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Some Extensions of Multiple Gaussian Hypergeometric Series

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Abstract: Some investigation of general multiple series identities which extend and generalize the theorems of Bailey [*1*] and Pathan [*2*] is done. Its special cases yield various new transformations and reduction formulae involving quadruple hypergeometric function $F_p^{(4)}$, and Srivastava's quadruple hypergeometric functions $F^{(4)}$ and triple hypergeometric function $F^{(3)}$.
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1. Introduction

Let (a_A) denote the sequences of A parameters given by $a_1, a_2, ..., a_A$ and $[(a_A)]_n$ denote the product of A Pochhamer symbols defined by

$$(b)_n = \frac{\Gamma(b+n)}{\Gamma(b)} = \begin{cases} 1, & \text{if } n = 0 \\ b(b+1)...(b+n-1), & \text{if } \quad n = 1, 2, 3, ... \end{cases}$$

In 1969, Srivastava and Daoust ([7, page 454], see also [8, page 37 (21,22)]) gave the following multivariable hypergeometric function,

$$F^{A:B^{(1)};...;B^{(n)}}_{D:E^{(1)};...;E^{(n)}} \left[\begin{array}{c} [(a_A) : \theta^{(1)}, ..., \theta^{(n)}] : [b^{(1)}_{B^{(1)}} : \Phi^{(1)}]; ...; [b^{(n)}_{B^{(n)}} : \Phi^{(n)}] z_1, ..., z_n \\ [(d_D) : \Psi^{(1)}, ..., \Psi^{(n)}] : [e^{(1)}_{E^{(1)}} : \delta^{(1)}]; ...[e^{(n)}_{E^{(n)}} : \delta^{(n)}] \end{array} \right]$$

$$= \sum_{m_1,...,m_n}^{\infty} \Xi(m_1, ..., m_n) \frac{z_1^{m_1}}{(m_1)!} ... \frac{z_n^{m_n}}{(m_n)!}$$

where for convenience,

$$\Xi(m_1,...,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}\prod_{j=1}^{B}{}^{(1)}(b_j^{(1)})_{m_1\Phi_j^{(1)}}\cdots\prod_{j=1}^{B}{}^{(n)}(b_j^{(n)})_{m_n\Phi_j^{(n)}}}{\prod_{j=1}^{D}(d_j)_{m_1\Psi_j^{(1)}+...+m_n\Psi_j^{(n)}}\prod_{j=1}^{E}{}^{(1)}(e_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{E}{}^{(n)}(e_j^{(n)})_{m_n\delta_j^{(n)}}}$$

The coefficients $\theta_j^{(k)}, j=1,...,A; \Phi_j^{(k)}, j=1,...,B^{(k)}; \Psi_j^k, j=1,...,D; \delta_j^{(k)}, j=1,...,E^{(k)}$ for all $k\in\{1,...,n\}$ are zero and real constants (positive, negative) [8] and $(b_{B^{(k)}}^{(k)})$ abbreviate the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,...,B^{(k)}$ for all $k\in\{1,...,n\}$ with similar interpretations for others.

The present paper is devoted to the investigation of general multiple series identities which extend and generalize the theorems of Bailey [1], Pathan [2]. The theorem given in section 2 will be seen extremely useful as it provides connections with various classes of well-known hypergeometric functions and even new representations of these functions. Some applications of this theorem are given in section 3. Also we deduce special cases in section 4.

2. General multiple series identities

We will establish the following theorems for multiple series which is more generalized than multiple Gaussian hypergeometric functions $F^{(3)}, F^{(4)}$ and $F_P^{(4)}$.

Theorems: Let $S(i,j,k,p)$ be the generalized coefficient of arbitrary complex numbers, where x, y, z be complex variables and c, f be arbitrary independent complex parameters (where $2f\neq 0,2,3,...$) and any values of numerator and denominator parameters and variables x, y, z leading to the results which do not make sense are tacitly excluded, then

$$\sum_{i,j,k,p=0}^{\infty}\frac{S(i+k,j,p)(-1)^k(1-c)_i(2f-c)_k x^j y^{i+k} z^p}{(2-2f)_i(2f)_k i!j!k!p!}$$

$$=\sum_{t=0}^{1}\frac{y^t}{t!}\Omega(t)\sum_{i,j,p=0}^{\infty}S(2i+t,j,p)\frac{\Lambda(t,i)(4y)^i x^j z^p}{j!p!} \tag{2.1}$$

$$=\sum_{t=0}^{1}\frac{y^t x^u}{t!u!}\Omega(t)\sum_{i,j,p=0}^{\infty}S(2i+t,2j+u,p)\frac{\Lambda(t,i)(4y)^i(4x)^j z^p}{((1+u)/2)_j(1+u/2)_j p!} \tag{2.2}$$

$$=\sum_{t=0}^{1}\frac{y^t x^u z^w}{t!u!w!}\Omega(t)\sum_{i,j,p=0}^{\infty}S(2i+t,2j+u,2p+w)$$

$$\times \frac{\Lambda(t,i)(4y)^i(4x)^j(4z)^p}{((1+u)/2)_j(1+u/2)_j((1+w)/2)_p(1+w/2)_p} \tag{2.3}$$

where

$$\Omega(t) = \frac{\Gamma(2f)\Gamma(c-t)\Gamma((2f+1-t)/2)\Gamma((2c+1+t)/2)\Gamma(2-2f)}{\Gamma(c)\Gamma(2f-t)\Gamma(2-2f+t)\Gamma((2c-2f+1-t)/2)\Gamma((2f+t+1)/2)} \tag{2.4}$$

$$\Lambda(t,i) = \frac{\left(\frac{1-c}{2}\right)_i \left(\frac{2-c}{2}\right)_i \left(\frac{2-2f+t}{2}\right)_i \left(\frac{1-2c+2f+t}{2}\right)_i \left(\frac{2-2c+2f+t}{2}\right)_i \Gamma(1+2i)}{\left(\frac{1-c+t}{2}\right)_i \left(\frac{2-c+t}{2}\right)_i \left(\frac{1+t}{2}\right)_i \left(\frac{2+t}{2}\right)_i (2i)!}$$

provided that each multiple series involved converges absolutely.

Proof of theorems (2.1)-(2.3):

Let L denote the LHS of equation (2.1), then using the series identity [4] defined as

$$\sum_{i,j,k,p=0}^{\infty} A(i,j,k,p) = \sum_{i,j,k,p=0}^{\infty} \sum_{k=0}^{i} A(i-k,j,k,p)$$

we may write,

$$L = \sum_{i,j,p=0}^{\infty} S(i,j,p) \sum_{k=0}^{i} \frac{(-1)^k (1-c)_{i-k}(2f-c)_k x^j y^i z^p}{(2-2f)_{i-k}(2f)_k (1-k)! j! k! p!} \tag{2.5}$$

$$= \sum_{i,j,p=0}^{\infty} S(i,j,p) \sum_{k=0}^{i} \frac{(2f-1-i)_k(-i)_k(1-c)_i(2f-c)_k x^j y^i z^p}{(c-i)_k(2-2f)_i(2f)_k i! j! k! p!}$$

$$= \sum_{i,j,p=0}^{\infty} \frac{S(i,j,p)(1-c)_i x^j y^i z^p}{(2-2f)_i i! j! p!} \sum_{k=0}^{\infty} \frac{(2f-1-i)_k(2f-c)_k(-i)_k}{(c-i)_k(2f)_k k!}$$

$$= \sum_{i,j,p=0}^{\infty} \frac{S(i,j,p)(1-c)_i x^j y^i z^p}{(2-2f)_i i! j! p!} \times {}_3F_2\left(\begin{array}{l} 2f-1-i, 2f-c, -i; 1 \\ c-i, 2f \end{array} \right) \tag{2.6}$$

Using Dixon's theorem [4] in equation (2.6) we may write,

$$L = \sum_{i,j,p=0}^{\infty} \frac{S(i,j,p)(1-c)_i x^j y^i z^p}{(2-2f)_i i! j! p!}$$

$$\times \frac{\Gamma((2f+1-i)/2)\Gamma((1+i-2f+2c)/2)\Gamma(2f)\Gamma(c-i)}{\Gamma((2c-2f+1-i)/2)\Gamma((2f+1+i)/2)\Gamma(c)\Gamma(2f-i)} \tag{2.7}$$

Now applying the following identity [9, page 214, 217] in equation (2.7) and using the series rearrangement technique we get,

$$\sum_{i=0}^{\infty} A(i) = \sum_{t=0}^{1}\sum_{i=0}^{\infty} A(2i+t)$$

$$L = \sum_{t=0}^{1}\sum_{i=0}^{\infty} \frac{S(2i+t,j,p)(1-c)_{2i+t}x^j y^{2i+t} z^p}{(2-2f)_i i! j! p!}$$
$$\times \frac{\Gamma((2f+1-2i-t)/2)\Gamma((1+2i+t-2f+2c)/2)\Gamma(2f)\Gamma(c-2i-t)}{\Gamma((2c-2f+1-2i-t)/2)\Gamma((2f+1+2i+t)/2)\Gamma(c)\Gamma(2f-2i-t)}$$
$$= \sum_{t=0}^{1} \frac{y^t\Gamma(1-c)\Gamma(c-t)\Gamma(2f)\Gamma(2-2f)\Gamma((2f+1-t)/2)\Gamma((1+t+2c)/2)}{\Gamma(1-c)\Gamma(c)\Gamma(2f-t)\Gamma(2-2f+t)\Gamma((2c-2f+1-t)/2)\Gamma((2f+t+1)/2)}$$
$$\times \sum_{i,j,p=0}^{\infty} \frac{S(2i+t,j,p)(1-c)_{2i}((2f+1-t)/2)_{-i}(c-t)_{-2i}((1+t+2c)/2)_i x^j y^{2i} z^p}{(1+2i)_t(2f-t)_{-2i}((2c-2f+1-t)/2)_{-2i}((2f+t+1)/2)_i (2i)! j! p!}$$
$$= \sum_{t=0}^{1} \frac{y^t}{t!}\Omega(t) \sum_{i,j,p=0}^{\infty} S(2i+t,j,p)\frac{\Lambda(t,i)(4y)^j x^j z^p}{j!p!}$$

where

$$\Omega(t) = \frac{\Gamma(2f)\Gamma(c-t)\Gamma((2f+1-t)/2)\Gamma((2c+1+t)/2)\Gamma(2-2f)}{\Gamma(c)\Gamma(2f-t)\Gamma(2-2f+t)\Gamma((2c-2f+1-t)/2)\Gamma((2f+t+1)/2)}$$

$$\Lambda(t,i) = \frac{\left(\frac{1-c}{2}\right)_i\left(\frac{2-c}{2}\right)_i\left(\frac{2-2f+t}{2}\right)_i\left(\frac{1-2c+2f+t}{2}\right)_i\left(\frac{2-2c+2f+t}{2}\right)_i\Gamma(1+2i)}{\left(\frac{1-c+t}{2}\right)_i\left(\frac{2-c+t}{2}\right)_i\left(\frac{1+t}{2}\right)_i\left(\frac{2+t}{2}\right)_i(2i)!}$$

Again applying the following identities [9, page 214, 217]

$$\sum_{i,j=0}^{\infty} B(i,j) = \sum_{t,u=0}^{1}\sum_{i,j=0}^{\infty} B(2i+t,2j+u)$$

$$\sum_{i,j,p=0}^{\infty} C(i,j,p) = \sum_{t,u,w=0}^{1}\sum_{i,j,p=0}^{\infty} C(2i+t,2j+u,2p+w)$$

in equation (2.6), then using the series rearrangement technique we get the RHS of equation (2.2)-(2.3).

3. Applications of theorems (2.1)-(2.3)

3.1. In theorem (2.1) and (2.2) setting

$$S(i+j+k)=\frac{[(a_A)]_{i+j+k}[(d_D)]_{i+k}[(g_G)]_j}{[(b_B)]_{i+j+k}[(e_E)]_{i+k}[(h_H)]_j}$$

and z=0 and using series rearrangment technique, we get

$$F^{(3)}\left[\begin{array}{l}(a_A)::-;(d_D);-:(g_G);1-c;2f-c;x,y,-y\\(b_B)::-;(e_E);-:(h_H);2-2f;2f\end{array}\right]$$

$$=\sum_{t=0}^{1}\frac{[(a_A)]_t[(d_D)]_t y^t}{[(b_B)]_t[(e_E)]_t t!}\Omega(t)\times$$

$$F^{A:2D+5;G}_{B:2E+6;H}\left[\begin{array}{l}[(a_A)+t:2,1]:[\Delta(2;(d_D)+t;1],\left[\frac{1+t}{2}+f-c:1\right],\left[\frac{2+t}{2}-f:1\right],\\ [(b_B)+t:2,1]:[\Delta(2;(e_E)+t:1],\left[\frac{1+t}{2}-c:1\right],\left[\frac{2+t}{2}-c:1\right],\end{array}\right.$$

$$\left.\begin{array}{l},\left[\dfrac{1-c}{2}:1\right],\left[\dfrac{2-c}{2}:1\right],\left[\dfrac{2+t}{2}+f-c:1\right];[(g_G):1]4^{1+E-D}y^2,x\\ ,\left[\dfrac{1+t}{2}:1\right],\left[\dfrac{2+t}{2}:1\right]\left[\dfrac{1+t}{2}+f-c:1\right],\left[\dfrac{1+t}{2}:1\right];[(h_H):1];\end{array}\right]$$

$$=\sum_{t,u=0}^{1}\frac{[(a_A)]_{t+u}[(d_D)]_t[(g_G)]_u y^t x^u}{[(b_B)]_{t+u}[(e_E)]_t[(h_H)]_u t!u!}\Omega(t)\times$$

$$F^{2A:2D+5;2G}_{2B:2E+2;2H+1}\left[\begin{array}{l}\Delta[2;(a_A)+t+u];\dfrac{1+t}{2}+f-c,\dfrac{2+t}{2}-f-c,\dfrac{2+t}{2}-f,\dfrac{1-c}{2},\\ \Delta[2;(b_B)+t+u];\dfrac{1}{2}-c+t,\dfrac{t-c}{2}+1,\Delta^*(2;1+t),\end{array}\right.$$

$$\left.\begin{array}{l},\dfrac{2-c}{2},\Delta[2;(d_D)+t];\Delta[2;(g_G)+u];\dfrac{4^{1+B+E}y^2}{4^{A+D}},\dfrac{4^{A+D}x^2}{4^{1+B+E}}\\ ,\Delta[2;(e_E)+t];\Delta^*(2;1+t),\Delta[2;(h_H)+u];\end{array}\right]$$

provided the denominator parameters are neither zero nor negative integers and for convenience, the symbol $\Delta(m;b)$ abbreviates the array of m parameters given by

$$\frac{b}{m},\frac{b+1}{m},\frac{b+2}{m},...,\frac{b+m-1}{m},$$

where $m = 1, 2, 3, ...$
The asterisk in $\Delta^*(N; j+1)$ represents the fact that the (denominator) parameter N/N is always omitted for $0 \leq j \leq (N-1)$ so the set $\Delta^*(N; j+1)$ contains only $N-1$ parameters [13 page 214].

3.2. In the theorem (2.3) setting

$$S(i,j,k,p) = \frac{[(a_A)]_{i+j+k+p}[(m_M)]_{j+p}[(d_D)]_{i+k}[(g_G)]_j[(q_Q)]_p}{[(b_B)]_{i+j+k}[(n_N)]_{j+p}[(e_E)]_{i+k}[(h_H)]_j[(r_R)]_p}$$

we get

$$F^{(4)}\left[\begin{array}{l} (a_A) :: 1-c; (d_D); (g_G); (m_M) : (d_D); (q_Q); (m_M); 2f-c; y, x, -y, z \\ \\ (b_B) :: 2-2f; (e_E); (h_H); (n_N) : 2f; (e_E); (r_R); (n_N); \end{array}\right]$$

$$= \sum_{t,u,w=0}^{1} \frac{[(a_A)]_{t+u+w}[(m_M)]_{u+w}[(d_D)]_t[(g_G)]_u[(q_Q)]_w y^t x^u z^w}{[(b_B)]_{t+u+w}[(n_N)]_{u+w}[(e_E)]_t[(h_H)]_u[(r_R)]_w t!u!w!}\Omega(t)\times$$

$$F^{(3)}\left[\begin{array}{l} \Delta[2; (a_A)+t+u+w] :: -; \Delta[2; (m_M)+u+w]; - : \frac{1+t}{2}+f-c, \\ \\ \Delta[2; (b_B)+t+u+w] :: -; \Delta[2; (n_N)+u+w]; - : \frac{3+t}{2}-f, \frac{1+t}{2}+f \end{array}\right.$$

$$\frac{1+t}{2}-f+c, \Delta[2; (d_D)+t]; \Delta[2; (g_G)+u]; \Delta[2; (q_Q)+w];$$

$$\Delta^*(2; 1+t), \Delta[2; (e_E)+t]; \Delta[2; (h_H)+u];$$

$$\left.\begin{array}{l} \frac{4^{1+B+E}y^2}{4^{A+D}} \frac{4^{1+B+H+M}x^2}{4^{A+G+M}} \frac{4^{1+B+N+R}z^2}{4^{A+M+Q}} \\ \\ \Delta^*(2; 1+w); \Delta[2; (r_R)+w]; \end{array}\right]$$

3.3. In the theorem (2.3), setting

$$S(i,j,k,p) = \frac{[(a_A)]_{j+p+i+k}[(m_M)]_j[(d_D)]_{p+i+k}[(g_G)]_{i+k+j}[(q_Q)]_p}{[(b_B)]_{j+p+i+k}[(n_N)]_j[(e_E)]_{p+i+k}[(h_H)]_{i+k+j}[(r_R)]_p}$$

we get

$$F^{(4)}\left[\begin{array}{l}(a_A)::1-c;(d_D);(g_G);(m_M);(q_Q);(m_M);2f-c;x,z,y,-y\\ \\ (b_B)::2-2f;(e_E);(h_H);(n_N):2f;(e_E);(r_R)\end{array}\right]$$

$$=\sum_{t,u,w=0}^{1}\frac{[(a_A)]_{t+u+w}[(m_M)]_u[(d_D)]_{t+w}[(g_G)]_{t+u}[(q_Q)]_w y^t x^u z^w}{[(b_B)]_{t+u+w}[(n_N)]_u[(e_E)]_{t+w}[(h_H)]_{t+u}[(r_R)]_w t!u!w!}\Omega(t)\times$$

$$F^{(3)}\left[\begin{array}{l}\Delta[2;(a_A)+t+u+w]::-;\Delta[2;(m_M)+u];-:\frac{1+t}{2}+f-c,\\ \\ \Delta[2;(b_B)+t+u+w]::-;\Delta[2;(n_N)+u];-:\frac{1+t-c}{2},\frac{2+t}{2}-\frac{c}{2}\end{array}\right.$$

$$\frac{2+t}{2}+f-c,,\frac{2+t}{2}-f,\frac{1-c}{2},\frac{2-c}{2},\Delta[2;(d_D)+t+w];$$

$$\Delta^*(2;1+t),\Delta[2;(e_E)+t+w];\Delta^*(2;1+u);$$

$$\left.\begin{array}{l}\Delta[2;(g_G)+t+u];\Delta[2;(q_Q)+w];\frac{4^{1+B+H+N}y^2}{4^{A+D+G}}\frac{4^{1+B+H+M}x^2}{4^{A+G+M}}\frac{4^{1+B+N+R}z^2}{4^{A+M+Q}}\\ \\ \Delta[2;(h_H)+t+u];\Delta^*(2;1+w)\Delta[2;(r_R)+w];\end{array}\right]$$

The double hypergeometric function $F^{A:D;G}_{B:E;H}$ is given by Kampe de Feriet, see [8, page 27], the triple hypergeometric function is given by $F^{(3)}$ is given by Srivastava [6, page 428] and quadruple hypergeometric functions $F^{(4)}$ and $F^{(4)}_P$ are given by Pathan [3, page 172].

4. Special Cases

(1). Setting $A=M=N=Q=R=1$ and $B=D=E=G=H=0$ in (3.2), we get

$$F^{(4)}\left[\begin{array}{l}a,c,q,m,1-c;r,n,2f-c;x,z,y,-y\\ 2-2f;n:2f;r;\end{array}\right]$$

$$=\sum_{t,u,w=0}^{1}\frac{(a)_{t+u+w}(m)_u(q)_w y^t x^u z^w}{(n)_u(r)_w t!u!w!}\Omega(t)\times$$

$$F^{(3)}\left[\begin{array}{l}\Delta[2;a+t+u+w]::-;\Delta[2;m+u];-;\frac{1+t}{2}+f-c,\frac{1+t}{2}-f+c,\frac{2+t}{2}-f\\ \\ \Delta[2;t+u+w]::-;\Delta[2;n+u];-;\frac{3+t}{2}-f,\frac{1+t}{2}+f,\Delta^*(2;1+t),\Delta[2;t];\end{array}\right.$$

$$\left.\begin{array}{l}\Delta[2;(d_D)+t+w];\Delta[2;(g_G)+t+u];\Delta[2;(q_Q)+w];y^2,x^2,z^2\\ \\ \Delta[2;t+u];\Delta^*(2;1+w)\Delta[2;r+w];\end{array}\right]$$

(2). In (2.1) setting $S(i,j,k)=\dfrac{(a)_{2j+i+k}}{(b)_j}$ and $z=0$, we get

$$X_8[a,(1-c),2f-c;b,2-2f,2f;x,y,-y]=\sum_{t=0}^{1}\frac{y^t(a)_t}{t!}\Omega(t)\times$$

$$F^{2:5;0}_{0:3;1}\left[\begin{array}{l}\Delta(2;a+t):\frac{1+t}{2}+f-c,\frac{2+t}{2}+f-c,\frac{2+t}{2}-f,\frac{1-c}{2},\frac{2-c}{2};y^2,x/4\\ \\ -:\frac{1-c}{2}+\frac{t}{2},\frac{2-c}{2}+\frac{t}{2},\Delta^*(2,1+t);b;\end{array}\right]$$

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