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NEW SORT OF NEIGHBOURHOOD AND DERIVED SETS IN MILLI TOPOLOGICAL SPACES

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Abstract: This manuscript aims to define several models in milli topological spaces. This approach extends the scope of applications in topology. Firstly, we generate a new class of milli neighbourhood of a point, milli neighbourhood of a set and milli neighbourhood system in milli topological spaces. We elucidate the relationship between them and investigate the conditions under which some of them are identical. Then we create the notion of milli limit points, milli derived sets and milli dense sets. We also introduce new types of sets namely milli clopen sets, milli extremely disconnected sets and milli door space in milli topological spaces. Moreover, we investigate some of their basic properties.

Keywords and Phrases: Milli limit point, milli dense set and milli door space.

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1. Introduction

While handling the real life problems having imprecision and ambiguity, deviations from conventional way of thinking or logic often resulted in fruitful structures. Rough-set theory is a novel mathematical approach originated by Pawlak [16] in the 1980s to manage inexplicit and uncertain data that cannot be addressed by the classical set theory. In recent years, rough set theory and its extended models have raised more and more scholars attention in various fields; especially, those who work in computer science and artificial intelligence. This theory was introduced by Pawlak, was an effective and robust tool to cope with imperfect knowledge problems. The key idea in this approach is the approximation space (AS) which comprises an equivalence relation R on a nonempty set U of objects. By Pawlak's approach, each subset of data can be approximated using approximation operators called lower approximation and upper approximation, which are defined by the equivalence classes induced by R. These operators categorize the knowledge obtained from the data into three main regions: positive, negative, and boundary.

In many real-life issues that humans deal with computer networks, economics, medical sciences, engineering, etc., the condition of an equivalence relation does not appear as a description for the relationship between the objects, which abolishes the ability of Pawlak's rough-set theory to deal with these problems [20]. To overcome this obstacle various frames of rough set theory defined with respect to non-equivalence relations, known as generalized rough-set theory or generalized AS, have been proposed.

The first generalized rough-set model constructed by a non-equivalence relation was introduced by Yao [27] in 1996. He defined the concepts of right neighborhood Nr and left neighborhood Nl of each object under arbitrary relation as alternatives to the equivalence class. That is, the granules or blocks that are used to approximate the knowledge obtained from the subset of data are these types of neighborhoods. Then, researchers have established other kinds of generalized ASs under specific relations like tolerance [20], similarity [1, 22], quasiorder [18, 29] and dominance [19, 30]. It has been introduced that many generalized ASs are produced by specific kinds of neighborhood systems; for example, Dai et al. [10] scrutinized some models of ASs using the maximal right neighborhoods defined over a similarity relation. Al-shami [3] completed studying the other kinds of maximal neighborhoods under any arbitrary relation and showed how they applied to classify patients suspected of infection with COVID-19. To improve the approximation operators by adding objects to the lower approximation and /or removing objects from the upper approximation, the concepts of core neighborhoods and remote neighborhoods were presented by Mareay [15] and Sun et al. [23], respectively. Also, Abu-Donia [2] adopted a new line of rough-set models depending on a finite family of arbitrary relations instead of one relation. Recently, Al-shami with his co-authors have displayed novel sorts of neighborhood systems and their generalized rough paradigms inspired by some relationships between $N\rho$-neighborhoods, such as $C\rho$-neighborhoods [4], $S\rho$-neighborhoods [5], and $E\rho$-neighborhoods [6].

Topology is another interesting orientation for studying rough-sets. The possibility of replacing roughest concepts with their topological counterparts follows from the similar behaviors of topological and rough-set concepts. Investigation of this link was started by Skowron [21] and Wiweger [26]. This domain attracted many scholars and researchers to initiate rough-set notions via their topological counterparts; for instance, Lashin et al. [14] suggested a family $N\rho$-neighborhood of each element as a subbase for topology, and then they coped with the notions of rough-set theory as topological concepts.

Tareq M. Al-shami [7], presented other types of generalized rough-set models directly defined by the concepts of subset neighborhoods and ideals. Tareq M. Al-shami [8], found novel rough-approximation operators inspired by an abstract structure called “supra-topology”. This approach is more relaxed than topological ones and extends the scope of applications because an intersection condition of topology is dispensed. Tareq M. Al-shami [9], initiate novel generalized rough set models using the concepts of maximal left neighborhoods and ideals. Their basic features are studied and the relationships between them are revealed.

Fuzzy sets, rough sets and soft sets are best examples of violating one or more orthodox schools of thinking [12, 28]. The notion of topology is a useful tool to identify the relationship between spatial objects and their characteristics. Also there are real world situations in which the boundaries of objects are vague. Generalized topological structures can analyse objects along with their vagueness. The notion of milli topology was introduced by Ittanagi et al. [11], which was defined in terms of approximations, boundary region, approximations of edges and approximations of boundary regions of a subset of a universe using an equivalence relation on it.

The milli topological space consists of maximum of nine elements. The authors also introduced some basic topological operators of the resultant milli topological space, such as milli open set, milli closed set, milli closure, milli interior and milli exterior.

This paper is organized as follows. In Section 2, some preliminary concepts of rough set theory, milli topological space and nano topological space are recalled. In Section 3, the concept of milli neighbourhood of a point, milli neighbourhood of a set and milli neighbourhood system in milli topological spaces are discussed. In Section 4, the notions of milli limit point, milli derived set and milli dense set are introduced and their properties are investigated. In Section 5, we defined milli clopen set, milli extremally disconnected set and milli door space in milli topological spaces and their properties are studied and the conclusion is given in Section 6.

2. Preliminaries

Definition 2.1. [22] *Let U be a non-empty set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. The pair (U, R) is called the approximation space. Let X be a subset of U.*
i). The lower approximation of X with respect to R is the set of all objects,

which can be for certain classified as X with respect to R and it is denoted by $\underline{R}(X)$ and is defined as $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$, where $[x]_R$ denotes the equivalence class determined by x.
ii). The upper approximation of X with respect to R is the set of all objects,

which can be possibly classified as X with respect to R and it is denoted by $\overline{R}(X)$ and is defined as $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$.

Definition 2.2. [1] *Let (U, R) be an approximation space and R be an equivalence relation on U. Let X be a subset of U.*
i). The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $BN(X)$ or $B(X)$. It is defined as $B(X) = \overline{R}(X) - \underline{R}(X)$.
ii). The set X is said to be rough with respect to R if $\overline{R}(X) \neq \underline{R}(X)$. That is, if $B(X) \neq \phi$.

Definition 2.3. [1] *Let (U, R) be an approximation space and R be an equivalence relation on U. Let X be a subset of U.*
i). The internal edge (lower edge) of X with respect to R and it is denoted by $\underline{E}(X)$. It is defined as $\underline{E}(X) = X - \underline{R}(X)$
ii). The external edge (upper edge) of X with respect to R and it is denoted by $\overline{E}(X)$. It is defined as $\overline{E}(X) = \overline{R}(X) - X$.

Definition 2.4. [16] *Let U be a non-empty set of objects called the universe and R an equivalence relation on U named as the indiscernibility relation. The pair (U, R) is called the approximation space. Let X be a subset of U.*
i). The lower boundary region of X with respect to R is denoted by $\underline{B}(X)$ and is defined as $\underline{B}(X) = \overline{R}(X) - \underline{E}(X)$.
ii). The upper boundary region of X with respect to R is denoted by $\overline{B}(X)$ and is defined as $\overline{B}(X) = \overline{R}(X) - \overline{E}(X)$.

Definition 2.5. [16] *Let U be a universe of objects and R an equivalence relation on U and $\tau_M(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X)\}$, where $X \subseteq U$ and $\tau_M(X)$ satisfies the following axioms:*
i) U and $\phi \in \tau_M(X)$
ii) The union of the elements of any sub-collection of $\tau_M(X)$ is in $\tau_M(X)$.
iii) The intersection of the elements of any finite sub-collection of $\tau_M(X)$ is in $\tau_M(X)$.

That is, $\tau_M(X)$ forms a topology on U called the milli topology on U with respect to X. We call $(U, \tau_M(X))$ as the milli topological space. The elements of $\tau_M(X)$ are called milli-open sets, denoted as $MO(X)$.

Definition 2.6. [16] *Types of milli topology: Let U be a non-empty finite universe and $X \subseteq U$.*
Type - 1: If $\underline{R}(X) = \overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi\}$, is the indiscrete milli topology on U.
Type - 2: If $\underline{R}(X) = \overline{R}(X) = X$ or $B(X) = \phi$, then $\tau_M(X) = \{U, \phi, X\}$.
Type - 3: If $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X)\}$.
Type - 4: If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X), \overline{R}(X)\}$.
Type - 5: If $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X\}$.
Type - 6: If $\underline{R}(X) \neq \overline{R}(X)$, where $\underline{R}(X) \neq \phi$ and $\overline{R}(X) \neq U$, then $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X, \overline{R}(X)\}$ is the discrete milli topological space on U.

Definition 2.7. [16] *Let $(U, \tau_M(X))$ be a milli topological space with respect to X where $X \subseteq U$ and let $A \subseteq U$, the milli closure of A is defined as the intersection of all milli closed sets containing A and it is denoted by $Mcl(A)$. That is $Mcl(A)$ is the smallest milli closed set containing A.*

Definition 2.8. [16] *Let $(U, \tau_R(X))$ be a milli topological space with respect to X where $X \subseteq U$ and let $A \subseteq U$, the milli interior of A is defined as the union of all milli open subsets contained in A and it is denoted by $Mint(A)$. That is $Mint(A)$*

is the largest milli open subset contained in A.

Definition 2.9. [18] *Let U be a non-empty, universe of objects and R an equivalence relation on U. Let $X \subseteq U$ and $\tau_R(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X)\}$. Then $\tau_R(X)$ is a topology on U called the nano topology with respect to X.*

Definition 2.10. [29] *In a nano topological space, a set is said to be nano clopen set if it is both nano open set and nano closed set.*

Definition 2.11. [27] *A topological space $(U, \tau_R(X))$ is said to be a door space if and only if every subset of X is either open or closed set.*

3. Milli Neighbourhood of a Set

In this section, the notion of milli neighbourhood of a point, milli neighbourhood of a set and milli neighborhood system are introduced in milli topological spaces and their properties are discussed.

Definition 3.1. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$.*

A subset N of U is said to be a milli neighbourhood of a point $x \in U$ if there exists a milli open set G containing x such that $x \in G \subseteq N$.

A subset N of U is called a milli neighbourhood of a set $A \subseteq U$ if there exists a milli open set G such that $A \subset G \subseteq N$.

The collection of all milli neighbourhood of $x \in U$ is called the milli neighborhood system at x and shall be denoted by $MN(x)$.

Remark 3.2. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. Then*
i) Every milli open set is a milli neighbourhood of each of its points.
ii) Milli neighbourhood of a point need not be a milli open set.
iii) Every milli open set containing x is a milli neighbourhood of a point x.

Example 3.3. Let $U = \{a, b, c, d\}$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{a, c\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a, c\}\}$. The milli neighbourhood of a point a consists of the sets: $\{\{a, c\}, \{a, b, c\}, \{a, c, d\}, U\}$. Hence $MN(a) = \{\{a, c\}, \{a, b, c\}, \{a, c, d\}, U\}$.

The set $\{a, b, c\}$ is milli neighborhood of a point a, but not milli open set in milli topological space U.

Example 3.4. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{a, c, d\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.
i) A subset $N = \{a, b, c\}$ of U, then for the point $a \in U$, there exists a milli open set $\{a\}$ containing a such that $a \in \{a\} \subseteq N$, for the point $b \in U$, there exists a milli open set $\{b\}$ containing b such that $b \in \{b\} \subseteq N$, also for the point $c \in U$,

there exists a milli open set $\{c, d\}$ containing c such that $c \in \{c, d\} \nsubseteq N$. Therefor the points a and b are milli neighbourhoods of N but c is not a milli neighbourhood of N.
ii) Let $N = \{b, c, d, e\} \subset U$ and $A = \{b, c, d\}$. Then there exists a milli open set $\{b, c, d\}$ such that $\{b, c, d\} \subset \{b, c, d\} \subseteq N$. Therefor the set $A = \{b, c, d\}$ is milli neighbourhood set of N.

Theorem 3.5. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. A subset A of a milli topological space U is milli open set if and only if it is a milli neighbourhood of each points.*
Proof. Let G be a milli open subset of a milli topological space U. Then for every $x \in G$, such that $x \in G \subseteq G$, and therefore G is a milli neighbourhood of each points.

Conversely let G be a milli neighbourhood of each points. If $G = \phi$, we are done. If $G \neq \phi$, then to each point $x \in G$ there exists a milli open set G_x suct that $x \in G_x \subseteq G$. It follows that $G = \bigcup\{G_x : x \in G\}$. Hence G is milli open, being a union of milli open sets.

Theorem 3.6. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$ and for each $x \in U$. Let $MN(x)$ be the collection of all milli neighbourhood x. Then*
i) $\forall\ x \in MN(x)$, $MN(x) \neq \phi$. i. e. every point x has at least one milli neighbourhood.
ii) $N \in MN(x) \Rightarrow x \in N$. i. e. every milli neighbourhood of x contains x. iii) $N \in MN(x)$, $N \subseteq M \Rightarrow M \in MN(x)$. i.e. every set containing a milli neighbourhood of x is a milli neighbourhood of x.
iv) $N \in MN(x)$, $M \in MN(x) \Rightarrow N \cap M \in MN(x)$. i. e. the intersection of two milli neighbourhoods of x is a milli neighbourhood of x.
v) $N \in MN(x) \Rightarrow \exists\ M \in MN(x)$ such that $M \subseteq N$ and $M \in MN(y)\ \ \forall\ y \in M$. i.e. if N is a milli neighbourhood of x, then there exists a milli neighbourhood M of x which is a subset of N such that M is a milli neighbourhood of each of its points.
Proof. i) Since U is a milli open set, it is a milli neighbourhood of every $x \in U$. Hence there exists at least one milli neighbourhood (namely U) for each $x \in U$. Hence $MN(x) \neq \phi$, $\forall\ x \in U$.
ii) If $N \in MN(x)$, then N is a milli neighbourhood of x. So by definition of milli neighbourhood, $x \in N$.
iii) If $N \in MN(x)$, then there is a milli open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, $x \in G \subseteq M$ and so M is milli neighbourhood of x. Hence $M \in MN(x)$.
iv) Let $N \in MN(x)$ and $M \in MN(x)$. Then by definition of milli neighbourhood, there exist milli open sets G_1 and G_2 such that $x \in G_1 \subseteq N$ and $x \in G_2 \subseteq M$.

Hence $x \in G_1 \cap G_2 \subseteq N \cap M$. Since $G_1 \cap G_2$ is milli open set in U. Therefore $N \cap M$ is a milli neighbourhood of x. Hence $N \cap M \in MN(x)$.
v) If $N \in MN(x)$, then there exists a milli open set M such that $x \in M \subseteq N$. Since M is milli open set, it is a milli neighbourhood of its points. Therefore $M \in MN(y)$, $\forall\, y \in M$.

Theorem 3.7. *Let U be a non-empty set and for each $x \in U$, let $MN(x)$ be a non-empty collection of subsets of U satisfying the following conditions:*
i) $N \in MN(x) \Rightarrow x \in N$
ii) $N \in MN(x)$, $N \subseteq M \Rightarrow M \in MN(x)$.
iii) Let $\tau_M(X)$ consist of the empty set and all non-empty subsets G and U having the property that $x \in G$ implies that there exists an $N \in MN(x)$ such that $x \in N \subseteq G$. Then $\tau_M(X)$ is a milli topology for U.
Proof. i) $\phi \in \tau_M(X)$, by definition. Now we show that $U \in \tau_M(X)$.
Let $x \in U$. Since $MN(x) \neq \phi$, there is a $N \in MN(x)$ and so $x \in N$ by (i). Since N is a subset of U, we have $x \in N \subseteq U$. Hence $U \in \tau_M(X)$.
ii) Let $G_1, G_2 \in \tau_M(X)$. If $x \in G_1 \cap G_2$, then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau_M(X)$ and $G_2 \in \tau_M(X)$, there exist $N \in MN(x)$ and $M \in MN(x)$ such that $x \in N \subseteq G_1$ and $x \in M \subseteq G_2$. Then $x \in N \cap M \subseteq G_1 \cap G_2$. But $N \cap M \in MN(x)$ by (ii). Hence $G_1 \cap G_2 \in \tau_M(X)$.
iii) Let $G_\lambda \in \tau_M(X)$, $\forall\, \lambda \in \Lambda$. If $x \in \bigcup\{G_\lambda : \lambda \in \Lambda\}$, then $x \in G_{\lambda_x}$ for some $\lambda_x \in \Lambda$.
iii) Let $G_\lambda \in \tau_M(X)$, $\forall\, \lambda \in \Lambda$. If $x \in \bigcup\{G_\lambda : \lambda \in \Lambda\}$, then $x \in G_{\lambda_x}$ for some $\lambda_x \in \Lambda$. Since $G_{\lambda_x} \in \tau_M(X)$, there exists an $N \in MN(x)$ such that $x \in N \subseteq G_{\lambda_x}$ and consequently $x \in N \subseteq \bigcup\{G_\lambda : \lambda \in \Lambda\}$. Hence $\bigcup\{G_\lambda : \lambda \in \Lambda\} \in \tau_M(X)$. It follows that $\tau_M(X)$ is a milli topology for U.

Theorem 3.8. *Let $(U, \tau_M(X)$ be a milli topological space where $X \subseteq U$. If A is milli closed set of U and $x \in Mcl(A)$ if and only if for any milli neighbourhood N of $x \in U$, $N \cap A \neq \phi$.*
Proof. On the contrary, let us assume that there exists milli neighbourhood N of $x \in U$ such that $N \cap A = \phi$. Then there exists a milli open set G of U such that $x \in G \subset N$. Therefore we have $G \cap A = \phi$ and so $x \in U - G$. Then $Mcl(A) \in U - G$ and therefore $x \notin Mcl(A)$ which is contradiction to the hypothesis $x \in Mcl(A)$. Therefore $N \cap A \neq \phi$.

Conversely, suppose that $x \notin Mcl(A)$. Then there exists a milli closed set F of U such that $A \subseteq F$ and $x \notin F$. Thus $x \in U - F$ and $U - F$ is a milli open set in U and hence $U - F$ is a milli neighbourhood of point $x \in U$. But $A \cap (U - F) = \phi$ which is contradiction to the hypothesis. Hence $x \notin Mcl(A)$.

Definition 3.9. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. A non-empty collection $\beta(x)$ of milli neighbourhood of x is called a milli base for the milli neighbourhood system of x if for every milli neighbourhood N of x there is $B \in \beta(x)$ such that $B \subseteq N$. Also we say that $\beta(x)$ is a milli local base at x or a fundamental system of milli neighbourhoods of x.*

If $\beta(x)$ is a milli local base at x, then the members of $\beta(x)$ are called milli basic neighbourhoods of x.

Example 3.10. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{a, c, e\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{e\}, \{a, c\}, \{b, d\}, \{a, c, e\}, \{a, b, d\}, \{a, b, c, d\}\}$. The milli local base at each of the points a, b, c, d, e is given by: $\beta(a) = \{a, c\}$, $\beta(b) = \{b, d\}$, $\beta(c) = \{a, c\}$, $\beta(d) = \{b, d\}$, $\beta(e) = \{e\}$. Observe that here a milli local base at each point consists of a single milli neighbourhood of the point.

Theorem 3.11. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$ and let $\beta(x)$ be a milli local base at any point x of U. Then $\beta(x)$ has the following properties:*
i) $\beta(x) \neq \phi$ for every $x \in U$.
ii) If $B \in \beta(x)$, then $x \in B$.
iii) If $A \in \beta(x)$ and $B \in \beta(x)$, then there exists a $C \in \beta(x)$ such that $C \subseteq A \cap B$.
iv) If $A \in \beta(x)$, then there exists a set B such that $x \in B \subseteq A$ and $\forall y \in B$, $\exists\ C \in \beta(y)$ satisfying $C \subseteq B$.

Definition 3.12. *A milli topological space $(U, \tau_M(X))$ is said to satisfy the first axiom of milli countability if each point of U possesses a countable milli local base. Such a milli topological space is said to be a first countable milli space.*

Example 3.13. By Example 3.10, the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{e\}, \{a, c\}, \{b, d\}, \{a, c, e\}, \{a, b, d\}, \{a, b, c, d\}\}$. The milli topological space $(U, \tau_M(X))$ is a first countable milli space. Since each point of U has countable milli local base.

Definition 3.14. *Let $(U, \tau_M(X))$ be a milli topological space. A collection $\beta(x)$ of subsets of U is said to form a milli base for $\tau_M(X)$ if and only if*
i) $\beta(x) \subseteq \tau_M(X)$,
ii) for each point $x \in U$ and each milli neighbourhood N of x there exists some $B \in \beta(x)$ such that $x \in B \subseteq N$.

Example 3.15. Let $U = \{a, b, c, d\}$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{a, b, c\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$. Then the collection $\beta = \{\{b\}, \{d\}, \{a, c\}\}$ is a milli base for $\tau_M(X)$. Since i) $\beta \subseteq \tau_M(X)$ and ii) Each milli neighbourhood of b contains {b} with a member of β containing b. Similarly each milli neighbourhood of d contains $\{d\} \in \beta$ and each milli neighbourhood of a or c contains $\{a, c\} \in \beta$.

Definition 3.16. *Let $(U, \tau_M(X))$ be a milli topological space. The space is said to be a second countable milli space if there exists a countable milli base for $\tau_M(X)$.*

Theorem 3.17. *Let $(U, \tau_M(X))$ be a milli topological space. A sub collection β of $\tau_M(X)$ is a milli base for $\tau_M(X)$ if and only if every milli open set can be expressed as the union of members of β.*

Proof. Let β be a milli base for $\tau_M(X)$ and $G \in \tau_M(X)$. Since G is milli open, it is a milli neighbourhood of each of its points. Hence by definition of milli base, $\forall\ G$, $\exists\ B \in \beta$ such that $x \in B \subseteq G$. It follows that $G = \bigcup\{B : B \in \beta \text{ and } B \subseteq G\}$.

Conversely, let $\beta \subseteq \tau_M(X)$ and every milli open set G be the union of members of β. We have to show that β is a milli base for $\tau_M(X)$. We have
i) $\beta(x) \subseteq \tau_M(X)$ (given),
ii) Let $x \in U$ and N be any milli neighbourhood of x. Then there exists a milli open set G such that $x \in G \subseteq N$. But G is the union of members of β. Hence there exists $B \in \beta$ such that $x \in B \subseteq G \subseteq N$. Thus β is a milli base for $\tau_M(X)$.

4. Milli Derived and Milli Dense Sets

In this section, the notion of milli limit points, milli derived sets and milli dense sets are introduced in milli topological spaces and their properties are investigated.

Definition 4.1. *Let $(U, \tau_M(X))$ be a milli topological space and A be a subset of U. A point $x \in U$ is called a milli limit point (or a milli cluster point) of A if and only if every milli neighbourhood N of x contains a point of A other than x. That is $[N - \{x\}] \cap A \neq \phi$.*

Equivalently if and only if every milli open set G containing x contains a point of A other than x. That is $G \cap (A - x) \neq \phi$.

The set of all milli limit points of A is called the milli derived set of A and denoted by $MD(A)$.

Example 4.2. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{a, c, d\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Let $A = \{a, c, e\} \subset U$. The point a is not a milli limit point of A, since the milli open set $\{a\}$ containing the point a, has no other point of A. That is $\{a\} \cap (A - \{a\}) = \phi$. The point b is not a milli limit point of A, since the milli open set $\{b\}$ containing the point b, has no other point of A. That is $\{b\} \cap (A - \{b\}) = \phi$. The point c is not a milli limit point of A, since the milli open set $\{c, d\}$ contains the point c. That is $\{c, d\} \cap (A - \{c\}) = \phi$.

But the point d is milli limit point of A, since the milli open set $\{c, d\}$ contains the point d. That is $\{c, d\} \cap (A - \{d\}) \neq \phi$. Also the point e is milli limit point of A, since $U \cap (A - \{e\}) \neq \phi$. Hence $MD(A) = \{d, e\}$.

Theorem 4.3. *Let $(U, \tau_R(X))$ and $(U, \tau_M(X))$ be nano topological and milli topological spaces, respectively, where $X \subseteq U$. Let $A \subseteq U$ then*
i) Every nano limit point is milli limit point of A.
ii) $ND(A) \subset MD(A)$.
Proof. Let $A \subseteq U$ and $x \in U$.
i) A point x is a nano limit point of A, for every nano open sets G containing x, such that $G \cap (A - \{x\}) \neq \phi$. Since every nano open set is milli open set. Therefore G is milli open set and G containing x, then $G \cap (A - \{x\}) \neq \phi$. Therefore x is a milli limit point of A. Hence x is nano limit point is milli limit point of A.
ii) If $x \in ND(A)$, for every nano open set G containing x such that $G \cap (A - \{x\}) \neq \phi$. Since every nano open set is milli open set. Therefore G is milli open set G is containing x. Then $G \cap (A - \{x\}) \neq \phi$, which implies $x \in MD(A)$. Hence $ND(A) \subset MD(A)$.

Theorem 4.4. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$.*
$Mcl(A) = A \cup MD(A)$, where $A \subseteq (U, \tau_R(X))$.
Proof. If $x \in Mcl(A)$, then $x \in A$ or $x \in MD(A)$. If $x \in A$ then $x \in Mcl(A)$. Therefore, let $x \notin A$. That is $x \in MD(A)$. Thus for every milli open set G containing x, $G \cap (A - \{x\}) \neq \phi$. Since $x \neq A$, $G \cap A \neq \phi$. Therefore $x \in Mcl(A)$. Thus $A \cup MD(A) \subseteq Mcl(A)$. If $x \in Mcl(A)$ and for every milli open set G containing x, we have $G \cap (A - \{x\}) \neq \phi$. Therefore, $x \in MD(A)$. This means $x \in A \cup MD(A)$. Thus $Mcl(A) \subseteq A \cup MD(A)$. Hence $Mcl(A) = A \cup MD(A)$.

Corollary 4.5. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. A is milli closed set in U if and only if $MD(A) \subseteq A$.*
Proof. A is milli closed set in U if and only if $Mcl(A) = A$, if and only if $A \cup MD(A) \subseteq A$, if and only if $MD(A) \subseteq A$.

Theorem 4.6. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. If $A, B \subseteq (U, \tau_M(X))$, then*
i) $MD(\phi) = \phi$
ii) $x \in MD(A) \Rightarrow x \in MD(A - \{x\})$
iii) $A \subseteq B \Rightarrow MD(A) \subseteq MD(B)$
iv) $MD(A \cup B) = MD(A) \cup MD(B)$
v) $MD(A \cap B) \subseteq MD(A) \cap MD(B)$.
Proof. i) Let $x \in U$. If $x \in MD(\phi)$, then every milli open set G containing x has

at least one point of ϕ other than x. But ϕ has no point. Therefore, no point of U can be a limit point of A. That is $MD(\phi) = \phi$.
ii) If $x \in MD(A)$, then $G \cap (A - \{x\}) \neq \phi$ for every milli open set G containing x. Since $(A - \{x\}) - \{x\} = A - \{x\}$, $G \cap ((A - \{x\}) - \{x\} \neq \phi$ for every milli open set G containing x. Therefore $x \in MD(A - \{x\})$. Thus $MD(A) \subseteq MD(A - \{x\})$
iii) Let $A \subseteq B$ and $x \in MD(A)$. Therefore for every milli open set G containing x, $G \cap (A - x) \neq \phi$. Since $A \subseteq B$, $G \cap (B - x) \neq \phi$ and hence $x \in MD(B)$. Thus $MD(A) \subseteq MD(B)$.
iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $MD(A) \subseteq MD(A \cup B)$ and $MD(B) \subseteq MD(A \cup B)$ and hence $MD(A) \cup MD(B) \subseteq MD(A \cup B)$. Let $x \notin MD(A) \cup MD(B)$. Then $x \notin MD(A)$ and $x \notin MD(B)$. Therefore, there exists milli open sets G_1 and G_1 containing x such that $G_1 \cap (A - \{x\}) = \phi$ and $G_2 \cap (B - \{x\}) = \phi$. Since $G_1 \cap G_2 \subseteq G_1$ and $G_1 \cap G_2 \subseteq G_2$, $(G_1 \cap G_2) \cap (A - \{x\}) = \phi$ and $(G_1 \cap G_2) \cap (B - \{x\}) = \phi$. Also $G_1 \cap G_2$ is a milli open set containing x. Therefore $(G_1 \cap G_2) \cap [(A \cup B) - \{x\}] = \phi$. That is, x is not a milli limit point of $A \cup B$. This implies $x \in MD(A \cup B)$. Thus $x \notin MD(A) \cup MD(B) \Rightarrow x \notin MD(A \cup B)$. Thus we have $x \notin MD(A \cup B) \Rightarrow x \notin MD(A) \cup MD(B)$. Therefore $MD(A \cup B) \subseteq MD(A) \cup MD(B)$. Hence $MD(A \cup B) = MD(A) \cup MD(B)$.
v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $MD(A \cap B) \subseteq MD(A)$ and $(A \cap B) \subseteq MD(B)$ and hence $MD(A \cap B) \subseteq MD(A) \cap MD(B)$.

Remark 4.7. *Theorem* 4.6(v)*, equality does not hold, as it is shown in the following example.*

Example 4.8. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$ and $X = \{a, c, d\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.

Equality does not hold in (v), if $A = \{a, e\}$ and $B = \{c, d\}$, $A \cap B = \phi$ and hence $MD(A \cap B) = \phi$, $MD(A) \cap MD(B) = \{e\} \cap \{c, d, e\} = \{e\}$. Thus $MD(A \cap B) \neq MD(A) \cap MD(B)$.

Theorem 4.9. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. If A is a singleton subset of a milli topological space U, then $MD(A) = Mcl(A) - A$.*
Proof. If $x \in MD(A)$, then for every milli open set G containing x, such that $G \cap (A - \{x\}) \neq \phi$. Then $x \notin A$. (otherwise, if $x \in A$, then $A = \{x\}$ and hence $G \cap (A - \{x\}) = \phi$). But $MD(A) \subseteq Mcl(A)$. Thus $x \in Mcl(A)$, but $x \notin A$, whenever $x \in MD(A)$. Therefore $MD(A) \subseteq Mcl(A) - A$. If $x \in Mcl(A) - A$, $x \in Mcl(A)$ but $x \notin A$. Therefore $G \cap A \neq \phi$ for every milli open set G containing x. That is $G \cap (A - \{x\}) \neq \phi$ for every milli open set G containing x. Therefore $x \in MD(A)$. Thus $Mcl(A) - A \subseteq MD(A)$. Hence $MD(A) = Mcl(A) - A$, if A is

a singleton set.

Definition 4.10. *Let $(U, \tau_M(X))$ be a milli topological space, where $X \subseteq U$.*
i) A subset D of U is said to be milli dense set if $Mcl(D) = U$.
ii A subset G of U is said to be milli nowhere dense set if $Mint[Mcl(G)] = \phi$.

Remark 4.11. *i) Trivially the entire set U is always milli dense set.*
ii) In type - 1, indiscrete milli topology, $\tau_M(X) = \{U, \phi\}$. Any non empty subset D of U is milli dense set of U, since $Mcl(D) = U$ and any subset G of U is milli nowhere dense set of U, since $Mint[Mcl(G)] = \phi$.
iii) In type - 2, milli topology, $\tau_M(X) = \{U, \phi, X\}$. Any non empty subset $D \neq X^C$ of U is milli dense set of U, since $Mcl(D) = U$. Suppose $D = X^C \subset U$ is not a milli dense set, since $Mcl(D) = Mcl(X^C) = X^C \neq U$, where X^C is milli closed set.

Theorem 4.12. *A subset D of a milli topological space $(U, \tau_M(X))$ is milli dense set of U if and only if for every non-empty milli open subset G of U, $D \cap G = \phi$.*
Proof. Suppose D is milli dense set of U and G is a non-empty milli open set of U. If $D \cap G = \phi$, then $D \subset (U - G)$ which implies $Mcl(D) \subset (U - G)$, since $U - G$ is milli closed set of U. But $U - G \subsetneq U$ contradicting that $Mcl(D) = U$. Since $Mcl(D) \subset (U - G) \subsetneq U$.

Conversely we assume that D meets every non-empty milli open subset of U. Thus the milli closed set containing D of U and consequently $Mcl(D) = U$. Hence D is milli dense set of U.

Theorem 4.13. *Let $(U, \tau_M(X))$ be a milli topological space where $X \subseteq U$. Any set C containing a milli dense set D is a milli dense set.*

5. Milli Extremally Disconnected Sets and Milli Door Space

In this section, milli clopen sets, milli extremally disconnected sets and milli door space in milli topological spaces are defined and their properties are studied.

Definition 5.1. *In a milli topological space, a set is said to be milli clopen set if it is both milli open set and milli closed set.*

Remark 5.2. *In a milli topological space U, U and ϕ are milli clopen sets.*

Example 5.3. Let $U = \{a, b, c, d\}$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{a, b, c\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a, c\}, \{b, d\}, \{b\}, \{d\}, \{a, c, d\}, \{a, b, c\}\}$. The milli open sets are: $U, \phi, \{a, c\}, \{b, d\}, \{b\}, \{d\}, \{a, c, d\}, \{a, b, c\}$ and the milli closed sets are: $\phi, U, \{b, d\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}, \{b\}, \{d\}$. Hence $U, \phi, \{a, c\}, \{b, d\}, \{b\}, \{d\}, \{a, c, d\}, \{a, b, c\}$ are milli clopen sets in U. Since every milli open set is milli closed set in U.

Theorem 5.4. *Every nano clopen set is milli clopen set, but converse need not be true.*

Proof. Let $(U, \tau_R(X))$ and $(U, \tau_M(X))$ be nano and milli topological spaces respectively, where $X \subseteq U$. The nano topological space, $\tau_R(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X)\}$ and the milli topological space, $\tau_M(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X)\}$. Since every nano open set is milli open set and every nano closed set is milli closed set. Hence every nano clopen set is milli clopen set of U.

Example 5.5. By Example 5.3, the nano topological space, $\tau_R(X) = \{U, \phi, \{a, c\}, \{c, d\}\}$ and the milli topological space, $\tau_M(X) = \{U, \phi, \{a, c\}, \{b, d\}, \{b\}, \{d\}, \{a, c, d\}, \{a, b, c\}\}$. Since $\{a, c, d\}$ is milli clopen set but not nano clopen set.

Theorem 5.6. *Let U be a non-empty finite universe and $X \subseteq U$, the following milli topological spaces are clopen sets.*
i) In type - 1: If $\underline{R}(X) = \overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi\}$
ii) In type - 3: If $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X)\}$.
iii) In type - 5: If $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$, then $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X)\}$.

Proof. Let U be a non-empty finite universe and $X \subseteq U$
i) Since $\underline{R}(X) = \overline{R}(X) = U$, that is $B(X) = \phi$, $\underline{E}(X) = \phi$, $\overline{E}(X) = \phi$, $\underline{B}(X) = U$, $\overline{B}(X) = U$. Thus $\tau_M(X) = \{U, \phi\}$. Hence U, ϕ are milli clopen sets in U. Since every milli open set is milli closed set in U.
ii) Since $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$, that is $B(X) = U$, $\underline{E}(X) = X - \phi = X$, $\overline{E}(X) = U - X = X^C$, $\underline{B}(X) = U - X = X^C$, $\overline{B}(X) = U - X^C = X$. Thus $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X)\}$, or $\tau_M(X) = \{U, \phi, X, X^C\}$. The milli open sets are $U, \phi, \underline{E}(X), \overline{E}(X)$ or U, ϕ, X, X^C and the milli closed sets are $\phi, U, [\underline{E}(X)]^C, [\overline{E}(X)]^C$ or ϕ, U, X^C, X. Hence $U, \phi, \underline{E}(X), \overline{E}(X)$ are milli clopen sets in U.
iii) Since $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$, that is $B(X) = [\underline{R}(X)]^C$, $\underline{E}(X) = \{X - \underline{R}(X)\}$, $\overline{E}(X) = X^C$, $\underline{B}(X) = [X - \underline{R}(X)]^C$, $\overline{B}(X) = X$. Thus $\tau_M(X) = \{U, \phi, \underline{R}(X), [\underline{R}(X)]^C, \{X - \underline{R}(X)\}, X^C, [X - \underline{R}(X]^C, X\}$. The milli open sets are $\{U, \phi, \underline{R}(X), [\underline{R}(X)]^C, \{X - \underline{R}(X)\}, X^C, [X - \underline{R}(X]^C, X\}$ and the milli closed sets are $\{\phi, U, [\underline{R}(X)]^C, \underline{R}(X), [X - \underline{R}(X)]^C, X, \{X - \underline{R}(X)\}, X^C\}$. Hence $U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X)\}$ are milli clopen sets in U.

Remark 5.7. *Let U be a non-empty finite universe and $X \subseteq U$, the following milli topological spaces are not clopen sets.*
i) In type - 2: If $\underline{R}(X) = \overline{R}(X) = X$ or $B(X) = \phi$, then $\tau_M(X) = \{U, \phi, X\}$.
ii) In type - 4: If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X), \overline{R}(X)\}$.
iii) In type - 6: If $\underline{R}(X) \neq \overline{R}(X)$, where $\underline{R}(X) \neq \phi$ and $\overline{R}(X) \neq U$, then

$\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X, \overline{R}(X)\}$. *As for the following example.*

Example 5.8. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c, d\}, \{e\}\}$, the family of equivalence classes of U by an equivalence relation R. We have the following
i) In type - 2: If $\underline{R}(X) = \overline{R}(X) = X$ or $B(X) = \phi$, then $\tau_M(X) = \{U, \phi, X\}$. Let $X = \{a, b\}$ then $\underline{R}(X) = \overline{R}(X) = \{a, b\} = X$, $B(X) = \phi$, $\underline{E}(X) = \phi$, $\overline{E}(X) = \phi$, $\underline{B}(X) = U$, $\overline{B}(X) = U$. Thus $\tau_M(X) = \{U, \phi, X\}$. The milli open sets are $U, \phi, \{a, b\}$ and the milli closed sets are $\phi, U, \{c, d, e\}$. Therefore $U, \phi, \{a, b\}$ are not milli clopen sets in U. Since the milli open set $\{a, b\}$ is not milli closed set in U.
ii) In type - 4: If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X), \overline{R}(X)\}$. Let $X = \{a, c\}$ then $\underline{R}(X) = \phi$, $\overline{R}(X) = \{a, b, c, d\} \neq U$, $B(X) = \overline{R}(X)$, $\underline{E}(X) = \{a, c\}$, $\overline{E}(X) = \{b, d\}$, $\underline{B}(X) = \{b, d\}$, $\overline{B}(X) = \{a, c\}\}$. Hence $\tau_M(X) = \{U, \phi, \{a, c\}, \{b, d\}, \{a, b, c, d\}\}$. The milli open sets are $U, \phi, \{a, c\}, \{b, d\}, \{a, b, c, d\}$ and the milli closed sets are $\phi, U, \{b, d, e\}, \{a, c, e\}, \{e\}$. Therefore $U, \phi, \{a, c\}, \{b, d\}, \{a, b, c, d\}$ are not milli clopen sets in U. Since the milli open set $\{a, c\}$ is not milli closed set in U.
iii) In type - 6: If $\underline{R}(X) \neq \overline{R}(X)$, then $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X, \overline{R}(X)\}$. Let $X = \{a, b, c\}$ then $\underline{R}(X) = \{a, b\}$, $\overline{R}(X) = \{a, b, c, d\}$, i.e. $\underline{R}(X) \neq \overline{R}(X)$, $\underline{R}(X) \neq \phi$, $\overline{R}(X) \neq U$, $B(X) = \{c, d\}$, $\underline{E}(X) = \{c\}$, $\overline{E}(X) = \{d\}$, $\underline{B}(X) = \{a, b, d\}$, $\overline{B}(X) = \{a, b, c\}\}$. Thus $\tau_M(X) = \{U, \phi, \{a, b\}, \{a, b, c, d\}, \{c, d\}, \{c\}, \{d\}, \{a, b, d\}, \{a, b, c\}\}$. The milli open sets are $U, \phi, \{a, b\}, \{a, b, c, d\}, \{c, d\}, \{c\}, \{d\}, \{a, b, d\}, \{a, b, c\}$ and the milli closed sets are $\phi, U, \{c, d, e\}, \{e\}, \{a, b, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{c, e\}, \{d, e\}$. Therefor $U, \phi, \{c\}, \{d\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$ are not milli clopen sets in U. Since the milli open set $\{a, b\}$ is not milli closed set in U.

Definition 5.9. *A milli topological space $(U, \tau_M(X))$ is called milli extremally disconnected, if the milli closure of each milli open set is milli open in U.*

Example 5.10. Let $U = \{a, b, c\}$, $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a, b\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a\}, \{b, c\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}\}$. The milli closed sets in U are: $\phi, U, \{b, c\}, \{a\}, \{a, c\}, \{a, b\}, \{b\}, \{c\}$. Therefore, $Mcl(U) = U$, $Mcl(\phi) = \phi$, $Mcl(\{a\}) = \{a\}$, $Mcl(\{b, c\}) = \{b, c\}$, $Mcl(\{b\}) = \{b\}$, $Mcl(\{c\}) = \{c\}$, $Mcl(\{a, c\}) = \{a, c\}$, $Mcl(\{a, b\}) = \{a, b\}$. That is, the milli closure of each milli open set in U is milli open set. Hence, $(U, \tau_M(X))$ is extremally disconnected.

Theorem 5.11. *Let U be a non-empty finite universe and $X \subseteq U$*
i) Type - 1: If $\underline{R}(X) = \overline{R}(X) = U$ then $(U, \tau_M(X))$ is extremally disconnected.

ii) Type - 2: If $\underline{R}(X) = \overline{R}(X) = X$ or $B(X) = \phi$ then $(U, \tau_M(X))$ is extremally disconnected.
iii) Type - 3: If $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$ then $(U, \tau_M(X))$ is extremally disconnected.
iv) Type - 5: If $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$ then $(U, \tau_M(X))$ is extremally disconnected.

Proof. Let U be a non-empty finite universe and $X \subseteq U$
i) Since $\underline{R}(X) = \overline{R}(X) = U$, that is $B(X) = \phi$, $\underline{E}(X) = \phi$, $\overline{E}(X) = \phi$, $\underline{B}(X) = U$, $\overline{B}(X) = U$. Therefore $\tau_M(X) = \{U, \phi\}$. The milli open sets are U, ϕ and milli closed sets are ϕ, U. Therefore $Mcl(U) = U$ and $Mcl(\phi) = \phi$. Thus the milli closure of each milli open set is milli open in U. Hence $(U, \tau_M(X))$, is extremally disconnected.
ii) Since $\underline{R}(X) = \overline{R}(X) = X$, that is $B(X) = \phi$, $\underline{E}(X) = \phi$, $\overline{E}(X) = \phi$, $\underline{B}(X) = X$, $\overline{B}(X) = X$. Therefore $\tau_M(X) = \{U, \phi, X\}$. The milli open sets are U, ϕ, X and milli closed sets are ϕ, U, X^C. Thus $Mcl(U) = U$, $Mcl(\phi) = \phi$, $Mcl(X) = U$. Therefore the milli closure of each milli open set is milli open in U. Hence, $(U, \tau_M(X))$, is extremally disconnected.
iii) Since $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$, that is $B(X) = U$, $\underline{E}(X) = X - \phi = X$, $\overline{E}(X) = U - X = X^C$, $\underline{B}(X) = U - X = X^C$, $\overline{B}(X) = U - X^C = X$. Therefore $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X)\}$, or $\tau_M(X) = \{U, \phi, X, X^C\}$. The milli open sets are $U, \phi, \underline{E}(X), \overline{E}(X)$ or U, ϕ, X, X^C and milli closed sets are $\phi, U, [\underline{E}(X)]^C, [\overline{E}(X)]^C$ or ϕ, U, X^C, X. Thus $Mcl(U) = U$ and $Mcl(\phi) = \phi$, $Mcl(X) = X$ and $Mcl(X^C) = X^C$. Therefore the milli closure of each milli open set is milli open in U. Hence $(U, \tau_M(X))$, is extremally disconnected.
iv) Since $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$, that is $B(X) = [\underline{R}(X)]^C$, $\underline{E}(X) = \{X - \underline{R}(X)\}$, $\overline{E}(X) = X^C$, $\underline{B}(X) = [X - \underline{R}(X)]^C$, $\overline{B}(X) = X$. Therefore $\tau_M(X) = \{U, \phi, \underline{R}(X), [\underline{R}(X)]^C, \{X - \underline{R}(X)\}, X^C, [X - \underline{R}(X]^C, X\}$. The milli open sets are $\{U, \phi, \underline{R}(X), [\underline{R}(X)]^C, \{X - \underline{R}(X)\}, X^C, [X - \underline{R}(X]^C, X\}$ and milli closed sets are $\{\phi, U, [\underline{R}(X)]^C, \underline{R}(X), [X - \underline{R}(X)]^C, X, \{X - \underline{R}(X)\}, X^C\}$. Therefore the milli closure of each milli open set is milli open in U. Hence $(U, \tau_M(X))$, is extremally disconnected.

Remark 5.12. *i) In type - 4: If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then $(U, \tau_M(X))$, is not extremally disconnected.*
ii) In type - 6: If $\underline{R}(X) \neq \overline{R}(X)$, then $(U, \tau_M(X))$, is not extremally disconnected. As for the following example.

Example 5.13. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c, d\}, \{e\}\}$, the family of equivalence classes of U by an equivalence relation R. We have the following

i) In type - 4: If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then $\tau_M(X) = \{U, \phi, \underline{E}(X), \overline{E}(X), \overline{R}(X)\}$. Let $X = \{a, c\}$ then $\underline{R}(X) = \phi$, $\overline{R}(X) = \{a, b, c, d\} \neq U$, $B(X) = U$, $\underline{E}(X) = \{a, c\}$, $\overline{E}(X)$={b, d}, $\underline{B}(X) = \{b, d\}$, $\overline{B}(X)$={a, c}. Hence $\tau_M(X) = \{U, \phi, \{a, c\}, \{b, d\}, \{a, b, c, d\}\}$. The milli open sets are $U, \phi, \{a, c\}, \{b, d\}, \{a, b, c, d\}$ and the milli closed sets are $\phi, U, \{b, d, e\}, \{a, c, e\}, \{e\}$. Then $Mcl(\{a, c\}) = \{a, c, e\}$ is not milli open set in U.
ii) In type - 6: If $\underline{R}(X) \neq \overline{R}(X)$, then $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X), \overline{R}(X)\}$. Let $X = \{a, b, c\}$ then $\underline{R}(X) = \{a, b\}$, $\overline{R}(X) = \{a, b, c, d\}$, $B(X) = \{c, d\}$, $\underline{E}(X) = \{c\}$, $\overline{E}(X)\{d\}$, $\underline{B}(X) = \{a, b, d\}$, $\overline{B}(X)\{a, b, c\}$. That is $\underline{R}(X) \neq \overline{R}(X)$, $\underline{R}(X) \neq \phi$, $\overline{R}(X) \neq U$. Hence $\tau_M(X) = \{U, \phi, \{a, b\}, \{a, b, c, d\}, \{c, d\}, \{c\}, \{d\}, \{a, b, d\}, \{a, b, c\}\}$. The milli open sets are $U, \phi, \{a, b\}, \{a, b, c, d\}, \{c, d\}, \{c\}, \{d\}, \{a, b, d\}, \{a, b, c\}$ and the milli closed sets are $\phi, U, \{c, d\}, \{e\}, \{a, b, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{c, e\}, \{d, e\}$. Then $Mcl(\{c\}) = \{c, e\}$ is not milli open set in U.

Definition 5.14. *Let $(U, \tau_M(X))$ be a milli topological space, where $X \subseteq U$. A milli topological space $(U, \tau_M(X))$ is said to be a milli door space if every subset of U is either milli open set or milli closed set.*

Example 5.15. Let $U = \{a, b, c, d\}$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$ and $X = \{a, b\}$. Then the milli topology on U with respect to X is $\tau_M(X) = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$. The milli open sets are: $U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}$ and the milli closed sets are: $\phi, U, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{a, c\}, \{c\}$. Thus all the subsets of U are either milli open or milli closed sets. Hence $(U, \tau_M(X))$ is a milli door space.

Theorem 5.16. *The discrete milli topological space in the type-6 of milli topology, the cordinality of U is less than or equal to four elements, i.e. $|U| \leq 4$. Then the discrete milli topology $\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X), \overline{R}(X)\}$ is a milli door space on U.*

Remark 5.17. *The discrete milli topological space in the type-6 of milli topology, the cordinality of $|U| \geq 5$. The discrete milli topology,*
$\tau_M(X) = \{U, \phi, \underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X, \overline{R}(X)\}$ is not a milli door space on U.

Example 5.18. Let $U = \{a, b, c, d, e\}$, $U/R = \{\{a, b\}, \{c, d\}, \{e\}\}$, the family of equivalence classes of U by an equivalence relation R. Then the milli topology on U with respect to X. The type - 6: If $X = \{a, b, c\}$ then $\underline{R}(X) = \{a, b\}$, $\overline{R}(X) = \{a, b, c, d\}$, i.e. $\underline{R}(X) \neq \overline{R}(X)$, $\underline{R}(X) \neq \phi$, $\overline{R}(X) \neq U$, $\tau_M(X) = \{U, \phi, \{c\}, \{d\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, i.e. $\tau_M(X) = \{U, \phi,$

$\underline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X, \overline{R}(X)\}$ is the discrete milli topological space on U. The milli open sets are: $U, \phi, \{c\}, \{d\}, \{c, d\}, \{a, b\}, \{a, b, c\}$ and the milli closed sets are: $\phi, U, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, e\}, \{c, d, e\}, \{d, e\}$.
Therefore $(U, \tau_M(X))$ is not a milli door space. Since the subset $\{a\}$ is neither milli open nor milli closed set.

6. Conclusion

In this paper, we have proposed a new concept of sets, which can be viewed as a set based on a generalization of milli topological spaces. We have explained the main advantages of improving previous model better than those existing in the literature. We presented basic properties based on milli topological spaces giving some examples when needed. Several new types of neighbourhood sets such as milli neighbourhood of a point, milli neighbourhood of a set, milli neighbourhood system, milli limit points, milli derived sets, milli dense sets, milli clopen set, milli extremely disconnected set and milli door space were discussed as well. We examined the relationship between these different sets. As future work, there is a lot of scope in this area. Also, we believe that topological structure is the appropriate umbrella covering many related concepts.

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