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CERTAIN RESULTS INVOLVING q-HYPERGEOMETRIC SERIES AND RAMANUJAN'S MOCK THETA FUNCTIONS

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Abstract: In this paper certain transformation formulas for q-hypergeometric series have been established. In another section of this paper, results involving mock theta functions have also been established.

Keywords and Phrases: q-hypergeometric series, transformation formula, mock theta functions, truncated hypergeometric series, summation formula.

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1. Introduction, Notations and Definitions

The $q-$shifted factorial for $|q| < 1$ is defined as,

$$(a;q)_0 = 1$$

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n \in N,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{r=0}^{\infty}(1-aq^r).$$

The basic hypergeometric series is defined as,

$$ {}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n}, \tag{1.1}$$

where max. $(|q|, |z|) < 1$ for the convergence of the series and

$$(a_1, a_2, ..., a_r; q)_n = (a_1; q)_n (a_2; q)_n ... (a_r; q)_n.$$

The truncated hypergeometric series is defined as,

$$ {}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right]_n = \sum_{k=0}^{n} \frac{(a_1, a_2, ..., a_r; q)_k}{(q, b_1, b_2, ..., b_s; q)_k} z^k. \tag{1.2}$$

In 1949 Bailey [2] established the following very useful transform known as Bailey's transform. It is defined as,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{1.3}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.4}$$

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.5}$$

where u_r, v_r, α_r and δ_r are arbitrary functions of r alone.
Taking $u_r = v_r = 1$ and $\delta_r = z^r$ in above transform, it takes the form,

If $\beta_n = \sum_{r=0}^{n} \alpha_r$, $\gamma_n = \frac{z^n}{1-z}$ then under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n z^n = (1-z) \sum_{n=0}^{\infty} \beta_n z^n = (1-z) \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} \alpha_r. \tag{1.6}$$

We shall make use of (1.6) in next sections and one can refereed the papers [4, 5, 6].

2. Summation formulas for truncated hypergeometric series

In this section we shall establish certain summation formulas for truncated $q-$series.

(i) Putting $c = abq$ in the summation formula [3; App II (II. 12) p. 355] we get

$$ {}_2\Phi_1 \left[\begin{array}{l} a, b; q; q \\ abq \end{array} \right]_n = \sum_{r=0}^{n} \frac{(a, b; q)_r q^r}{(q, abq; q)_r} = \frac{(aq, bq; q)_n}{(q, abq; q)_n}. \tag{2.1} $$

As $n \to \infty$, (2.1) yields

$$ {}_2\Phi_1 \left[\begin{array}{l} a, b; q; q \\ abq \end{array} \right] = \frac{(aq, bq; q)_\infty}{(q, abq; q)_\infty}. \tag{2.2} $$

(ii) Putting $b = aq^{1+n}$ in [3; App II (II. 14)] we find,

$$ {}_2\Phi_1 \left[\begin{array}{l} a, -q\sqrt{a}; q; \frac{1}{\sqrt{a}} \\ -\sqrt{a} \end{array} \right]_n = \frac{(aq; q)_n}{(q; q)_n (\sqrt{a})^n}. \tag{2.3} $$

(iii) Putting $c = aq^{1+n}$ in [3; App II (II. 21)] we get,

$$ {}_4\Phi_3 \left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; \frac{1}{b} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b} \end{array} \right]_n = \frac{(aq, bq; q)_n}{(q, aq/b, ; q)_n b^n}. \tag{2.4} $$

(iii) Taking $e = aq^{1+n}$ in [3; App II (II. 22)] we get,

$$ {}_6\Phi_5 \left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{array} \right]_n = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \quad a = bcd. \tag{2.5} $$

[3; App II (II. 34)] can be put as

$$ \sum_{k=0}^{n} \frac{(apq; pq)_k (a; p)_k (c; q)_k q^k}{(a; pq)_k (q; q)_k (ap/c; p)_k} = \frac{(ap; p)_n (cq; q)_n}{(q; q)_n (ap/c; p)_n}. \tag{2.6} $$

3. Transformation Formulas

In this section we shall establish transformation formulas by making use of the summation formulas established in the previous section.

(i) Putting $\alpha_n = \dfrac{(a, b; q)_n q^n}{(q, abq; q)_n}$ in (1.6) we get

$$ \sum_{n=0}^{\infty} \frac{(a, b; q)_n z^n q^n}{(q, abq; q)_n} = (1 - z) \sum_{n=0}^{\infty} \frac{(aq, bq; q)_n z^n}{(q, abq; q)_n}, $$

which can be written as

$$(zq;q)_\infty\, {}_2\Phi_1\left[\begin{array}{l} a,b;q;zq \\ abq \end{array}\right] = (z;q)_\infty\, {}_2\Phi_1\left[\begin{array}{l} aq,bq;q;z \\ abq \end{array}\right] \tag{3.1}$$

(ii) Putting $\alpha_n = \dfrac{(a,-q\sqrt{a};q)_n}{(q,-\sqrt{a};q)_n a^{n/2}}$ in (1.6) we get,

$$(zq;q)_\infty\, {}_2\Phi_1\left[\begin{array}{l} a,-q\sqrt{a};q;\frac{z}{\sqrt{a}} \\ -\sqrt{a} \end{array}\right] = (z;q)_\infty\, {}_1\Phi_0\left[\begin{array}{l} aq;q;\frac{z}{\sqrt{a}} \\ - \end{array}\right] = \frac{(z;q)_\infty(zq\sqrt{a};q)_\infty}{(z/\sqrt{a};q)_\infty}. \tag{3.2}$$

(iii) Taking $\alpha_n = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b;q)_n b^n}$ in (1.6) we get,

$$(zq;q)_\infty\, {}_4\Phi_3\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b;q;\frac{z}{b} \\ \sqrt{a},-\sqrt{a},aq/b \end{array}\right] = (z;q)_\infty\, {}_2\Phi_1\left[\begin{array}{l} aq,bq;q;\frac{z}{b} \\ aq/b \end{array}\right]. \tag{3.3}$$

(iv) Taking $\alpha_n = \dfrac{(a,q\sqrt{a},-q\sqrt{a},b,c,d;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d;q)_n} q^n$ in (1.6) and using (2.5) we have,

$$(zq;q)_\infty\, {}_6\Phi_5\left[\begin{array}{l} a,q\sqrt{a},-q\sqrt{a},b,c,d;q;zq \\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d \end{array}\right]$$
$$= (z;q)_\infty\, {}_4\Phi_3\left[\begin{array}{l} aq,aq/bc,aq/bd,aq/cd;q;z \\ aq/b,aq/c,aq/d,aq/bcd \end{array}\right]. \tag{3.4}$$

(v) Putting $\alpha_n = \dfrac{(apq;q)_n(a;p)_n(c;q)_n q^n}{(q;q)_n(a;pq)_n(ap/c;p)_n}$ in (1.6) we get,

$$(zq;q)_\infty \sum_{n=0}^{\infty} \frac{(apq;pq)_n(a;p)_n(c;q)_n z^n q^n}{(q;q)_n(a;pq)_n(ap/c;p)_n} = (z;q)_\infty \sum_{n=0}^{\infty} \frac{(ap;p)_n(cq;q)_n z^n}{(q;q)_n(ap/c;p)_n}. \tag{3.5}$$

4. Main Results

In this section we shall establish certain results involving Ramanujan's mock theta functions.
Let us define as,
If $A(q) = \sum_{n=0}^{\infty} B_n$ is a mock theta function, then $A_m(q) = \sum_{n=0}^{m} B_n$ is called partial mock theta function.
Also, $A(z,q) = \sum_{n=0}^{\infty} B_n z^n$ is a mock theta function having one more variable z.

For the definitions of mock theta functions of order three, five and seven one is refereed chapter 2 and 3 of the book [1, Agarwal].

(i) Putting $\alpha_n = \dfrac{q^{n^2}}{(-q;q)_n^2}$ in (1.6) we get,

$$(zq;q)_\infty f(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n f_n(q). \tag{4.1}$$

(ii) Putting $\alpha_n = \dfrac{q^{n^2}}{(-q^2;q^2)_n}$ in (1.6) we get,

$$(zq;q)_\infty \Phi(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n \Phi_n(q). \tag{4.2}$$

(iii) For $\alpha_n = \dfrac{q^{(n+1)^2}}{(q;q^2)_{n+1}}$ in (1.6) we have,

$$(zq;q)_\infty \Psi(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n \Psi_n(q). \tag{4.3}$$

(iv) Taking $\alpha_n = \dfrac{q^{n^2}}{(-\omega q, -\omega^2 q;q)_n}$ in (1.6) we get,

$$(zq;q)_\infty \chi(z;q) = (z;q)_n \sum_{n=0}^{\infty} z^n \chi_n(q). \tag{4.4}$$

(v) For $\alpha_n = \dfrac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}$ in (1.6) we have,

$$(zq;q)_\infty \omega(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n \omega_n(q). \tag{4.5}$$

(vi) Taking $\alpha_n = \dfrac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$ in (1.6) we find,

$$(zq;q)_\infty \nu(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n \nu_n(q). \tag{4.6}$$

(vii) Taking $\alpha_n = \dfrac{q^{2n(n+1)}}{(\omega q, \omega^2 q; q^2)_n}$ in (1.6) we have,

$$(zq;q)_\infty \rho(z;q) = (z;q)_\infty \sum_{n=0}^{\infty} z^n \rho_n(q). \tag{4.7}$$

By proper choice of α_n one can establish similar results for mock theta functions of five and seven order.

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