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APPROXIMATION OF FIXED POINTS FOR CLASS OF GENERALIZED NONEXPANSIVE MAPPING VIA NEW ITERATION PROCESS

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Abstract: In this paper, we establish strong and Δ-converges theorem for the class of generalized nonexpansive mapping via new iteration process (SRJ-iteration) in CAT(0) space. Our result generalizes and extends the results of Varatechakongka et al. [31] and Ullah et al. [30].

Keywords and Phrases: CAT(0) space, condition $(B_{\gamma,\mu})$, strong and Δ- convergence theorems.

2020 Mathematics Subject Classification: 47H10, 54H25, 54E50.

1. Introduction

The concept of fixed points theory and its application has proven to be a vital tool in the study of nonlinear functional analysis, as well as a very useful tool in establishing existence and uniqueness theorems for nonlinear ordinary, partial and random differential and integral equations in various abstract spaces. We recall the following: Let E be a nonempty subset of a Banach space X and $\Im\colon E \to E$ a self-mapping. A point $x \in X$ is said to be a fixed point of $\Im$ if $\Im x = x$.

Remember that a selfmap $\Im$ on a metric space subset E is called nonexpansive if

$$d(\Im x, \Im y) \leq d(x, y), \quad for\ all\ x, y \in E. \tag{1.1}$$

When a member p is available in the set E such that $p = \Im p$, it is referred as a fixed point of $\Im$. Throughout this study, the notation $F(\Im)$ will represent the fixed point set of $\Im$. Nowadays, the study of fixed points for nonexpansive operators is an important and active research field. One of Gohde's [10] earlier results states that nonexpansive operators always admit a fixed point on closed bounded and convex subsets in the framework of uniform convexity of Banach space. Kirk [13, 14] was the first to introduce fixed point theory of nonexpansive operators in the context of nonlinear CAT(0) spaces. Suzuki's [23] made a significant breakthrough in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a selfmap $\Im$ of a metric space subset E is said to satisfy Condition (C) (also known as Suzuki's map) if for any $x, y \in E$, we have

$$\frac{1}{2}d(\Im x, \Im y) \leq d(x, y)\ implies\ d(\Im x, \Im y) \leq d(x, y), \quad for\ all\ x, y \in E. \tag{1.2}$$

Many researchers [6, 27] have studied the class of Suzuki's nonexpansive mappings in linear and nonlinear settings. In 2018, Patir et al. [19] proposed a two-parameter condition, which they called Condition $(B_{\gamma,\mu})$. They demonstrated that Condition $(B_{\gamma,\mu})$ is weaker than the corresponding condition (C). Recently, Varatechakongka and Phuengrattana [31] studied Condition $(B_{\gamma,\mu})$ in the setting of CAT(0) spaces and proved the demiclosed principle for this class of mappings. A selfmap $\Im$ of a subset E of a metric space is said to satisfy Condition $(B_{\gamma,\mu})$ (or called Patir map) if there are some $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$, with $2\mu \leq \gamma$ such that for all $x, y \in E$,

$$\begin{aligned} &\gamma d(x, \Im x) \leq d(x, y) + \mu d(y, \Im y) \\ &\ implies\ d(\Im x, \Im y) \leq (1 - \gamma)d(x, y) + \mu(d(x, \Im y) + d(y, \Im x)). \end{aligned} \tag{1.3}$$

Iterative techniques for finding fixed points are an important and active research area in nonlinear analysis with numerous applications in computers, applied economics, physics and many other applied sciences [1, 18]. Because the Picard iteration $x_{n+1} = \Im x_n$ does not always converge to a fixed point of a given nonexpansive operator, we will present here some other well-known processes that not only converge to a fixed point of a given nonexpansive operator but also have a higher rate of convergence than the Picard iteration. Let we assume E be a nonempty convex subset of a Banach space, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\Im\colon E \to E$ be a given operator.

Over the last few years many iterative processes have been obtained in different domains to approximate fixed points of various classes of mappings. Mann iteration [16], Ishikawa iteration [12], Thakur et al. [25] and Ullah et al. [29] are the few basic iteration processes.

Mann [16] described one of the earlier iteration processes as follows:

$$\begin{aligned} &x_1 \in E, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Im x_n, \quad n \geq 1. \end{aligned} \tag{1.4}$$

The Mann iteration can be seen as a subset of the Ishikawa iteration process, which was described by Ishikawa in [12] as follows:

$$\begin{aligned} &x_1 \in E, \\ &y_n = (1-\beta_n)x_n + \beta_n \Im x_n, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Im y_n, \quad n \geq 1. \end{aligned} \tag{1.5}$$

Agarwal et al. [4] is the slightly modified the Ishikawa iteration and defined as follows:

$$\begin{aligned} &x_1 \in E, \\ &y_n = (1-\beta_n)x_n + \beta_n \Im x_n, \\ &x_{n+1} = (1-\alpha_n)\Im x_n + \alpha_n \Im y_n, \quad n \geq 1. \end{aligned} \tag{1.6}$$

We can infer from [4] that the Agarwal iterative process is superior to the earlier processes, namely the Picard, Mann and Ishikawa iterative processes, by a significant margin.

In 2016, Thakur et al. [25] proposed the iterative process listed below:

$$\begin{aligned} &x_1 \in E, \\ &z_n = (1-\beta_n)x_n + \beta_n \Im x_n, \\ &y_n = \Im((1-\alpha_n)x_n + \alpha_n z_n), \\ &x_{n+1} = \Im y_n, \quad n \geq 1. \end{aligned} \tag{1.7}$$

Thakur et al. [25] demonstrated that the sequence $\{x_n\}$ defined by the iterative process (1.7) converges (in certain circumstances) to a fixed point of a given Suzuki's map. Furthermore, they built a new example of Suzuki's mappings $\Im$ and demonstrated that the iterative process (1.7) converges to a fixed point faster than earlier iterative processes proposed by Mann [16], Ishikawa [12], Noor [17], S-iteration [4] and Abbas [1].

In 2020, Abedeljawad et al. [3] introduced a new iterative process, which they call it JA iteration process, as follows:

$$\begin{aligned} &x_1 \in E, \\ &z_n = (1-\beta_n)x_n \oplus \beta_n \Im x_n, \\ &y_n = \Im z_n, \\ &x_{n+1} = \Im((1-\alpha_n)\Im x_n \oplus \alpha_n \Im y_n), \quad n \geq 1. \end{aligned} \tag{1.8}$$

Question: Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

To answer this, we introduce the new iteration process.

Let E be a nonempty, closed and convex subset of a complete CAT(0) space X and $\Im\colon E \to E$ be a mapping. Let $x_1 \in E$ be arbitrary and the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} &x_1 \in E, \\ &z_n = \Im((1-\alpha_n)x_n \oplus \alpha_n \Im x_n), \\ &y_n = \Im((1-\beta_n)z_n \oplus \beta_n \Im z_n), \\ &x_{n+1} = \Im((1-\gamma_n)y_n \oplus \gamma_n \Im y_n), \quad n \geq 1 \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). We created a new example of Patir maps $\Im$ and demonstrated that the iterative process (1.9) converges to a fixed point faster than the leading iterative processes proposed by Agarwal [4], Thakur et al. [25] and Abedeljawad et al. [3]. We improve and extend their results to the nonlinear setting of CAT(0) spaces in this paper.

2. Preliminaries

We give some elementary properties about CAT(0) spaces as follows:

Lemma 2.1. [7] *Let M be a CAT(0) space, $x, y, z \in M$ and $t \in [0, 1]$. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z).$$

Let $\{x_n\}$ be a bounded sequence in M, complete CAT(0) spaces. For $x \in X$ set:

$$r(x, \{x_n\}) = \lim_{n\to+\infty} sup\ d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = inf\{r(x, x_n) \colon x \in E\},$$

and the asymptotic center $A(\{x_n\})$ *of* $\{x_n\}$ *is defined as:*

$$A(\{x_n\}) = \{x \in E \colon r(x, x_n) = r(\{x_n\})\}.$$

Remark 2.2. *The cardinality of the set* $A(\{x_n\})$ *in any CAT(0) space is always equal to one, (see e.g.,* [7]*).*

The ([7], Proposition 2.1) tells us that in the setting of CAT(0) spaces, for every bounded sequence, namely, $\{x_n\} \subset E$, the set $A(\{x_n\})$ is essentially the subset of E provided that E is convex and bounded. It is well-known that $\{x_n\}$ has a subsequence which Δ-converges to some point provided that the sequence is bounded.

Definition 2.3. [14] *A sequence* $\{x_n\}$ *in CAT(0) space is said to be* Δ*-converges to* $x \in E$ *if* x *is the unique asymptotic center for every subsequence* $\{a_n\}$ *of* $\{x_n\}$*. In this case we write* $\Delta - lim_n x_n = x$ *and read as* x *is the* $\Delta - limit$ *of* $\{x_n\}$*.*

Notice that a bounded sequence $\{x_n\}$ in a CAT(0) space is known as regular if and only if for every subsequence, namely, $\{a_n\}$ of $\{x_n\}$ one has $r(\{x_n\}) = r\{a_n\}$. It is wellknown that, in the setting of CAT(0) spaces each regular sequence Δ-converges and consequently each bounded sequence has a Δ-convergent subsequence.

Definition 2.4. [22] *Let* $\Im$ *be a selfmap on a subset* E *of a given CAT(0) space and* f *be a selfmap of* $[0, \infty)$*. We say that* $\Im$ *has condition* (I) *if the following holds:*

1. $f(g) = 0$ *if and only if* $g = 0$*.*
2. $f(g) > 0$ *for every* $g > 0$*.*
3. $d(x, \Im x) \geq f(d(x, f(\Im)))$*.*

Definition 2.5. *Suppose* E *is a nonempty subset of a given CAT(0) space. If* $\Im \colon E \to E$ *has condition* $(B_{\gamma,\mu})$*. Then for every fixed point* p *of* $\Im$*, one has*

$$d(p, \Im x) \leq d(p, x), \tag{2.1}$$

for each $x \in E$*.*

Lemma 2.6. [14] *Suppose* E *is nonempty closed convex subset of a given CAT(0) space. If* $\Im \colon E \to E$ *has condition* $(B_{\gamma,\mu})$ *and the sequence* $\{x_n\} \subseteq E$ *satisfy* $\lim_{n\to\infty} d(\Im x_n, x_n) = 0$ *and* $\Delta - \lim_n x_n = p$*, then* $p = \Im p$*.*

Lemma 2.7. [2] *Let* E *be a nonempty subset of a given CAT(0) space. If* $\Im \colon E \to$

E has the condition $(B_{\gamma,\mu})$. Then the set $F(\Im)$ always closed.

Lemma 2.8. *(see [31], Lemma 3.5) Suppose E be nonempty subset of a given CAT(0) space. If $\Im\colon E \to E$ has condition $(B_{\gamma,\mu})$. Then for $x, y \in E$ and $m \in [0,1]$, the following hold:*

1. $d(\Im x, \Im^2 x) \le d(x, \Im x)$,
2. *Either (i) or (ii)satisfy:*
   (i) $(\frac{m}{2})d(x, \Im x) \le d(x, y)$,
   (ii) $(\frac{m}{2})d(\Im x, \Im^2 x) \le d(\Im x, y)$,
3. $d(x, \Im y) \le (3 - m + 2\mu)d(x, \Im x) + (1 - \frac{m}{2})d(x, y) + \mu(d(x, \Im y) + d(y, \Im x) + 2d(\Im x, \Im^2 x))$.

Lemma 2.9. [15] *Let M be a CAT(0) space and $\{t_n\}$ be any real sequence such that $0 < a \le a_n \le b < 1$ for $n \ge 1$. Let $\{y_n\}$ and $\{z_n\}$ be any two sequences of M such that $\lim_{n\to+\infty} sup\ d(y_n, x) \le q$, $\lim_{n\to+\infty} sup\ d(z_n, x) \le q$ and $\lim_{n\to+\infty} d(a_n y_n \oplus (1-a_n)z_n, x) = p$ hold for some $q \ge 0$. Then $\lim_{n\to+\infty} d(y_n, z_n) = 0$.*

3. Main result

This section establishes some significant strong and Δ-convergence results for operators with condition $(B_{\gamma,\mu})$. Our result generalizes the results of Varatechakongka et al. [31] and Ullah et al. [30].

Theorem 3.1. *Let $\Im\colon E \to E$ be a condition $(B_{\gamma,\mu})$ defined on a nonempty closed convex subset E of a complete CAT(0) space M such that $F(\Im) \ne \emptyset$. If $\{x_n\}$ is a sequence defined as (1.9) then $\lim_{n\to+\infty} d(x_n, p)$ exists for all $p \in F(\Im)$.*

Proof. For any $p \in F(\Im)$, By Definition 2.5, $\Im$ is quasi-nonexpansive, so we have

$$\begin{aligned} d(z_n, p) &= d(\Im((1-\alpha_n)x_n \oplus \alpha_n \Im x_n), p) \\ &\le ((1-\alpha_n))d(x_n, p) + \alpha_n d(\Im x_n, p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{3.1}$$

By Definition 2.5 and (3.1), we get

$$\begin{aligned} d(y_n, p) &= d(\Im((1-\beta_n)z_n \oplus \beta_n \Im z_n), p) \\ &\le (1-\beta_n)d(z_n, p) + \beta_n d(\Im z_n, p) \\ &\le (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{3.2}$$

By Definition 2.5, (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1},p) &= d(\Im((1-\gamma_n)y_n \oplus \gamma_n \Im y_n),p) \\ &\le (1-\gamma_n)d(y_n,p)+\gamma_n d(\Im y_n,p) \\ &\le (1-\gamma_n)d(y_n,p)+\gamma_n d(y_n,p) \\ &\le (1-\gamma_n)d(x_n,p)+\gamma_n d(x_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.3}$$

Thus, $\{d(x_n,p)\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to+\infty} d(x_n,p)$ exists $forall\ p \in F(\Im)$.

Theorem 3.2. *Let $\Im\colon E \to E$ satisfies the Condition $(B_{\gamma,\mu})$ defined on a nonempty closed convex subset E of a complete CAT(0) space M. If $\{x_n\}$ is a sequence generated by in (1.9), then $F(\Im) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and* $\lim_{n\to+\infty} d(\Im x_n, x_n) = 0$.

Proof. Suppose $F(\Im) \neq \emptyset$ and $p \in F(\Im)$. Then, by Theorem 3.1, $\lim_{n\to+\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to+\infty} d(x_n,p) = c, \tag{3.4}$$

By the proof of Theorem 3.1 and (3.4), we have

$$\lim_{n\to+\infty} sup\ d(z_n,p) < \lim_{n\to+\infty} sup\ d(x_n,p) = c. \tag{3.5}$$

By Definition 2.5, we have

$$\lim_{n\to+\infty} sup\ d(\Im x_n,p) \le \lim_{n\to+\infty} sup\ d(x_n,p) = c. \tag{3.6}$$

Consequence of inequalities. By using (1.9), (3.1), (3.2), (3.3), we have

$$d(x_{n+1},p) \le (1-\gamma_n)d(x_n,p)+\gamma_n d(z_n,p). \tag{3.7}$$

It follows that,

$$\begin{aligned} d(x_{n+1},p)-d(x_n,p) &\le \frac{d(x_{n+1},p)-d(x_n,p)}{\gamma_n} \\ &\le d(y_n,p)-d(x_n,p) \\ &\le (1-\beta_n)d(z_n,p)+\beta_n d(z_n,p)-d(x_n,p) \\ &\le d(z_n,p)-d(x_n,p). \end{aligned} \tag{3.8}$$

Therefore, $c \leq \lim_{n\to+\infty} inf\ d(\Im z_n, p)$.

$$c = \lim_{n\to+\infty} d(z_n, p) = \lim_{n\to+\infty} d(\Im((1-\alpha_n)x_n \oplus \alpha_n \Im x_n), p). \tag{3.9}$$

Applying Lemma 2.9, we obtain

$$\lim_{n\to+\infty} d(\Im x_n, x_n) = 0. \tag{3.10}$$

Conversely, let $p \in A(\{x_n\})$. By Lemma 2.8 (iii) for $\gamma = \frac{m}{2}$, $m \in (0,1)$, we have

$$\begin{aligned} d(x_n, \Im p) &\leq (3 - m + 2\mu) d(x_n, \Im x_n) + (1 - \frac{m}{2}) d(x_n, p) \\ &\quad + \mu(d(x_n, \Im p) + d(p, \Im x_n) + 2d(\Im x_n, \Im^2 x_n)). \end{aligned} \tag{3.11}$$

So, by Definition 2.5 and Lemma 2.8 (I), we get

$$\begin{aligned} d(x_n, \Im p) &\leq (3 - m + 4\mu) d(x_n, \Im x_n) \\ &\quad + (1 - \frac{m}{2} + \mu) d(x_n, p) + \mu d(x_n, \Im p). \end{aligned} \tag{3.12}$$

Then, we have

$$d(x_n, \Im p) \leq \frac{3 - m + 4\mu}{1-\mu} d(x_n, \Im x_n) + \frac{1 - \frac{m}{2} + \mu}{1-\mu} d(x_n, p). \tag{3.13}$$

This implies that

$$\begin{aligned} r(x_n, \Im p) &= \lim_{n\to+\infty} sup\ d(x_n, \Im p) \\ &\leq \frac{1 - \frac{m}{2} + \mu}{1-\mu} \lim_{n\to+\infty} sup\ d(x_n, p) \\ &\leq \lim_{n\to+\infty} sup\ d(x_n, p) = r(p, x_n). \end{aligned} \tag{3.14}$$

So $\Im p \in A(\{x_n\})$. By the uniqueness of asymptotic centers, one can conclude that $\Im p = p$.

The following results establishes the Δ-convergence for operators with condition $(B_{\gamma,\mu})$ in CAT(0) spaces under new iterations. This improves [30, 31], by increasing the rate of convergence.

Theorem 3.3. *Let $\Im\colon E \to E$ satisfies the Condition $(B_{\gamma,\mu})$ defined on a nonempty closed convex subset E of a complete CAT(0) space M such that $F(\Im) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by* (1.9)*. Then $\{x_n\}$ Δ-converges to a fixed point of $\Im$.*

Proof. By Theorem 3.2, the sequence $\{x_n\}$ is bounded. Hence one can take $A(\{x_n\}) = \{c\}$ for some $c \in X$. We are going to prove $A(\{x_n\}) = \{c\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_n\}) = \{c\}$. Since $\{x_{n_k}\}$ is bounded, one can find a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_j}\}$ Δ-converges to p for some $p \in E$. By Theorem 3.2 and Lemma 2.9, one has $p \in F(\Im)$ and hence $\lim_{n\to\infty} sup\ d(x_n, p)$ exists. If $p \neq x$, then the singletoness of the cardinality of the asymptotic centers allows us the following

$$\begin{aligned}\lim_{n\to\infty} sup\ d(x_n, p) &= \lim_{j\to\infty} sup\ d(x_{n_j}, p) < \lim_{j\to\infty} sup\ d(x_{n_j}, x)\\ &\leq \lim_{k\to\infty} sup\ d(x_{n_k}, x) < \lim_{k\to\infty} sup\ d(x_{n_k}, p)\\ &= \lim_{n\to\infty} sup\ d(x_n, p),\end{aligned} \tag{3.15}$$

which is contradiction. Therefore, $x = p \in F(\Im)$. Suppose that $x \neq c$. Then

$$\begin{aligned}\lim_{n\to\infty} sup\ d(x_n, x) &= \lim_{k\to\infty} sup\ d(x_{n_k}, x) \leq \lim_{k\to\infty} sup\ d(x_{n_k}, c)\\ &\leq \lim_{n\to\infty} sup\ d(x_m, c) < \lim_{n\to\infty} sup\ d(x_m, x)\\ &= \lim_{n\to\infty} sup\ d(x_n, x).\end{aligned} \tag{3.16}$$

Thus $\{x_n\}$ Δ-converges to an element $c \in F(\Im)$.

The following result establishes the strong–convergence for operators having condition $(B_{\gamma,\mu})$ under SRJ iterations in CAT(0) spaces. One may notice that it is that analog of ([3], Theorem 20).

Theorem 3.4. *Let $\Im\colon E \to E$ satisfies the Condition $(B_{\gamma,\mu})$ defined on a nonempty closed convex subset E of a complete CAT(0) space M such that $F(\Im) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by* (1.9)*, then $\{x_n\}$ strongly converges to a fixed point of $\Im$ if and only if* $\lim_{n\to\infty} d(x_n, F(\Im)) = 0$.

Proof. If the sequence $\{x_n\}$ converges to a point $p \in F(\Im)$, then

$$\lim_{n\to\infty} inf d(x_n, p) = 0,$$

so

$$\lim_{n\to\infty} d(x_n, F(\Im)) = 0.$$

For converse part, assume that $\lim_{n\to\infty} inf d(x_n, F(\Im)) = 0$. From Theorem 3.1, we have

$$d(x_{n+1}, p) \leq d(x_n, p)\ for\ any\ p \in F(\Im),$$

so we have,

$$d(x_{n+1}, F(\Im)) \leq d(x_n, F(\Im)). \tag{3.17}$$

Thus, $d(x_n, F(\Im))$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} d(x_n, F(\Im))$ exists. Since, $\lim_{n\to\infty} inf d(x_n, F(\Im)) = 0$ so $\lim_{n\to\infty} d(x_n, F(\Im)) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $F(\Im)$ such that $d(x_{n_j}, x_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 3.1, we have

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{n_j}, x_j) \\ &\leq \frac{1}{2^j}. \end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{j+1}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \\ &\to 0 \ as \ j \to \infty. \end{aligned}$$

So, $\{x_j\}$ is a cauchy sequence in $F(\Im)$. From Lemma 2.7 $F(\Im)$ is closed, so $\{x_j\}$ converges to some $x \in F(\Im)$.

Again, owing to triangle inequality, we have

$$d(x_{n_j}, x) \leq d(x_{n_j}, x_j) + d(x_j, x).$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $x \in F(\Im)$.

Since $\lim_{n\to\infty} inf d(x_n, x)$ exists by Theorem 3.1, therefore $\{x_n\}$ converges to $x \in F(\Im)$.

Eventually, we discuss the strong convergence for our scheme (1.9) by using the condition(I) given by definition 2.4.

Theorem 3.5. *Let $\Im \colon E \to E$ satisfies the Condition $(B_{\gamma,\mu})$ defined on a nonempty closed convex subset E of a complete CAT(0) space M such that $F(\Im) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (1.9), then $\{x_n\}$ converges to an element of $F(\Im)$ provided that $\Im$ has condition (I).*

Proof. It follows from Theorem 3.2 the $\lim_{n\to\infty} inf d(x_n, \Im x_n) = 0$. By condition (I), $\lim_{n\to\infty} inf d(x_n, F(\Im))$. Theorem 3.4 leads to the conclusions.

4. Numerical example

In this section, we are interested in the rate of convergence. We first create a illustration of a mapping $\Im$ that is Patir mapping (Condition $(B_{\gamma,\mu})$) but not Suzuki's.

Example. Let $E = [0, 4]$ which is a closed and convex subset of the CAT(0) space. Define an operator $\Im\colon [1, 4] \to [1, 4]$ as follows

$$Tx = \begin{cases} \frac{x+2}{2}, & if \quad x \neq 4 \\ 2, & if \quad x = 4, \end{cases}$$

for all $x \in E$.

To show that $\Im$ is not Suzuki's mapping let $x = \frac{35}{10}$, $y = 4$. We see that $\frac{1}{2}d(x, \Im x) = \frac{3}{8} < \frac{1}{2} = d(x, y)$. Thus $\Im$ does not satisfy condition (C). Choose $\gamma = 1$ and $\mu = \frac{1}{2}$, we prove that $\Im$ has the $(B_{1,\frac{1}{2}})$ condition.

(i) If we take $x, y \in [1, 4)$, then

$$\begin{aligned} (1-\gamma)d(x,y) + \mu(d(x,\Im y) + d(y,\Im x)) &= \frac{1}{2}(|x - \Im y| + |y - \Im x|) \\ &\geq \frac{1}{2}|\frac{3x}{2} - \frac{3y}{2}| \\ &= \frac{3}{4}|x-y| \geq \frac{1}{2}|x-y| \\ &= d(\Im x, \Im y). \end{aligned}$$

(ii) If we take $x \in [1, 4)$ and $y = 4$, then

$$\begin{aligned} (1-\gamma)d(x,y) + \mu(d(x,\Im y) + d(y,\Im x)) &= \frac{1}{2}(|x - \Im y| + |y - \Im x|) \\ &= \frac{1}{2}(|x-2| + |y - \frac{x+2}{2}|) \\ &= \frac{1}{2}|x-2| + \frac{1}{2}|y - \frac{x+2}{2}| \\ &\geq \frac{1}{2}|x-2| = d(\Im x, \Im y). \end{aligned}$$

(iii) If we take $x = 4 = y$, then we have

$$(1-\gamma)d(x,y) + \mu(d(x,\Im y) + d(y,\Im x)) \geq 0 = d(\Im x, \Im y).$$

Hence, $\Im$ satisfies the $(B_{1,1/2})$ condition. Table 1 and Figure 1 show the strong convergence of leading iterations to the fixed point 2 of the mapping $\Im$. In Table 1 and Figure 1, it is clear that the new iteration (1.9) converges faster than the leading JA iteration (1.8), Thakur (1.7) and Agarwal (1.6) iterative processes. Strong convergence of leading iterative processes under $\alpha_n = 0.85, \beta_n = 0.75, \gamma_n = 0.65$ and $x_1 = 2.4$.

Table 1: Convergence of new iteration (1.9) for fixed point 2.

No. of iteration	Agarwal iteration	Thakur iteration	JA iteration	New iteration
1	2.4	2.4	2.4	2.4
2	2.136250000	2.068125000	2.041562500	2.012128906
3	2.046410156	2.011602539	2.004318604	2.000367776
4	2.015808459	2.001976057	2.000448730	2.000011152
5	2.005384757	2.000336547	2.000046626	2.000000338
6	2.001834183	2.000057318	2.000004845	2.000000010
7	2.000624768	2.000009762	2.000000503	2
8	2.000212812	2.000001663	2.000000052	2
9	2.000072489	2.000000283	2.000000005	2
10	2.000024692	2.000000048	2.000000001	2
11	2.000008411	2.000000008	2	2
12	2.000002865	2.000000001	2	2
13	2.000000976	2	2	2
14	2.000000332	2	2	2
15	2.000000113	2	2	2
16	2.000000039	2	2	2

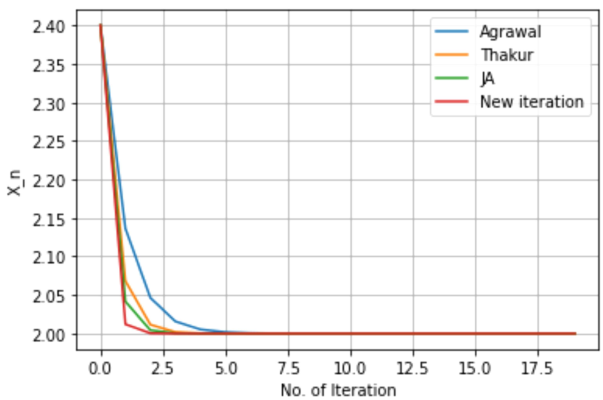

Figure 1: Convergence of Agarwal, Thakur, JA and New iteration

5. Conclusion

In this paper, we introduced a new type of iteration procedure called SRJ iteration for the class of generalized nonexpansive mapping in CAT(0) spaces. Our result generalizes results of Varatechakongka et al. [31] and Ullah et al. [30] in the sense of faster iteration process.

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