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ON 2-ABSORBING IDEALS IN COMMUTATIVE $\Gamma-$ SEMIRINGS

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Abstract: In this paper, our aim is to study the results of 2-absorbing ideals and weakly 2-absorbing ideals in a commutative $\Gamma-$ semirings which is a generalization of prime ideals of a commutative $\Gamma-$ semirings. Finally, we prove a characterization theorem for 2-absorbing and weakly 2-absorbing ideals in terms of k- extension of an ideal in a commutative $\Gamma-$ semiring.

Keywords and Phrases: k-ideals, strong ideal, Q-ideal, 2-absorbing ideals, weakly 2-absorbing ideals.

2020 Mathematics Subject Classification: 16Y60, 46J20, 12K10.

1. Introduction

The concept of semiring was first introduced by Vandiver [16] in 1934. After that, various studies have been done and it plays a prominent role in various branches of mathematics as well as in diverse areas of applied science. Several authors have used this concept in various disciplines in many ways. The structure of prime ideals in semiring theory has gained importance and many mathematicians have exploited its usefulness in algebraic systems. Badawi [1] and Badawi and Darani [2] introduced the concept of 2-absorbing and weakly 2-absorbing ideals of a commutative ring with non-zero unity, which are generalizations of prime and weakly prime ideals in a commutative ring. Darani [4] has examined these concepts in commutative semirings and characterized several results in terms of 2-absorbing and weakly 2-absorbing ideals in commutative semirings. The concept of subtractive extension of an ideal has been introduced by Chaudhary and Bonde [3] to study the ideal theory in quotient semiring. Rao [10] introduced the notion of $\Gamma-$ semirings in 1995 and also discussed some properties of ideals and k-ideals in $\Gamma-$ semirings. The definition of prime ideals in $\Gamma-$ semirings was introduced by Dutta and Sardar [5] and studied some of their properties. Sangjaer and Pianskool [12] gave the definition of 2-absorbing and weakly 2-absorbing ideals of commutative $\Gamma-$ semirings and studied various properties of 2-absorbing primary ideals in commutative $\Gamma-$ semirings.

As a continuation of the previous paper "On 2-absorbing Primary Ideals in Commutative $\Gamma-$ semirings" [12], here we study the consequences of imposing the results and their characterizations of 2-absorbing and weakly 2-absorbing ideals in commutative $\Gamma-$ semirings that are extended from those in semirings.

2. Preliminaries

In this section, we examine some of the basic definitions and fundamental concepts that are important to this paper. R represents a $\Gamma-$ semiring throughout this paper.

Definition 2.1. [10] *Let R and Γ be two additive commutative semi-group. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying the following conditions:*

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$.

(ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$.

(iii) $x\alpha(y + z) = x\alpha y + x\alpha z$.

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$ *for all* $x, y, z \in R$ *and* $\alpha, \beta \in \Gamma$.

The set $R = Z_0^+$, which represent the set of non-negative integers.

Example 2.2. [12] For each $n \in N$, let $nZ_0^+ = \{nx | x \in Z_0^+\}$ is a commutative semigroup under the usual addition of integers. Then nZ_0^+ is an mZ_0^+- semiring for all $m, n \in N$ where $x\alpha y$ is the usual multiplication of integers for all $x, y \in nZ_0^+$ and $\alpha \in mZ_0^+$.

Example 2.3. Let $R = Z_0^+$ be the commutative semigroup of positive integers and $\Gamma = 2Z_0^+$ be the commutative semigroup of positive integers. Then Z_0^+ is a $2Z_0^+-$ semiring with $x\alpha y$ usual multiplication of integers for all $x, y \in Z_0^+$ and $\alpha \in 2Z_0^+$.

Definition 2.4. [13] *A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$.*

Definition 2.5. [13] *A $\Gamma-$ semiring R is said to have an identity element if for all $x \in R$ there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.*

Definition 2.6. [10] *A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$.*

Definition 2.7. [14] *An element x in a $\Gamma-$ semiring R is said to be nilpotent if there exists a positive integer n (depending on x) such that $(x\alpha)^{n-1}x = 0$, $\alpha \in \Gamma$. The set of all nilpotent element of R is denoted by Nil(R).*

Definition 2.8. [10] *A $\Gamma-$ semiring R is said to be regular if for each $x \in R$ there exists $y \in R$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$.*

Definition 2.9. [6] *A non empty subset I of R is said to be left (right) ideal of R if I is sub semi-group of $(R, +)$ and $x\alpha y \in I (y\alpha x \in I)$ for all $y \in I, x \in R$ and $\alpha \in \Gamma$. If I is both left and right ideal of R, then I is known to be an ideal of R.*

If R is a $\Gamma-$ semiring with zero element then it is easy to verify that every ideal of R has zero element.

Definition 2.10. [6] *An ideal I of a $\Gamma-$ semiring R is called k-ideal if for $x, y \in R, x + y \in I$ and $y \in I$ implies that $x \in I$.*

An ideal $(J : r)$ is defined as $(J : r) = \{x \in R : r\alpha x \in J, \alpha \in \Gamma\}$. It is easy to see that if J is a k- ideal of R, then $(J : r)$ is a k-ideal of R.

Definition 2.11. [15] *An ideal I of a $\Gamma-$ semiring R is called a strong ideal if and only if $x + y \in I$ implies that $x \in I$ and $y \in I$.*

Definition 2.12. [6] *Let R be a $\Gamma-$ semiring. An ideal P of R is prime ideal if for any two ideals A and B of R such that $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.*

Definition 2.13. [6] *An ideal J of a $\Gamma-$ semiring R is said to be irreducible if for ideals K and L of R, $J = K \cap L$ implies that $J = K$ or $J = L$.*

Definition 2.14. [12] *Let R be a $\Gamma-$ semiring and J be an ideal in R. Then $\sqrt{J} = \{x \in R|$ there exists $n \in N$ such that $(x\alpha)^{n-1}x \in J$ for all $\alpha \in \Gamma\}$ is an ideal in R containing J. The ideal $\sqrt{J}$ is called the radical ideal of J and is denoted by Rad(J).*

Remark 2.15. *All through here, R will signify with "0" and "1" as zero and identity element except if in any case expressed.*

3. 2- absorbing ideals in commutative $\Gamma-$ semirings

In this section, the properties of 2- absorbing ideals and weakly 2- absorbing ideals in commutative $\Gamma-$ semirings are examined and various results are proved.

Definition 3.1. [12] *Let R be a commutative* $\Gamma-$ *semiring. A proper ideal J is said to be 2- absorbing ideal in R if whenever* $x, y, z \in R$, $\alpha, \beta \in \Gamma$ *with* $x\alpha y\beta z \in J$ *implies* $x\alpha y \in J$ *or* $x\beta z \in J$ *or* $y\beta z \in J$.

Definition 3.2. *Let R be a* $\Gamma-$ *semiring. A proper ideal J is said to be weakly 2-absorbing ideal in R if whenever* $x, y, z \in R$, $\alpha, \beta \in \Gamma$ *and* $0 \neq x\alpha y\beta z \in J$ *implies* $x\alpha y \in J$ *or* $x\beta z \in J$ *or* $y\beta z \in J$.

Every 2− absorbing ideal of a $\Gamma-$ semiring R is a weakly 2− absorbing ideal of R but the converse need not be true. To learn more about the characteristics of 2-absorbing ideals in commutative $\Gamma-$ semirings, refer [12].

Theorem 3.3. *Let R be a* $\Gamma-$ *semiring and J be a* 2−*absorbing ideal of R. Then* $(J : x)$ *is a* 2− *absorbing ideal of R for all* $x \in R \setminus J$.
Proof. Let $x \in R \setminus J$ and $y, z, w \in R$, $\alpha, \beta \in \Gamma$ be such that $y\alpha z\beta w \in (J : x)$. Then $x\gamma y\alpha z\beta w \in J$, $\gamma \in \Gamma$. Since J is a 2− absorbing ideal of R, either $x\gamma y \in J$ or $x\gamma z\beta w \in J$ or $y\alpha z\beta w \in J$. If either $x\gamma y \in J$ or $x\gamma z\beta w \in J$, we are done. If $y\alpha z\beta w \in J$, then $y\alpha z \in J$ or $y\beta w \in J$ or $z\beta w \in J$, which implies $x\gamma y\alpha z \in J$ or $x\gamma y\beta w \in J$ or $x\gamma z\beta w \in J$. Hence, $(J : x)$ is a 2− absorbing ideal of R.

Theorem 3.4. *Let R be a* $\Gamma-$ *semiring and J be a* 2− *absorbing k-ideal of R with* $\sqrt{J} = K$ *and* $K\Gamma K \subseteq J$. *If* $J \neq K$, *then* $(J : x)$ *is a prime ideal of R containing J with* $K \subseteq (J : x)$ *for all* $x \in K \setminus J$.
Proof. Let $y, z \in R$, $\beta \in \Gamma$ such that $y\beta z \in (J : x)$. Then $x\alpha y\beta z \in J$. So $x\alpha y \in J$ or $x\beta z \in J$ or $y\beta z \in J$, since J is a 2− absorbing ideal of R. If $x\alpha y \in J$ or $x\beta z \in J$, then $y \in (J : x)$ or $z \in (J : x)$, hence there is no necessity to prove. If $y\beta z \in J$ and $x\gamma x \in K\Gamma K \subseteq J$, $\gamma \in \Gamma$. For a particular $z \in R$, this gives $x\beta z \in (J : x)$. We have $(x + y)\beta z \in (J : x)$, which implies that $x\gamma(x + y)\beta z \in J$. Since J is a 2− absorbing ideal of R, so either $x\beta z \in J$ or $(x + y)\beta z \in J$ or $x\gamma(x + y) \in J$. If $x\beta z \in J$ then $z \in (J : x)$. If $(x + y)\beta z \in J$ and $y\beta z \in J$, then $x\beta z \in J$ (as J is a k-ideal). This gives $z \in (J : x)$, hence $(J : x)$ is prime. And finally, if $x\gamma(x+y) \in J$ and $x\gamma x \in K\Gamma K \subseteq J$. This gives $x\gamma y \in J$ implies $y \in (J : x)$. Hence, $(J : x)$ is a prime ideal of R.

Converse of the above theorem need not be true in general.

Example 3.5. By Example 2.3, Z_0^+ is a $2Z_0^+-$ semiring. Let $J = 12Z_0^+$ and $K = 3Z_0^+$ be ideals of Z_0^+ with $K\Gamma K \subseteq J$. Then $(J : x) = (12Z_0^+ : 3) = 2Z_0^+$, as $(2)(2)(3) \in 12Z_0^+$, where $2 \in Z_0^+, 3 \in K/J$ and $2 \in 2Z_0^+$. Hence, $(J : x)$ is a prime ideal. While $J = 12Z_0^+$ is not 2-absorbing ideal, since $1, 3 \in Z_0^+$ and $2 \in 2Z_0^+$ such that $(1)(2)(3)(2)(1) \in 12Z_0^+$ but neither $(1)(2)(3) \in 12Z_0^+$ nor $(1)(2)(1) \in 12Z_0^+$ nor $(3)(2)(1) \in 12Z_0^+$.

Corollary 3.6. *Let* R *be a* $\Gamma-$ *semiring and* J *be a* $2-$ *absorbing k-ideal of* R *with* $\sqrt{J} = K$ *and* $K\Gamma K \subseteq J$*. If* $J \neq K$ *and for all* $x \in K \setminus J$*, then* $(J : x)$ *is a* $2-$ *absorbing ideal of* R *with* $K \subseteq (J : x)$*.*

The converse of Theorem 3.4 is true in the case of $\sqrt{J}$ is a prime ideal of R.

Theorem 3.7. *Let* R *be a* $\Gamma-$ *semiring,* J *be a k-ideal of* R *such that* $J \neq \sqrt{J}$ *and* $\sqrt{J}$ *is a prime ideal of* R *with* $\sqrt{J}\Gamma\sqrt{J} \subset J$*. Then* J *is a 2-absorbing ideal of* R *if and only if* $(J : x) = \{y \in R : x\alpha y \in J,\ \alpha \in \Gamma\}$ *is a prime ideal of* R *for all* $x \in \sqrt{J} \setminus J$*.*

Proof. By Theorem 3.4, if J is a 2-absorbing ideal of R, then $(J : x)$ is a prime ideal of R. Conversely, let $u\alpha v\beta w \in J$ for some $u, v, w \in R$ and $\alpha, \beta \in \Gamma$. Since $J \subseteq \sqrt{J}$ and $\sqrt{J}$ is a prime ideal of R, we can assume that $u \in \sqrt{J}$. If $u \in J$, then $u\alpha v \in J$, $\alpha \in \Gamma$, which gives J is a 2-absorbing ideal of R. Let $v\beta w \in (J : u)$ for all $u \in \sqrt{J} \setminus J$, $\beta \in \Gamma$ and by assumption $(J : u)$ is a prime ideal of R, therefore either $v \in (J : u)$ or $w \in (J : u)$. This implies that either $u\alpha v \in J$ or $u\beta w \in J$. Hence, J is a 2-absorbing ideal of R.

Lemma 3.8. *Let* R *be a* $\Gamma-$ *semiring. If* J *and* K *are two k-ideals in* R*. Then* $J \cup K$ *is a k-ideal of* R *if and only if* $J \cup K = J$ *or* $J \cup K = K$*.*

Theorem 3.9. *Let* R *be a* $\Gamma-$ *semiring and* J *be a 2-absorbing k-ideal of* R *with* $\sqrt{J} = K$*. If* $J \neq K$*,* K *is a prime ideal of* R *and for all* $x \in R \setminus K$*. Then* $T = \{(J : x) : x \in R\}$ *is a totally ordered set.*

Proof. Since K is a prime ideal of R, therefore $x\alpha y \in R \setminus K$ for all $x, y \in R \setminus K$ and $\alpha \in \Gamma$. Clearly, $x\alpha y \notin J$ and $(J : x) \subseteq (J : x\alpha y)$ and $(J : y) \subseteq (J : x\alpha y)$ implies that $(J : x) \cup (J : y) \subseteq (J : x\alpha y)$. Let $z \in (J : x\alpha y)$ then $x\alpha y\beta z \in J$ implies that either $x\beta z \in J$ or $y\beta z \in J$ as $x\alpha y \notin J$, $\alpha, \beta \in \Gamma$. Thus, $(J : x\alpha y) \subseteq (J : x) \cup (J : y)$. Therefore, either $(J : x\alpha y) = (J : x)$ or $(J : x\alpha y) = (J : y)$, by Lemma 3.8. This implies that either $(J : x) \subseteq (J : y)$ or $(J : y) \subseteq (J : x)$. Hence $T = \{(J : x) : x \in R \setminus K\}$ is a totally ordered set. Moreover, we show that $(J : y) \subseteq (J : x)$ for some $x, y \in K \setminus J$. Let $x, y \in K \setminus J$. Since $K \subseteq (J : y)$, then for all $u \in (J : x) \setminus (J : y)$, we may assume that $u \in (J : x) \setminus K$. Similarly, for all $v \in (J : y) \setminus (J : x)$, we may assume that $v \in (J : y) \setminus K$. As a result, since $u \notin K$, $v \notin K$ and $\beta \in \Gamma$ such that $u\beta v \notin K$. Furthermore, $u\alpha(x + y)\beta v \in J$ and $u\beta v \notin J$, $\alpha, \beta \in \Gamma$, therefore we have $u\alpha(x + y) \in J$ or $(x + y)\beta v \in J$, which gives either $u\alpha y \in J$ or $x\beta v \in J$ implies that either $u \in (J : y)$ or $v \in (J : x)$. We get a contradiction in each case. Therefore, either $(J : x) \subseteq (J : y)$ or $(J : y) \subseteq (J : x)$ for $x, y \in K \setminus J$. Hence $T = \{(J : x) : x \in R\}$ is a totally ordered set.

Theorem 3.10. *Let* R *be a* $\Gamma-$ *semiring,* J *be an irreducible k-ideal of* R *and* K

be an ideal of R such that $\sqrt{J} = K$ *and* $K\Gamma K \subseteq J$. *Then J is 2-absorbing if and only if* $(J : x) = (J : x\alpha x)$ *for all* $x \in R \setminus K$ *and* $\alpha \in \Gamma$.

Proof. Let J be a 2-absorbing ideal of R. For $x \in R \setminus K$, $\alpha \in \Gamma$, $x\alpha x \notin J$ if $x\alpha x \in J$ then $x \in \sqrt{J} = K$, which is a contradiction and $(J : x) \subseteq (J : x\alpha x)$ is clear. Thus, for any $y \in (J : x\alpha x)$, we have $x\alpha x\beta y \in J$, $\alpha, \beta \in \Gamma$. Therefore, either $x\beta y \in J$ or $x\alpha x \in J$, since J is a 2- absorbing ideal of R. So $x\beta y \in J$ such that $y \in (J : x)$ because $x\alpha x \notin J$. Hence, $(J : x) = (J : x\alpha x)$. Conversely, let $x\alpha y\beta z \in J$ for some $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $x\alpha y \notin J$. We prove that either $x\beta z \in J$ or $y\beta z \in J$. Since $x\alpha y \notin J$, we have $x \notin K$ or $y \notin K$. If $x \in K$ and $y \in K$, then $x\alpha y \in K\Gamma K \subseteq J$, $\alpha \in \Gamma$, which is a contradiction. By assumption, we have either $(J : x) = (J : x\alpha x)$ or $(J : y) = (J : y\alpha y)$. If $(J : x) = (J : x\alpha x)$ and we also assume that $x\beta z \notin J$ and $y\beta z \notin J$, then we prove this result by contradiction. Let $u \in (J + (x\beta z)) \cap (J + (y\beta z))$. Then $u = u_1 + x_1\alpha x\beta z = u_2 + x_2\alpha y\beta z$ for some $u_1, u_2 \in J$, $x_1, x_2 \in R$ and $\alpha, \beta \in \Gamma$. Thus, $u\gamma x = u_1\gamma x + x_1\gamma x\alpha x\beta z = u_2\gamma x + x_2\gamma x\alpha y\beta z$, $\gamma \in \Gamma$. Since $x\alpha y\beta z \in J$, therefore $x_1\gamma x\alpha x\beta z \in J$ (as J is a k- ideal of R). This implies $x_1\gamma x\beta z \in J$ since $(J : x) = (J : x\alpha x)$. As a result, $u = u_1 + x_1\alpha x\beta z$. This shows that $(J + (x\beta z)) \cap (J + (y\beta z)) \subseteq J$ and thus $(J + (x\beta z)) \cap (J + (y\beta z)) = J$, a contradiction, since J is irreducible. So, we have $x\beta z \in J$ or $y\beta z \in J$. Hence, J is a $2-$ absorbing ideal of R.

Theorem 3.11. *Let R be a regular* $\Gamma-$ *semiring. Then every irreducible ideal J of R is 2-absorbing ideal of R.*

Proof. Let R be a regular $\Gamma-$ semiring and J be an irreducible ideal of R. If $x\alpha y\beta z \in J$ for some $x, y, z \in R$, $\alpha, \beta \in \Gamma$ and $x\alpha y \notin J$, then we have to show that $x\beta z \in J$ or $y\beta z \in J$. On the other hand, we assume that $x\beta z \notin J$ or $y\beta z \notin J$. Then $A = J + (x\beta z)$ and $B = J + (y\beta z)$ are two ideals of R that properly contain J. Since J is irreducible, therefore $J \neq A \cap B$. Thus, there exists $v \in R$ such that $v \in (J + (x\beta z)) \cap (J + (y\beta z)) \setminus J$. Since R is regular, we have $A \cap B = A\Gamma B$, therefore $v \in (J + (x\beta z))\Gamma(J + (y\beta z)) \setminus J$. Then, there are $v_1, v_2 \in J$, $x_1, x_2 \in R$ and $\beta, \gamma, \delta \in \Gamma$ such that $v = (v_1 + x_1\gamma x\beta z)\delta(v_2 + x_2\gamma y\beta z) = v_1\delta v_2 + v_1\delta x_2\gamma y\beta z + x_1\gamma x\beta z\delta v_2 + x\gamma y\beta x_1\delta x_2\gamma z\beta z$. This implies that $v \in J$, which is a contradiction. Hence, J is a 2-absorbing ideal of R.

Theorem 3.12. *Let R be a* $\Gamma-$ *semiring,* $r \in R$ *and J be an ideal of R. Then the following holds:*

(i) If $R\Gamma r$ *is a k- ideal of R and* $(0 : r) \subseteq R\Gamma r$*, then the ideal* $R\Gamma r$ *is 2-absorbing if and only if it is weakly 2-absorbing.*

(ii) If J is a k-ideal of R and $(0 : r) \subseteq J\Gamma r$*, then the ideal* $J\Gamma r$ *is 2-absorbing if*

and only if it is weakly 2-absorbing.

Proof.

(i) Let $R\Gamma r$ be weakly 2-absorbing ideal of R and $x\alpha y\beta z \in J$ for some $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. If $x\alpha y\beta z \neq 0$, then $x\alpha y \in R\Gamma r$ or $x\beta z \in R\Gamma r$ or $y\beta z \in R\Gamma r$. Then we are done. Let $x\alpha y\beta z = 0$. Clearly, $x\alpha(y+r)\beta z = x\alpha y\beta z + x\alpha r\beta z \in R\Gamma r$. If $x\alpha(y+r)\beta z \neq 0$, then $x\alpha(y+r) \in R\Gamma r$ or $x\beta z \in R\Gamma r$ or $(y+r)\beta z \in R\Gamma r$, as $R\Gamma r$ is a weakly 2-absorbing ideal of R. Since $R\Gamma r$ is a k- ideal of R, so $x\alpha y \in R\Gamma r$ or $y\beta z \in R\Gamma r$ or $x\beta z \in R\Gamma r$. Thus, we may assume that $x\alpha(y+r)\beta z = 0$ then $x\alpha r\beta z = 0$, since $x\alpha y\beta z = 0$. So $x\beta z \in (0 : r) \subseteq R\Gamma r$. Consequently, $x\beta z \in R\Gamma r$. Hence, $R\Gamma r$ is a 2-absorbing ideal of R.

(ii) The proof is similar to (i).

Theorem 3.13. *Let R be a commutative $\Gamma-$ semiring. If J is a weakly 2-absorbing k-ideal of R, then either $J\Gamma J\Gamma J = 0$ or J is 2-absorbing.*

Proof. Let us assume that $J\Gamma J\Gamma J \neq 0$. We show that J is 2-absorbing. Let $x\alpha y\beta z \in J$ for some $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. If $x\alpha y\beta z \neq 0$, then J is weakly 2-absorbing gives either $x\alpha y \in J$ or $x\beta z \in J$ or $y\beta z \in J$. So we assume that $x\alpha y\beta z = 0$. First, suppose that $x\alpha y\Gamma J \neq 0$, such that $x\alpha y\beta j \neq 0$ for some $j \in J$ then $0 \neq x\alpha y\beta j = x\alpha y\beta(z+j) \in J$. Since J is weakly 2-absorbing, either $x\alpha y \in J$ or $x\beta(z+j) \in J$ or $y\beta(z+j) \in J$. Hence, $x\alpha y \in J$ or $x\beta z \in J$ or $y\beta z \in J$. So we can assume that $x\alpha y\Gamma J = 0$. Similarly, we can assume that $x\beta z\Gamma J = 0$ and $y\beta z\Gamma J = 0$. Since $J\Gamma J\Gamma J \neq 0$, there exist $x_1, y_1, z_1 \in J$ and $\alpha, \beta \in \Gamma$ with $x_1\alpha y_1\beta z_1 \neq 0$. If $x\alpha y_1\beta z_1 \neq 0$, then $0 \neq x\alpha y_1\beta z_1 = x\alpha(y+y_1)\beta(z+z_1) \in J$ implies that $x\alpha(y+y_1) \in J$ or $x\beta(z+z_1) \in J$ or $(y+y_1)\beta(z+z_1) \in J$. Hence, $x\alpha y \in J$ or $x\beta z \in J$ or $y\beta z \in J$. So we can assume that $x\alpha y_1\beta z_1 = 0$. Similarly, we can assume that $x_1\alpha y_1\beta z = 0$ and $x_1\alpha y\beta z_1 = 0$. Then $0 \neq x_1\alpha y_1\beta z_1 = (x+x_1)\alpha(y+y_1)\beta(z+z_1) \in J$ we get $(x+x_1)\alpha(y+y_1) \in J$ or $(x+x_1)\beta(z+z_1) \in J$ or $(y+y_1)\beta(z+z_1) \in J$. Therefore, $x\alpha y \in J$ or $x\beta z \in J$ or $y\beta z \in J$, and so J is 2-absorbing.

Theorem 3.14. *Let R be a $\Gamma-$ semiring and J be a weakly 2-absorbing k- ideal of R but not a 2-absorbing ideal of R. Then*

(i) if $x \in Nil(R)$ and $\alpha \in \Gamma$, then either $x\alpha x \in J$ or $x\alpha x\Gamma J = x\Gamma J\Gamma J = \{0\}$.

(ii) $Nil(R)\Gamma Nil(R)\Gamma J\Gamma J = \{0\}$.

Proof.

(i) Let $x \in Nil(R)$ and $\alpha \in \Gamma$. We assert that if $x\alpha x\Gamma J \neq \{0\}$, then $x\alpha x \in J$. Assume that $x\alpha x\Gamma J \neq \{0\}$. Let n be the least positive integer such that $(x\alpha)^{n-1}x = 0$, then for $n \geq 3$, we have $0 \neq x\alpha x\alpha y = x\alpha x\alpha(y+(x\alpha)^{n-3}x) \in J$ for some $y \in J$ and $\alpha \in \Gamma$. Since J is a weakly 2-absorbing ideal of R, so either $x\alpha x \in J$ or $x\alpha y + (x\alpha)^{n-2}x \in J$. If $x\alpha x \in J$, we have nothing to prove. Let $x\alpha x \notin J$. Then $x\alpha y+(x\alpha)^{n-2}x \in J$, gives that $(x\alpha)^{n-2}x \in J$ and $(x\alpha)^{n-2}x \neq 0$, and thus $x\alpha x \in J$. Thus, we have either $x\alpha x \in J$ or $x\alpha x\Gamma J = \{0\}$ for all $x \in Nil(R)$. If $y\alpha y \notin J$ for some $y \in Nil(R)$ and $\alpha \in \Gamma$, then by using the previous argument, we have $y\alpha y\Gamma J = \{0\}$. We claim that $y\Gamma J\Gamma J = \{0\}$. Assume that, for some $j, k \in J$ and $\alpha \in \Gamma$ we have $y\alpha j\alpha k \neq 0$. Let $m \geq 3$ be the least positive integer such that $(y\alpha)^{m-1}y = 0$. Since $y\alpha y \notin J$, for $m \geq 3$ and $y\alpha y\Gamma J = \{0\}$, thus we have $y\alpha(y + j)\alpha((y\alpha)^{m-3}y + k)) = y\alpha j\alpha k \neq 0$. Since $0 \neq y\alpha(y + j)\alpha((y\alpha)^{m-3}y + k)) \in J$ and J is a weakly 2-absorbing ideal of R, so either $y\alpha y \in J$ or $0 \neq (y\alpha)^{m-2}y \in J$ (as J is a k- ideal of R). Therefore, we have $y\alpha y \in J$, a contradiction. Hence, $y\Gamma J\Gamma J = \{0\}$.

(ii) Let $u, v \in Nil(R)$. If either $u\alpha u \notin J$ or $v\alpha v \notin J$, $\alpha \in \Gamma$, then from (i), we have $u\alpha v\Gamma J\Gamma J = \{0\}$ and hence the result. Let us assume that $u\alpha u \in J$ or $v\alpha v \in J$, then $u\alpha v\alpha(u + v) \in J$. If $0 \neq u\alpha v\alpha(u + v) \in J$ and since J is a weakly 2-absorbing k-ideal of R, then $u\alpha v \in J$. Therefore, $u\alpha v\Gamma J\Gamma J = \{0\}$ by Theorem 3.13. Furthermore, if $0 = u\alpha v\alpha(u+v) \in J$ and $0 \neq u\alpha v\alpha i \in J$ for some $i \in J$, then $0 \neq u\alpha v\alpha(u+v+i) \in J$ implies that either $u\alpha(u+v+i) \in J$ or $v\alpha(u+v+i) \in J$ or $u\alpha v \in J$. Since J is a weakly 2-absorbing ideal but not a 2-absorbing ideal of R, we have $u\alpha v \in J$ in each case, which is a contradiction. Thus, we have $u\alpha v\Gamma J = \{0\}$. Hence, $u\alpha v\Gamma J\Gamma J = \{0\}$.

Definition 3.15. [12] *Let R_1 and R_2 be $\Gamma-$ semirings (not necessary commutative) and $f : R_1 \to R_2$ be a $\Gamma-$ homomorphism. Then f is called an $\Gamma-$ epimorphism if f is surjective.*

Theorem 3.16. *Let $f : R \to R_1$ be a $\Gamma-$ epimorphism of $\Gamma-$ semirings R and R_1 such that $f(0) = 0$ and J be a strong k-ideal of R. Then the following holds:*

(i) If J is a weakly 2-absorbing ideal of R such that $kerf \subseteq J$, then $f(J)$ is a weakly 2-absorbing ideal of R_1.

(ii) If J is a 2-absorbing ideal of R such that $kerf \subseteq J$, then $f(J)$ is a 2-absorbing ideal of R_1.

Proof.

(i) Let $0 \neq x\alpha y\beta z \in f(J)$ for some $x, y, z \in R_1$ and $\alpha, \beta \in \Gamma$. Then there exists $m \in J$ such that $0 \neq x\alpha y\beta z = f(m)$. Since f is an $\Gamma-$ epimorphism, then $f(u) = x, f(v) = y, f(w) = z$ for some $u, v, w \in R$. Also, J is a strong ideal of R and $m \in J$, therefore there exists $n \in J$ such that $m + n = 0$. This implies $f(m+n) = 0$ that is, $f(u\alpha v\beta w+n) = 0$ implies that $u\alpha v\beta w+n \in kerf \subseteq J$, $\alpha, \beta \in \Gamma$. Since J is k-ideal so, $0 \neq u\alpha v\beta w \in J$. If $u\alpha v\beta w = 0$, then $f(m) = 0$, which is a contradiction. Since J is a weakly 2-absorbing ideal of R, therefore either $u\alpha v \in J$ or $v\beta w \in J$ or $u\beta w \in J$. Thus, $x\alpha y \in f(J)$ or $y\beta z \in f(J)$ or $x\beta z \in f(J)$. Hence, $f(J)$ is a weakly 2-absorbing ideal of R_1.

(ii) It follows form (i).

Theorem 3.17. [11] *Let* R_1 *and* R_2 *be* Γ_1 *and* Γ_2 *semirings respectively. If we define*

(i) $(x, y) + (z, w) = (x + z, y + w)$

(ii) $(x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w)$, *for all* $(x, y), (z, w) \in R_1 \times R_2$ *and* $(\alpha, \beta) \in \Gamma_1 \times \Gamma_2$. *Then* $R_1 \times R_2$ *is a* $\Gamma_1 \times \Gamma_2-$ *semirings.*

Theorem 3.18. *Let* $R = R_1 \times R_2$ *be a commutative* $\Gamma = \Gamma_1 \times \Gamma_2-$ *semirings. If* J *is a proper ideal of a* $\Gamma-$ *semiring* R_1. *Then the following statements are equivalent:*

(i) J *is a 2-absorbing ideal of* R_1.

(ii) $J \times R_2$ *is a 2-absorbing ideal of* $R = R_1 \times R_2$.

(iii) $J \times R_2$ *is a weakly 2-absorbing ideal of* $R = R_1 \times R_2$.

Proof. $(i) \Rightarrow (ii)$. Let $(x_1, x_2)(\alpha, \beta)(y_1, y_2)(\gamma, \delta)(z_1, z_2) \in J \times R_2$ for some (x_1, x_2), $(y_1, y_2), (z_1, z_2) \in R$ and $(\alpha, \beta), (\gamma, \delta) \in \Gamma$. Then $(x_1\alpha y_1\gamma z_1, x_2\beta y_2\delta z_2) \in J \times R_2$. Hence, $x_1\alpha y_1\gamma z_1 \in J$. Since J is a 2-absorbing ideal of R_1, $x_1\alpha y_1 \in J$ or $y_1\gamma z_1 \in J$ or $x_1\gamma z_1 \in J$. If $x_1\alpha y_1 \in J$, then $(x_1, x_2)(\alpha, \beta)(y_1, y_2) \in J \times R_2$. Similarly, we can prove the other cases. Hence, $J \times R_2$ is a 2-absorbing ideal of R.

$(ii) \Rightarrow (iii)$. It is obvious.

$(iii) \Rightarrow (i)$. Let $x, y, z \in R_1$ and $(\alpha, \beta) \in \Gamma$ be such that $x\alpha y\beta z \in J$. Then, we have $(0, 0) \neq (x, 1)(\alpha, \beta)(y, 1)(\gamma, \delta)(z, r) \in J \times R_2$ for all $0 \neq r \in R_2$. Since $J \times R_2$ is a weakly 2-absorbing ideal of R, either $(x, 1)(\alpha, \beta)(y, 1) \in J \times R_2$ or $(y, 1)(\gamma, \delta)(z, r) \in J \times R_2$ or $(x, 1)(\gamma, \delta)(z, r) \in J \times R_2$. This implies that, either

$x\alpha y \in J$ or $y\beta z \in J$ or $x\beta z \in J$. Hence, J is a 2-absorbing ideal of R_1.

Definition 3.19. [6] *Let R be a $\Gamma-$ semiring. An ideal J of R is said to be partitioning ideal ($Q-ideal$) if there exists a subset Q of R such that:*

(i) $R = \cup\{q + J : q \in Q\}$.

(ii) If $q_1, q_2 \in Q$, then $(q_1 + J) \cap (q_2 + J) \neq \phi$ if and only if $q_1 = q_2$.

Let J be a Q-ideal of $\Gamma-$ semiring R and let $R/J_{(Q)} = \{q + J : q \in Q\}$, then $R/J_{(Q)}$ form a $\Gamma-$ semiring under the binary operations $\oplus, \odot$ defined as follows: $((q_1 + J) \oplus (q_2 + J) = q_3 + J$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + J \subseteq q_3 + J$. $((q_1 + J) \odot \alpha \odot (q_2 + J) = q_4 + J$, where $q_4 \in Q$ is the unique element such that $q_1\alpha q_2 + J \subseteq q_4 + J$ for all $\alpha \in \Gamma$. This $\Gamma-$ semiring $R/J_{(Q)}$ is called the quotient $\Gamma-$ semiring of R by J and denoted by $(R/J_{(Q)}), \oplus, \odot)$ or simply $R/J_{(Q)}$. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + J \subseteq q_{(Q)} + J$. Then $q_0 + J$ is a zero element of $R/J_{(Q)}$. Clearly, if R is commutative, then so is $R/J_{(Q)}$ [6].

Definition 3.20. *Let R be a $\Gamma-$ semiring and J be an ideal of R. An ideal A of R with $J \subseteq A$ is said to be k- extension of J if $x \in J, x + y \in A, y \in R$, then $y \in A$.*

Further, we give some characterization of 2-absorbing and weakly 2-absorbing ideals in terms of k- extension of an ideal of a $\Gamma-$ semiring R.

Theorem 3.21. *Let R be a $\Gamma-$ semiring, J be a Q-ideal of R and S a k- extension of J. Then S is 2-absorbing ideal of R if and only if $S/J_{(Q\cap S)}$ is a 2-absorbing ideal of $R/J_{(Q)}$.*

Proof. Let S be a 2-absorbing ideal of R. Assume that $q_1+J, q_2+J, q_3+J \in R/J_{(Q)}$ and $\alpha, \beta \in \Gamma$ such that $(q_1 + J) \odot \alpha \odot (q_2 + J) \odot \beta \odot (q_3 + J) = q_4 + J \in S/J_{(Q\cap S)}$ where $q_4 \in Q \cap S$ is a unique element such that $q_1\alpha q_2\beta q_3 \subseteq q_4 + J \in S/J_{(Q\cap S)}$. Thus $q_1\alpha q_2\beta q_3 = q_4 + j$ for some $j \in J$. This gives either $q_1\alpha q_2 \in S$ or $q_2\beta q_3 \in S$ or $q_1\beta q_3 \in S$, since S is a 2-absorbing ideal of R and $q_1\alpha q_2\beta q_3 \in S$. Let $q_1\alpha q_2 \in S$. If $(q_1 + J) \odot \alpha \odot (q_2 + J) = q + J$, where $q \in Q$ is a unique element such that $q_1\alpha q_2 + J \subseteq q + J$. So $q + k = q_1\alpha q_2 + l$ for some $k, l \in J$. Since S is a k-extension of J, we get $q \in S$, therefore $q \in Q \cap S$. Hence $S/J_{(Q\cap S)}$ is a 2-absorbing ideal of $R/J_{(Q)}$. Conversely, let $S/J_{(Q\cap S)}$ be a 2-absorbing ideal of $R/J_{(Q)}$. Assume that $x\alpha y\beta z \in S$ for some $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. There are $q_1, q_2, q_3, q_4 \in Q$ such that $x \in q_1 + J, y \in q_2 + J, z \in q_3 + J$ and $x\alpha y\beta z \in (q_1 + J) \odot \alpha \odot (q_2 + J) \odot \beta \odot (q_3 + J) = q_4 + J$, since J is a Q-ideal of R. Thus, $x\alpha y\beta z = q_4 + i \in S$ for some $i \in J$. Since S is a k- extension of J, we have $q_4 \in S$. Therefore, $(q_1 + J) \odot \alpha \odot (q_2 + J) \odot \beta \odot (q_3 + J) = q_4 + J \in S/J_{(Q\cap S)}$, since $S/J_{(Q\cap S)}$ is a 2-absorbing ideal of $R/J_{(Q)}$, gives either $(q_1 + J) \odot \alpha \odot (q_2 + J) \in S/J_{(Q\cap S)}$ or $(q_2 + J) \odot \beta \odot (q_3 + J) \in S/J_{(Q\cap S)}$ or $(q_1 + J) \odot \beta \odot (q_3 + J) \in S/J_{(Q\cap S)}$. If $(q_1 + J) \odot \alpha \odot (q_2 + J) \in S/J_{(Q\cap S)}$, then there exists $q_5 \in Q \cap S$ such that $x\alpha y \in (q_1 + J) \odot \alpha \odot (q_2 + J) = q_5 + J$. which gives $x\alpha y = q_5 + h$ for some $h \in J$ implies that $x\alpha y \in S$. Hence, S is a 2-absorbing ideal of R.

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