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ON A MODULAR EQUATION OF DEGREE 23

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Abstract: On page 249 of his second notebook, Ramanujan incorrectly recorded and crossed out a modular equation of degree 23. B. C. Berndt has corrected and proved the same using the theory of modular forms. The main objective of this article is to prove the modular equation of degree 23 corrected by Berndt by employing the method of parametrization. Using a similar technique, we also prove a modular equation of degree 11 due to Ramanujan.

Keywords and Phrases: Modular equations, hypergeometric series.

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1. Introduction

The Gauss series or the ordinary hypergeometric series ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1 $$

where

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1 $$

and

$$(a)_0 = 1.$$

In the above definition, a, b, c, d and z are complex numbers with $|z| < 1$ and $c \neq 0, -1, -2, \cdots$.

Suppose that

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}$$

holds for some positive integer n and $0 < \alpha < 1,\ 0 < \beta < 1$. The relation between α and β induced by the above is called a modular equation of degree n and we say that β has degree n over α. The multiplier m is defined by

$$m = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}.$$

Ramanujan recorded many modular equations in his notebooks. Most of them are found in Chapters 18-20 of his second notebook [4]. The proofs of all these modular equations can be found in B. C. Berndt [1]. In fact, Berndt proved most of the modular equations by using elementary algebraic techniques or through theta function identities, and a few by employing the theory of modular forms. On page 326 of [1], Berndt mentioned that the primary disadvantage of employing the theory of modular forms is that the modular equations must be known in advance and hence the proofs are more aptly called verifications.

The main aim of this article is to prove a modular equation of degree 23 using the method of parametrization. In the same method, we also prove a modular equation of degree 11. In Section 2 of this article, we develop a parametrization theory for a modular equation of degree 23. In Section 3, we prove a modular equation of degree 23 due to Berndt which was incorrectly recorded and crossed out by Ramanujan. In Section 4, we prove a modular equation of degree 11 by using the parametrization theory developed by K. R. Vasuki and M. V. Yathirajsharma in [7].

2. On parameter a

Through out this section and in Section 3, let β has degree 23 over α and m be a multiplier of degree 23. Let the parameter a be defined by

$$a = 2^{\frac{2}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{24}}.$$

By the definition of a, it is clear that a is positive. On page 248 of his second notebook [4], Ramanujan recorded

$$(\alpha\beta)^{\frac{1}{8}} + \{(1-\alpha)(1-\beta)\}\}^{\frac{1}{8}} = 1 - a. \tag{2.1}$$

Berndt proved this by using one of the Schröter's formula [1, p. 69, Eqn. (36.11)]. Schröter has recorded (2.1) in [5] and [6].

We now express $\alpha, \beta, 1-\alpha$ and $1-\beta$ in terms of the parameter a. Squaring (2.1) on both sides, we obtain

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}\}^{\frac{1}{4}} = 1 - 2a + a^2 - \frac{a^3}{2}. \tag{2.2}$$

Squaring (2.2) on both sides, we obtain

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} = \left(1 - 2a + a^2 - \frac{a^3}{2}\right)^2 - \frac{a^6}{8}. \tag{2.3}$$

Squaring (2.3) on both sides, we find that

$$(\alpha\beta) + (1-\alpha)(1-\beta) = \left(\left(1 - 2a + a^2 - \frac{a^3}{2}\right)^2 - \frac{a^6}{8}\right)^2 - \frac{a^{12}}{128}. \tag{2.4}$$

For convenience, we set

$$s = \left(\left(1 - 2a + a^2 - \frac{a^3}{2}\right)^2 - \frac{a^6}{8}\right)^2 - \frac{a^{12}}{128}. \tag{2.5}$$

From (2.4), it follows that

$$(\alpha\beta) - (1-\alpha)(1-\beta) = \pm\frac{\sqrt{(256s)^2 - 4a^{24}}}{256}. \tag{2.6}$$

Since the left hand side of the above is $\alpha + \beta - 1$, we have

$$(\alpha\beta) - (1-\alpha)(1-\beta) = \frac{\sqrt{(256s)^2 - 4a^{24}}}{256}, \quad \alpha + \beta > 1 \tag{2.7}$$

and

$$(\alpha\beta) - (1-\alpha)(1-\beta) = -\frac{\sqrt{(256s)^2 - 4a^{24}}}{256}, \quad \alpha + \beta \leq 1. \tag{2.8}$$

We now proceed by assuming that $\alpha+\beta>1$. Adding (2.4) and (2.7), we obtain

$$\alpha\beta=\frac{256s+\sqrt{(256s)^2-4a^{24}}}{512}. \tag{2.9}$$

From (2.7), it follows that

$$\alpha+\beta=\frac{256+\sqrt{(256s)^2-4a^{24}}}{256}. \tag{2.10}$$

From (2.9) and (2.10), we have

$$\alpha,\beta=\frac{256+\sqrt{(256s)^2-4a^{24}}\pm\sqrt{(256s-256)^2-4a^{24}}}{512}.$$

Since $\alpha>\beta$, we have

$$\alpha=\frac{256+\sqrt{(256s)^2-4a^{24}}+\sqrt{(256s-256)^2-4a^{24}}}{512}$$

and

$$\beta=\frac{256+\sqrt{(256s)^2-4a^{24}}-\sqrt{(256s-256)^2-4a^{24}}}{512}.$$

One can see that

$$\begin{aligned}(256s)^2-4a^{24}=256(1-a)^2(2-4a+a^2)^2(2-4a+2a^2-a^3)^2\\ (4-8a+6a^2-4a^3+a^4)^2(1-2a+a^2-a^3).\end{aligned}$$

The left hand side of the above is positive (follows from (2.9)) and hence we must have

$$1-2a+a^2-a^3>0.$$

We observe that

$$2-4a+2a^2-a^3=2(1-2a+a^2-a^3)+a^3>0,$$

and

$$4-8a+6a^2-4a^3+a^4=4(1-2a+a^2-a^3)+a^2(2+a^2)>0.$$

It is not difficult to see that

$$2-4a+a^2>0.$$

Now, we have

$$\begin{aligned}(256s-256)^2-4a^{24} = \; & 256a^2(2-a)^2(2-a+a^2)^2(4-4a+4a^2-a^3)^2\\ & (8-4a+4a^2-a^3)(8-20a+28a^2-25a^3+14a^4-5a^5+a^6).\end{aligned}$$

We observe that the left hand side of the above is positive and hence we must have

$$(8-4a+4a^2-a^3)(8-20a+28a^2-25a^3+14a^4-5a^5+a^6)>0.$$

Clearly, we have

$$2-a>0,$$

$$2-a+a^2>0,$$

and

$$4-4a+4a^2-a^3>0.$$

From the above discussion, we have

$$\alpha=\frac{16+A\sqrt{p}+B\sqrt{qr}}{32}, \tag{2.11}$$

$$\beta=\frac{16+A\sqrt{p}-B\sqrt{qr}}{32}, \tag{2.12}$$

$$1-\alpha=\frac{16-A\sqrt{p}-B\sqrt{qr}}{32}, \tag{2.13}$$

and

$$1-\beta=\frac{16-A\sqrt{p}+B\sqrt{qr}}{32} \tag{2.14}$$

where

$$\begin{aligned}A &= (1-a)(2-4a+a^2)(2-4a+2a^2-a^3)(4-8a+6a^2-4a^3+a^4),\\ B &= a(2-a)(2-a+a^2)(4-4a+4a^2-a^3),\\ p &= 1-2a+a^2-a^3,\\ q &= 8-4a+4a^2-a^3,\end{aligned}$$

and

$$r=8-20a+28a^2-25a^3+14a^4-5a^5+a^6.$$

From (2.1), we obtain

$$(\alpha\beta)^{\frac{1}{8}}-\{(1-\alpha)(1-\beta)\}^{\frac{1}{8}}=1-2a+a^2-a^3. \tag{2.15}$$

3. A modular equation of degree 23

In this section, we prove the following modular equation of degree 23.

Theorem 3.1. *If m is a multiplier of degree 23 and β has degree 23 over α, then*

$$\begin{aligned} m-\frac{23}{m} = \; & 2((\alpha\beta)^{\frac{1}{8}}-\{(1-\alpha)(1-\beta)\}^{\frac{1}{8}})(11-13\cdot 4^{\frac{1}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{24}} \\ & +18\cdot 2^{\frac{1}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}-14\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{8}} \\ & +2^{\frac{5}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}}). \end{aligned}$$

The above theorem is due to Berndt [1, p. 411]. This was incorrectly recorded and crossed out by Ramanujan in [4, p. 249] as follows:

$$\begin{aligned} m-\frac{23}{m} = \; & 2\left((\alpha\beta)^{\frac{1}{4}}-\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\right) \\ & \left(11-2\cdot 4^{\frac{1}{8}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{24}}+14\cdot 2^{\frac{1}{3}}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}\right). \end{aligned}$$

Proof of Theorem 3.1. Let

$$x = x(a) := (\alpha\beta)^{1/8}$$

and

$$y = y(a) := \{(1-\alpha)(1-\beta)\}^{1/8}.$$

From (2.1) and (2.15), we find that

$$x = (\alpha\beta)^{1/8} = \frac{1-a+\sqrt{p}}{2} \tag{3.1}$$

and

$$y = \{(1-\alpha)(1-\beta)\}^{1/8} = \frac{1-a-\sqrt{p}}{2}. \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$\dot{x}(a) = -\frac{1}{2}\left[1+\frac{2-2a+3a^2}{2\sqrt{p}}\right] \tag{3.3}$$

and

$$\dot{y}(a) = -\frac{1}{2}\left[1-\frac{2-2a+3a^2}{2\sqrt{p}}\right], \tag{3.4}$$

where $\dot{x}(a)$ and $\dot{y}(a)$ denote the derivatives $\dfrac{dx}{da}$ and $\dfrac{dy}{da}$ respectively.
From (2.11), (2.13), (3.1), (3.2), (3.3), and (3.4), we have

$$\alpha y\dot{x}(a) + (1-\alpha)x\dot{y}(a) = \frac{a^2}{128\sqrt{p}}\left((48-(3-a)A)\sqrt{p} - (3-a)B\sqrt{qr}\right). \tag{3.5}$$

From (2.12), (2.14), (3.1), (3.2), (3.3), and (3.4), we have

$$\beta y\dot{x}(a) + (1-\beta)x\dot{y}(a) = \frac{a^2}{128\sqrt{p}}\left((48-(3-a)A)\sqrt{p} + (3-a)B\sqrt{qr}\right). \tag{3.6}$$

From [1, Entry 24(vi), p. 217]), we have

$$23\frac{d\alpha}{d\beta} = \frac{\alpha(1-\alpha)}{\beta(1-\beta)}m^2. \tag{3.7}$$

Using the definition of x and y in (3.7), we find that

$$\frac{23}{m^2} = \frac{\alpha(1-\alpha)\frac{d\beta}{da}}{\beta(1-\beta)\frac{d\alpha}{da}} = -\frac{\alpha y\dot{x}(a) + (1-\alpha)x\dot{y}(a)}{\beta y\dot{x}(a) + (1-\beta)x\dot{y}(a)}. \tag{3.8}$$

Employing (3.5) and (3.6) in (3.8), we obtain

$$\begin{aligned}\frac{23}{m} = \pm((22 - 26a + 18a^2 - 7a^3 + a^4)\sqrt{p} \\ - (3-a)(4-4a+4a^2-a^3)\sqrt{qr}).\end{aligned} \tag{3.9}$$

From the above, it follows that

$$\begin{aligned}m = \mp((22 - 26a + 18a^2 - 7a^3 + a^4)\sqrt{p} \\ +(3-a)(4-4a+4a^2-a^3)\sqrt{qr}).\end{aligned} \tag{3.10}$$

We observe that

$$\begin{aligned}22-&26a + 18a^2 - 7a^3 + a^4 \\ &= (4 - 8a + 6a^2 - 4a^3 + a^4) + 3(2 - 4a + 2a^2 - a^3) + 6(2 - a + a^2) > 0.\end{aligned}$$

Since $m > 0$, (3.9) and (3.10) implies that

$$m = (22 - 26a + 18a^2 - 7a^3 + a^4)\sqrt{p} + (3-a)(4-4a+4a^2-a^3)\sqrt{qr} \tag{3.11}$$

and

$$\frac{23}{m} = (3-a)(4-4a+4a^2-a^3)\sqrt{qr} - (22-26a+18a^2-7a^3+a^4)\sqrt{p}. \quad (3.12)$$

From (3.11) and (3.12), we obtain

$$m - \frac{23}{m} = (22-26a+18a^2-7a^3+a^4)\sqrt{p}. \quad (3.13)$$

From (2.15), we have

$$p = (\alpha\beta)^{\frac{1}{8}} - \{(1-\alpha)(1-\beta)\}^{\frac{1}{8}}.$$

Using the above and the definition of a in (3.13), we obtain the required result. Similarly we have deduced (3.13) for $\alpha+\beta \leq 1$. This completes the proof.

4. A modular equation of degree 11

In this section, we prove the following modular equation of degree 11.

Theorem 4.1. *If m is a multiplier of degree 11 and β has degree 11 over α, then*

$$m - \frac{m}{11} = 2\left((\alpha\beta)^{\frac{1}{4}} - \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\right)\left(4 + (\alpha\beta)^{\frac{1}{4}} - \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\right).$$

Ramanujan has recorded the above modular equation in his second notebook [4, p. 243].
Let

$$a = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}.$$

We suppose that $\alpha+\beta > 1$.
In [4, p. 243], Ramanujan has recorded the following modular equation:

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1-2a. \quad (4.1)$$

From [7], we have

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} = 1-4a+4a^2-a^3, \quad (4.2)$$

$$\alpha\beta + (1-\alpha)(1-\beta) = s, \quad (4.3)$$

$$\alpha\beta - (1-\alpha)(1-\beta) = \frac{\sqrt{4s^2-a^{12}}}{2}, \quad (4.4)$$

$$(\alpha\beta)^{\frac{1}{4}} = \frac{1-2a+\sqrt{p}}{2}, \quad (4.5)$$

$$\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = \frac{1-2a-\sqrt{p}}{2}, \tag{4.6}$$

$$\alpha = \frac{2+\sqrt{4s^2-a^{12}}+\sqrt{4+4s^2-a^{12}-8s}}{4}, \tag{4.7}$$

and

$$\beta = \frac{2+\sqrt{4s^2-a^{12}}-\sqrt{4+4s^2-a^{12}-8s}}{4} \tag{4.8}$$

where

$$s = 1-8a+24a^2-34a^3+24a^4-8a^5+\frac{a^6}{6}$$

and

$$p = 1-4a+4a^2-2a^3.$$

Thus, we have

$$\alpha = \frac{1+A\sqrt{p}+B\sqrt{2q}}{2} \tag{4.9}$$

and

$$\beta = \frac{1+A\sqrt{p}-B\sqrt{2q}}{2}, \tag{4.10}$$

where

$$A = (1-a)(1-2a)(1-3a+a^2),$$
$$B = a(2-a)(2-3a+2a^2),$$

and

$$q = 2-4a+4a^2-a^3.$$

We have noted that

$$p = 1-4a+4a^2-2a^3 > 0,$$
$$1-2a > 0,$$
$$1-a > 0,$$
$$1-3a+a^2 > 0,$$
$$2-3a+2a^2 > 0,$$

and

$$q = 2-4a+4a^2-a^3 > 0.$$

Proof of Theorem 4.1. Let

$$x = x(a) := (\alpha\beta)^{1/4}$$

and

$$y = y(a) := \{(1-\alpha)(1-\beta)\}^{1/4}.$$

From (4.5) and (4.6), we have

$$x = \frac{1-2a+\sqrt{p}}{2} \tag{4.11}$$

and

$$y = \frac{1-2a-\sqrt{p}}{2}. \tag{4.12}$$

From (4.11) and (4.12), it follows that

$$\dot{x}(a) = -\left[1 + \frac{2-4a+3a^2}{2\sqrt{p}}\right] \tag{4.13}$$

and

$$\dot{y}(a) = -\left[1 - \frac{2-4a+3a^2}{2\sqrt{p}}\right]. \tag{4.14}$$

From (4.12) and (4.13), we obtain

$$y\dot{x}(a) = \frac{-3a^2+2a^3+3a^2\sqrt{p}}{4\sqrt{p}}. \tag{4.15}$$

From (4.11) and (4.14), we obtain

$$x\dot{y}(a) = \frac{3a^2-2a^3+3a^2\sqrt{p}}{4\sqrt{p}}. \tag{4.16}$$

From (4.9), (4.15) and (4.16), we find that

$$\begin{aligned} &\alpha y\dot{x}(a) + (1-\alpha)x\dot{y}(a) = \frac{a^3}{4\sqrt{p}}\times \\ &\left((5-2a)(2-a)(2-3a+2a^2)\sqrt{p} - (3-2a)(2-a)(2-3a+a^2)\sqrt{2q}\right). \end{aligned} \tag{4.17}$$

From (4.10), (4.15) and (4.16), we find that

$$\begin{aligned} &\beta y\dot{x}(a) + (1-\beta)x\dot{y}(a) = \frac{a^3}{4\sqrt{p}}\times \\ &\left((5-2a)(2-a)(2-3a+2a^2)\sqrt{p} + (3-2a)(2-a)(2-3a+a^2)\sqrt{2q}\right). \end{aligned} \tag{4.18}$$

From [1, Entry 24(vi), p. 217]), we have

$$11\frac{d\alpha}{d\beta}=\frac{\alpha(1-\alpha)}{\beta(1-\beta)}m^2. \tag{4.19}$$

Using the definition of x and y in (4.19), we get

$$\frac{11}{m^2}=\frac{\alpha(1-\alpha)\frac{d\beta}{da}}{\beta(1-\beta)\frac{d\alpha}{da}}=-\frac{\alpha y\dot{x}(a)+(1-\alpha)x\dot{y}(a)}{\beta y\dot{x}(a)+(1-\beta)x\dot{y}(a)}. \tag{4.20}$$

Utilizing (4.17) and (4.18) in (4.20), we get

$$\frac{11}{m^2}=\frac{(5-2a)\sqrt{p}-(3-a)\sqrt{2q}}{11}.$$

Thus we have,

$$\frac{11}{m}=\pm((5-2a)\sqrt{p}-(3-a)\sqrt{2q}). \tag{4.21}$$

From (4.21), it follows that

$$m=\mp((5-2a)\sqrt{p}+(3-a)\sqrt{2q}). \tag{4.22}$$

Since $m>0$, (4.21) and (4.22) implies that

$$m=(5-2a)\sqrt{p}+(3-a)\sqrt{2q}, \tag{4.23}$$

and

$$\frac{11}{m}=(3-a)\sqrt{2q}-(5-2a)\sqrt{p}. \tag{4.24}$$

From (4.23) and (4.24), we have

$$m-\frac{11}{m}=2(5-2a)\sqrt{p}. \tag{4.25}$$

From (4.5) and (4.6), we find that

$$(\alpha\beta)^{\frac{1}{4}}-\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}=\sqrt{p}.$$

Using the above and the definition of a in (4.25), we obtain the required result. Similarly, we have deduced (4.25) for $\alpha+\beta\leq 1$. This completes the proof.

In [7], Vasuki and Yathirajsharma proved the above theorem by making use of the three identities of which one is found in Liu [3] and the other two are found in [8]. All these three identities can also be found in Cooper [2].

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