Since g is $[J]$-quotient map, U is open in (P, μ). We arrive at $g \circ e : (R, \zeta) \to (P, \mu)$ is a $[J]$-quotient map.

Proposition 4.28. *If $e : (R, \zeta) \to (Z, \sigma)$ and $g : (Z, \sigma) \to (P, \mu)$ are $[J]$-quotient maps, then $g \circ e : (R, \zeta) \to (P, \mu)$ is a $[J]$-quotient map.*

Proof. Let U be a J-open set in (P, μ). By the given condition, g is a $[J]$-quotient map, $g^{-1}(U)$ is a J-open set in (Z, σ). Since e is a $[J]$-quotient map, $e^{-1}(g^{-1}(U)) = (g \circ e)^{-1}(U)$ is J-open in (R, ζ). This implies $(g \circ e)$ is a J-irresolute function —–(1). Consider $(g \circ e)^{-1}(U) = e^{-1}(g^{-1}(U))$ is J-open in (R, ζ)—–(2) Since e is a $[J]$-quotient map, $g^{-1}(U)$ is an open set in (Z, σ). By Remark 2.20.(iv).,every open is J-open. $g^{-1}(U)$ is a J-open set in (Z, σ). Since g is a $[J]$-quotient map, U is an open set in (P, μ) —–(3). From (1),(2) and (3), $g \circ e : (R, \zeta) \to (P, \mu)$ is a $[J]$-quotient map.

Proposition 4.29. *If a function $e : (R, \zeta) \to (Z, \sigma)$ is a J-quotient map where (R, ζ) and (Z, σ) are JTC-spaces. Then $g : (Z, \sigma) \to (P, \mu)$ is quasi J-continuous the composite map $g \circ e : (R, \zeta) \to (P, \mu)$ is quasi J-continuous.*

Proof. Let U be a J-open set in (P, μ). Using the quasi J-continuity of g, g^{-1} (U) is open in (Z, σ). Now e^{-1} $(g^{-1}(U)) = (g \circ e)^{-1}(U)$ is J-open in (R, σ), using the J-continuity part of e being J-quotient. Since (R, ζ) is a JTC -space, $e^{-1}(g^{-1}(U))$ is open in (R, ζ). Thus $g \circ e$ is quasi J-continuous.

Conversely, let $g \circ e$ be quasi J-continuous, then for every J-open set U in (P, μ). e^{-1} $(g^{-1}(U))$ is open in (R, σ). Since e is a J-quotient map, $g^{-1}(U)$ is J-open in (Z, σ). Since (Z, σ) is a JTC -space, $g^{-1}(U)$ is open in (Z, σ). Hence g is quasi J-continuous.

Proposition 4.30. *An injective $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a J-open function.*

Proof. Let U be an open set in (R, ζ). By Remark 2.20.(iv)., U is an J-open set in (R, ζ). As e is injective $[J]$-quotient, $e^{-1}(e(U)) = U$ is an J-open set in (R, ζ) implies that $e(U)$ is an open set in (Z, σ). Again by Remark 2.20.(iv)., $e(U)$ is a J-open set in (Z, σ). Therefore e is a J-open function.

Theorem 4.31. *If $e : (R, \zeta) \to (Z, \sigma)$ is a function from a JTC-space to another JTC-space, then the following conditions are equivalent.*

(i) e is a $[J]$-quotient map

(ii) e is a strongly J-quotient map

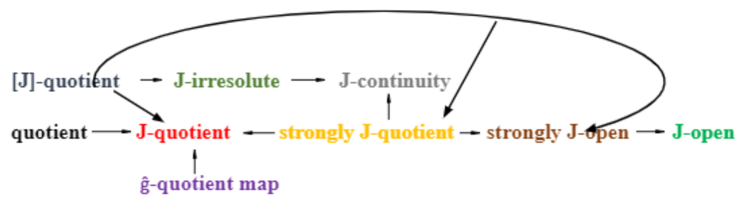
(iii) e is a J-quotient map.

Proof. $(i) \to (ii), (ii) \to (iii)$ By Proposition 4.15., Proposition 4.16. Now to prove $(iii) \to (i)$. Since e is a J-quotient map, e is a J-continuous function in (Z, σ).(Z, σ) is a JTC - space, by Proposition 2.22., e is a J - irresolute function in (Z, σ). Let $e^{-1}(U)$ be a J-open set in (R, ζ). Since (R, ζ) is a JTC - space, $e^{-1}(U)$ is an open set in (R, ζ). By (iii), U is a J-open set in (Z, σ).(Z, σ) is JTC-space, U is an open set in (Z, σ). Hence $(iii) \to (i)$.

Proposition 4.32. *An injective $[J]$-quotient map $e : (R, \zeta) \to (Z, \sigma)$ is a J-closed function.*

Proof. Let U be a closed set in (R, ζ). By Remark 2.20.(iv), U is a J-closed set in (R, ζ). Then $R - U$ is a J-open set in (R, ζ). As e is injective $[J]$-quotient, $e^{-1}(e(R-U)) = R-U$ is an J-open set in (R, ζ) implies that $e(R-U)$ is an open set in (Z, σ). Again by Remark 2.20.(iv)., $e(R-U)$ is a J-open set in (Z, σ) implies $e(U)$ is a J-closed set in (Z, σ). Therefore e is a J-closed function.

From the above deliberations so get the following picture.



5. Conclusion

In this literature of Mathematics, quotient maps are generally called strong continuous maps or identification maps, because of strong conditions of continuity i.e. $e^{-1}(U)$ is open in (Y, ζ) iff U is open in (Z, σ) of a surjective function $e : (Y, \zeta) \to (Z, \sigma)$ and their importance for the philosophy of gluing. Three types of quotient maps namely J - quotient maps, Strongly J - quotient maps and $[J]$ - quotient maps are introduced and their properties are established. In JTC - space, J - quotient maps, Strongly J - quotient maps and $[J]$ - quotient maps are equivalent.

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