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ON TRANSFORMATION OF BASIC HYPERGEOMETRIC FUNCTIONS APPLYING FRACTIONAL q-DIFFERENTIAL OPERATOR

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Abstract: The object of this paper is to establish transformations in which a basic hypergeometric functions can be expressed as an infinite sum of functions of higher order by the application of fractional q-differential operator.Several well known hypergeometric functions and basic Kampė de Fėriet function are expressed as an infinite sum of basic hypergeometric functions.Some special cases can be deduced as the application of the main results.

Keywords and Phrases: Basic Hypergeometric Functions, fractional q–differential operator.

2020 Mathematics Subject Classification: 33D15, 26A33.

1. Introduction

The theory of Fractional Calculus operators has been developed widely and extensively by various mathematicians such as R. K. Yadav, S. D. Purohit [8]. In the present context, we propose to derive several transformations of Basic hypergeometric series by appropriately applying certain fractional q-operators in order to achieve our goals. Recently the authors (see [1], [8], [6]) have investigated several results on q-differential operator associated with basic hypergeometric functions. These results are new contributions to the theory of q-differential operator.

In the theory of q-calculus [2], for $0 < q < 1$ and real or complex a, the q-shifted factorial is defined by

$$(a, q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}) & \text{if } n \in N. \end{cases} \tag{1.1}$$

The q-gamma function is given as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \tag{1.2}$$

In terms of gamma the equation (1.1) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}(1-q)^n \tag{1.3}$$

For arbitrary β, the q-binomial coefficient [2, I.42, p.235] is given by

$$\begin{bmatrix} \beta \\ n \end{bmatrix}_q = \frac{(q^{-\beta};q)_n}{(q;q)_n}(-q^\beta)^n q^{-\binom{n}{2}} \tag{1.4}$$

The general basic hypergeometric Series [3, (3.2.1.11), p. 90] is defined as

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\dots(b_s;q)_n} z^n \tag{1.5}$$

In which there are always r of the numerator parameters and s of the denominator parameters. Recently authors ([6], [7], [9]) have derived several transformation formulae in the case of basic hypergeometric series using some known results.
Let us consider a regular function $f(z)$ defined in terms of a sequence $\{A_n\}$ as

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \tag{1.6}$$

The q-series identity due to Yadav, Purohit and Vyas [8] is given as

$$\sum_{n=0}^{\infty} \frac{(\alpha q^\beta;q)_n}{(\alpha;q)_n} A_n z^n = \sum_{n-0}^{\infty} \begin{bmatrix} \beta \\ n \end{bmatrix}_n q^{n^2+n\alpha-n} \frac{z^n(1-q)^n}{(\alpha;q)_n} D_{z,q}^n\{f(z)\} \tag{1.7}$$

where $Re(\alpha q^{\beta}) > 0$, the arbitrary sequence of real or complex number is bounded in the domain $|z| < R$.The fractional q-derivative operator $D^{\beta}_{z,q}$ for arbitrary operator β due to Agarwal [1] is given as

$$D^{\beta}_{z,q}\{f(z)\} = \frac{1}{\Gamma_q(-\beta)} \int_0^z (z - tq)_{-\beta-1} f(t) d(t;q) \tag{1.8}$$

The fractional q-derivative of generalized basic hypergeometric function due to Yadav and Purohit [8] is given by

$$D^{n}_{z,q}\left\{ {}_r\phi_s \left[\begin{array}{c} a_1, a_2, \ldots, a_r \\ ; q, z \\ b_1, b_2, \ldots, b_s \end{array} \right] \right\}$$

$$= \frac{(1-q)^{-n}(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \ldots (b_s;q)_n} {}_r\phi_s \left[\begin{array}{c} a_1 q^n, a_2 q^n, \ldots, a_r q^n \\ ; q, z \\ b_1 q^n, b_2 q^n, \ldots, b_s q^n \end{array} \right] \tag{1.9}$$

In order to prove the main result, we will be using the following transformation formulae due to Srivastava and Jain [4].

$${}_4\phi_3 \left[\begin{array}{c} ab, -ab, abq^{\frac{1}{2}}, -abq^{\frac{1}{2}}; \\ ; q, z \\ a^2b^2, -a^2, -b^2q; \end{array} \right] = {}_2\phi_1 \left[\begin{array}{c} a^2, a^2q; \\ ; q^2, z \\ a^4; \end{array} \right] {}_2\phi_1 \left[\begin{array}{c} b^2, b^2q; \\ ; q^2, z \\ b^4; \end{array} \right] \tag{1.10}$$

$${}_4\phi_3 \left[\begin{array}{c} a, b, \sqrt{ab}, -\sqrt{ab}; \\ ; q, z \\ ab, \sqrt{abq}, -\sqrt{abq}; \end{array} \right] = {}_2\phi_1 \left[\begin{array}{c} a, b; \\ ; q^2, z \\ abq; \end{array} \right] {}_2\phi_1 \left[\begin{array}{c} a, b; \\ ; q^2, qz \\ abq; \end{array} \right] \tag{1.11}$$

$${}_3\phi_2 \left[\begin{array}{c} a, bc; \\ ; q, z \\ d; \end{array} \right] = \phi^{1;1;1}_{1;0;0;} \left[\begin{array}{c} a; b; c; \\ q; bz, z \\ d; -; -; \end{array} \right] \tag{1.12}$$

2. Main Results

$${}_8\phi_7 \left[\begin{array}{c} \lambda, a, b, c, -c, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda q/c^2; \\ ; q, z \\ \lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, \sqrt{\lambda}, -\sqrt{\lambda}, c^2; \end{array} \right]$$

$$= \sum_{n-0}^{\infty} \left[\begin{array}{c} \beta \\ n \end{array} \right]_n q^{n^2+n\alpha-n} \frac{(\lambda, a, b, c, -c, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda q/c^2; q)_n}{(\lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, \sqrt{\lambda}, -\sqrt{\lambda}, \alpha q^{\beta}; q)_n}$$

$$\times\, {}_9\phi_8\left[\begin{matrix}\lambda q^n, aq^n, bq^n, cq^n, -cq^n, q^{n+1}\sqrt{\lambda}, -q^{n+1}\sqrt{\lambda}, \lambda q^{n+1}/c^2, \alpha q^n; \\ \lambda q^{n+1}/a, \lambda q^{n+1}/b, \lambda q^{n+1}/c, -\lambda q^{n+1}/c, \sqrt{\lambda}q^n, -\sqrt{\lambda}q^n, q^n c^2, \alpha q^{\beta+n};\end{matrix} ; q, z\right] \tag{2.1}$$

$${}_2\phi_1\left[\begin{matrix}a^2, a^2q; \\ a^4;\end{matrix} ; q^2, z\right] {}_2\phi_1\left[\begin{matrix}b^2, b^2q; \\ b^4;\end{matrix} ; q^2, z\right]$$

$$= \sum_{n-0}^{\infty} \left[\begin{matrix}\beta \\ n\end{matrix}\right]_n q^{n^2+n\alpha-n} \frac{z^n(1-q)^n(ab, -ab, ab\sqrt{q}, -ab\sqrt{q}, \alpha; q)_n}{(a^2b^2, -a^2, -b^2q, \alpha q^\beta; q)_n}$$

$$\times {}_5\phi_4\left[\begin{matrix}abq^n, -abq^n, abq^{n+\frac{1}{2}}, -abq^{n+\frac{1}{2}}, \alpha q^n; \\ a^2b^2q^n, -a^2q^n, -b^2q^{n+1}, \alpha q^{\beta+n};\end{matrix} ; q, z\right] \tag{2.2}$$

$${}_2\phi_1\left[\begin{matrix}a, b; \\ abq;\end{matrix} ; q^2, z\right] {}_2\phi_1\left[\begin{matrix}a, b; \\ abq;\end{matrix} ; q^2, qz\right]$$

$$= \sum_{n-0}^{\infty} \left[\begin{matrix}\beta \\ n\end{matrix}\right]_n q^{n^2+n\alpha-n} \frac{z^n(1-q)^n(a, b, \sqrt{ab}, -\sqrt{ab}, \alpha; q)_n}{(ab, \sqrt{ab}q, -\sqrt{ab}q, \alpha q^\beta; q)_n}$$

$$\times {}_5\phi_4\left[\begin{matrix}aq^n, bq^n, \sqrt{ab}q^n, -\sqrt{ab}q^n, \alpha q^n; \\ abq^n, \sqrt{ab}q^{n+\frac{1}{2}}, -\sqrt{ab}q^{n+\frac{1}{2}}, \alpha q^{\beta+n};\end{matrix} ; q, z\right] \tag{2.3}$$

$$\phi^{1;1;1}_{1;0;0;}\left[\begin{matrix}a; b; c; \\ d; -; -;\end{matrix} \; q; bz, z\right] = \sum_{n=0}^{\infty} \left[\begin{matrix}\beta \\ n\end{matrix}\right]_n q^{n^2+n\alpha-n} \frac{z^n(1-q)^n(a, bc; q)_n}{(d, \alpha q^\beta; q)_n}$$

$$\times\, {}_3\phi_2\left[\begin{matrix}aq^n, bcq^n, \alpha q^n; \\ dq^n, \alpha q^{n+\beta};\end{matrix} ; q, z\right] \tag{2.4}$$

Proof of (2.1). Choosing $A_n = \frac{(a,b,c,-c,\lambda,q\sqrt{\lambda},-q\sqrt{\lambda},\lambda q/c^2,\alpha;q)_n}{(q,\sqrt{\lambda},-\sqrt{\lambda},\lambda q/c,-\lambda q/c,c^2;q)_n}$ in (1.6) we get,

$$f(z) = {}_9\phi_8\left[\begin{matrix}\lambda, a, b, c, -c, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda q/c^2, \alpha; \\ \lambda q/a, \lambda q/b, \lambda q/c, -\lambda/c, \sqrt{\lambda}, -\sqrt{\lambda}, c^2, \alpha q^\beta;\end{matrix} ; q, z\right].$$

Using this in equation (1.7) and after some simplification with the help of the equation (1.9), finally we get (2.1).

Proof of (2.2). Choosing $A_n = \dfrac{(ab, -ab, ab\sqrt{q}, -ab\sqrt{q}, \alpha; q)_n}{(q, a^2b^2, -a^2, -b^2q, \alpha q^\beta; q)_n}$ in (1.6) we get,

$$ {}_5\phi_4 \left[\begin{array}{c} ab, -ab, ab\sqrt{q}, -ab\sqrt{q}, \alpha; \\ \\ a^2b^2, -a^2, -b^2q, \alpha q^\beta; \end{array} ; a, z \right] $$

Using this in equation (1.7) and after simplification with the help of the equations (1.9) and (1.10), finally we get (2.2).
In the similar way results (2.3) and (2.4) can be proved by choosing A_n.

3. Conclusion

The author concludes with the remark that one can deduce several results involving fractional q-differential operators associate with Basic hypergeometric functions. The findings of the present work can be referred by several researchers and mathematicians in this field and encourages them to take up the further challenging works in the same field.

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