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ON THE DISTRIBUTION OF ZEROS OF BICOMPLEX POLYNOMIALS

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Abstract: Bicomplex numbers are a modern generalization of complex numbers in four-dimensional settings. In this study, we derive a region containing all zeros of a bicomplex polynomial. Furthermore, we provide some examples to validate the obtained results.

Keywords and Phrases: Bicomplex number, zero, polynomial.

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1. Introduction

The process of finding all the zeros of a higher-degree polynomial is much more difficult; therefore, it is desirable to find a region where the zeros will lie. This study in the field of complex numbers began a long time ago with the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra only gives information about the number of zeros in a polynomial but not about the location of the zeros. The problem of finding a region containing all the zeros of a polynomial in the field of complex numbers $\mathbb{C}$ have a rich old history [26]. In 1829, Cauchy [26] introduced the following classical result.

Theorem A. [26] *If $P(z) = \sum_{t=0}^{n} a_t z^t$ is a polynomial of degree n in $\mathbb{C}$ with complex coefficients, then all the zeros of $P(z)$ lie in $|z| \leq 1 + \max\limits_{0\leq t\leq (n-1)} |\frac{a_t}{a_n}|$.*

Several improvements of Theorem A by many researchers were found in [12, 15, 19, 25]. Joyal et al. [16] proved the following theorem as an improvement of Theorem A:

Theorem B. [16] *If* $P(z)=\sum_{t=0}^{n}a_tz^t$ $(a_n=1)$ *is a polynomial of degree* n *and* $\beta=\max\limits_{0\le t<n-1}|a_t|$, *then all the zeros of* $P(z)$ *lie in*

$$|z|\le\frac{1}{2}\left\{1+|a_{n-1}|+\sqrt{(1-|a_{n-1}|)^2+4\beta}\right\}.$$

This theorem does not improve Theorem A when $\beta=|a_{n-1}|$. Datt and Govil [11] proved the following result, which is an improvement of Theorem A even if $\beta=|a_{n-1}|$.

Theorem C. [11] *If* $P(z)=\sum_{t=0}^{n}a_tz^t$ $(a_n=1)$ *is a polynomial of degree* n *and if* $A=\max\limits_{0\le t\le n-1}|a_t|$, *then* $P(z)$ *has all its zeros in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}.(nA+1)}\le|z|\le1+\left(1-\frac{1}{(1+A)^n}\right)A.$$

Molla and Datta [24] introduced the subsequent theorem, enhancing both Theorems A and B and Theorem C in specific circumstances.

Theorem D. [24] *Let* $P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$ *be a polynomial of degree* $n>1$ *with* $M_1=\max\{|a_2-a_1|,|a_3-a_2|,...,|a_n-a_{n-1}|,\ |a_n|\}$ *and* $M_2=\max\{|a_0|,|a_1-a_0|,...,|a_{n-1}-a_{n-2}|\}$. *Then all the zeros of* $P(z)$ *are contained in the ring shaped region* $R_1\le|z|\le R_2$ *where*

$$R_1=\frac{2|a_0|}{|a_0|+|a_1-a_0|+\sqrt{(|a_0|-|a_1-a_0|)^2+4|a_0|M_1}},$$
$$R_2=\frac{1}{2|a_n|}\left\{|a_n|+|a_n-a_{n-1}|+\sqrt{(|a_n|-|a_n-a_{n-1}|)^2+4|a_n|.M_2}\right\}.$$

In a different manner, G. Enström and S. Kakeya [14] introduced the following result known as Enström-Kakeya theorem.

Theorem E. [14] *If* $P(z)=\sum_{t=0}^{n}a_tz^t$ *is a complex polynomial of degree* n *with real coefficients satisfying* $0\le a_0\le a_1\le...\le a_n$, *then all the zeros of* $P(z)$ *lie in* $|z|\le1$.

Many results on the generalization of Theorem E for complex polynomials can be found in [2-4, 8, 13, 14, 16, 17]. Although full-fledged extensions of Theorems

A and E are not available in bicomplex settings, a partial reflection of the same is seen in [10, 23].

The primary concern of this paper is to revisit Theorems A and E for bicomplex polynomials. Further, some examples are provided to validate the results obtained. We do not explain the standard definitions and notations in the theory of bicomplex analysis because they are available in [20, 21, 27].

2. Preliminary Definitions and Notations

To think of the extension of complex numbers in four-dimensional settings to solve various problems in physics that could not be solved in three dimensions, W.R. Hamilton introduced quaternions in 1844 [22] by considering three anticommuting imaginary units i, j & k such that $ij = k$. However, because of a lack of commutativity, many difficulties arise when one attempts to extend theories of the holomorphicity of one complex variable to the skew field of quaternions. Bicomplex numbers are compatible four-dimensional extensions of complex numbers and have been used in numerous fields such as digital image processing, geometry, and theoretical physics [1, 5, 29]. In 1892, Segre [27] defined set of bicomplex numbers as $\mathbb{C}_2 = \{z : z = z_1 + jz_2, z_1, z_2 \in \mathbb{C}\}$ where $ij = ji = k$ and $i^2 = j^2 = -k^2 = -1$.

Defining addition and multiplication on $\mathbb{C}_2$ in a manner similar to that on $\mathbb{C}$, it is observed that multiplication is commutative, associative, and distributive over addition and makes $\mathbb{C}_2$ a commutative algebra [27]. However, $\mathbb{C}_2$ is not a field because of the presence of zero-divisors [27], namely the set

$$\mathcal{O} = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}\}.$$

An element $z_1 + jz_2 \in \mathbb{C}_2$ is called non-singular if $z_1 + jz_2 \notin \mathcal{O}$ [22].

2.1. Idempotent Representation [27]

The bicomplex numbers $e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}$ are linearly independent in the $\mathbb{C}$- linear space $\mathbb{C}_2$ and $e_1 + e_2 = 1$, $e_1 - e_2 = ij$, $e_1.e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$. Any number $z = z_1 + jz_2 \in \mathbb{C}_2$ can be written uniquely as $z = \omega_1 e_1 + \omega_2 e_2$ where $\omega_1 = z_1 - iz_2$ & $\omega_2 = z_1 + iz_2$. This representation is called the idempotent representation of z.

2.2. Auxiliary Complex Spaces [20]

The complex spaces $\boldsymbol{A_1} = \{\omega_1 : \omega_1 = z_1 - iz_2,\ z_1,\ z_2 \in \mathbb{C}\}$ and $\boldsymbol{A_2} = \{\omega_2 : \omega_2 = z_1 + iz_2,\ z_1,\ z_2 \in \mathbb{C}\}$ are called auxiliary complex spaces. Spaces A_1 and A_2 contain the same elements as in $\mathbb{C}$. However, the convenient notations A_1 and A_2 are used for the special representation of elements. Each point $z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2$ associates the points $\omega_1 \in A_1$ and $\omega_2 \in A_2$. In addition, for each pair of points $(\omega_1, \omega_2) \in A_1 \times A_2$ there is a unique point in $\mathbb{C}_2$.

2.3. Cartesian Product [20]

$\mathbb{C}_2$-cartesian set determined by $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ is defined as follows:

$$X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2,\ (\omega_1, \omega_2) \in X_1 \times X_2\}.$$

2.4. Euclidean Norm [27]

The Euclidean norm function $\|\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$ denote the set of all non negative real numbers) is defined as follows:

If $z = z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2$, then

$$\|z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\omega_1|^2 + |\omega_2|^2}{2}\right\}^{\frac{1}{2}}.$$

This norm function has the following properties [27]:

If $a \in \mathbb{C},\ z = z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2$ & $w = w_1 + jw_2 \in \mathbb{C}_2$, then

i. $\|az\| = |a|.\|z\|$.

ii. $\|w + z\| \leq \|w\| + \|z\|$.

iii. $\|wz\| \leq \sqrt{2}\|w\|.\|z\|$.

iv.
$$\left\{ \begin{array}{l} |\omega_1| \\ |\omega_2| \end{array} \right. \leq \sqrt{2}\|z\| \leq |\omega_1| + |\omega_2|.$$

2.5. $\mathbb{C}_2$-Open Discus [20]

An open discus $D(a; r_1, r_2)$ with centre $a = a_1 e_1 + a_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined as

$$D(a; r_1, r_2) = B_1(a_1, r_1) \times_e B_1(a_2, r_2)$$

$$= \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - a_1| < r_1,\ |\omega_2 - a_2| < r_2\}$$

where $B_1(z, r)$ is an open ball with center $z \in \mathbb{C}$ and radius $r > 0$.

2.6. $\mathbb{C}_2$-Closed Discus [20]

A closed discus $\bar{D}(a; r_1, r_2)$ with centre $a = a_1 e_1 + a_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined by

$$\bar{D}(a; r_1, r_2) = \bar{B}_1(a_1, r_1) \times_e \bar{B}_1(a_2, r_2)$$

$$= \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - a_1| \leq r_1,\ |\omega_2 - a_2| \leq r_2\}$$

where $\bar{B}_1(z, r)$ is a closed ball with center $z \in \mathbb{C}$ and radius $r > 0$.

2.7. Zeros of a bicomplex polynomial

Let $P(z) = \sum_{t=0}^{n} a_t z^t$ be a bicomplex polynomial of degree n with $z = z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2$ and $a_t = \alpha_t e_1 + \beta_t e_2, t = 0, 1, 2, ..., n$. Therefore, $P(z)$ can be written as [28]

$$\begin{aligned} P(z) &= \sum_{t=0}^{n} \alpha_t \omega_1^t e_1 + \sum_{t=0}^{n} \beta_t \omega_2^t e_2 \\ &= \phi(\omega_1) e_1 + \psi(\omega_2) e_2. \end{aligned}$$

The set S of zeros of $P(z)$ is fully described by the sets S_1 & S_2 of distinct zeros of respective polynomials $\phi(\omega_1)$ & $\psi(\omega_2)$ i.e, $S = S_1 e_1 + S_2 e_2$ [28].

Example 2.1. Let $P(z) = z^3 - 1,\ z = z_1 + jz_2$.
The associated complex polynomials are $\phi(\omega_1) = \omega_1^3 - 1$ & $\psi(\omega_2) = \omega_2^3 - 1$.
The set of zeros of $\phi(\omega_1)$ & $\psi(\omega_2)$ are

$$\begin{aligned} S_1 &= \{\omega_1^1 = 1,\ \omega_1^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega_1^3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}, \\ S_2 &= \{\omega_2^1 = 1,\ \omega_2^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega_2^3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}. \end{aligned}$$

Hence, the set of 9 zeros of the bicomplex polynomial $P(z)$ of degree 3 is $S = \{\omega_1^t e_1 + \omega_2^t e_2 : t = 1,\ 2,\ 3\}$.

Bicomplex polynomials can exhibit unusual behavior, such as having no zeros within the bicomplex domain or having a finite or infinite number of zeros. To address this complexity, it is crucial to first state an analogue of the Fundamental Theorem of Algebra tailored for bicomplex polynomials [6].

Theorem 2.1. [6] *Consider a bicomplex polynomial* $P(z) = \sum_{t=0}^{n} a_t z^t$ *of degree* n *with* $a_t = \alpha_t e_1 + \beta_t e_2,\ t = 0,\ 1,\ 2, ..., n$. *If all the coefficients* a_j *with the exception of the free term* $a_0 = \alpha_0 e_1 + \beta_0 e_2$ *are complex multiples of* e_1 *(respectively* e_2*), but* a_0 *has* $\beta_0 \neq 0$ *(respectively* $\alpha_0 =\neq 0$*), then* $P(z)$ *has no roots. In all other cases,* $P(z)$ *has at least one root.*

Examining the distribution of zeros in bicomplex polynomials, especially compared to complex polynomials, poses unique challenges. Motivated by this, the study presents results that predict the distribution of these zeros.

3. Theorems

In this section, the main results of the study are presented.

Theorem 3.1. *Let* $P(z) = \sum_{t=0}^{n} (a_t + jb_t) z^t$ *be a bicomplex polynomial of degree* n *with* $z = \omega_1 e_1 + \omega_2 e_2$ *and at least one* $a_t + jb_t \notin \mathcal{O}$ *for* $t \geq 1$. *If* $|a_p - ib_p| \neq 0$,

$|a_q + ib_q| \neq 0$ for the greatest positive integers $p,\ q \leq n$, then all the zeros of $P(z)$ lie in the annular region
$\left\{z = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : \frac{|a_0 - ib_0|}{|a_0 - ib_0| + M_2} \leq |\omega_1| \leq 1 + \frac{M_1}{|a_p - ib_p|},\ \frac{|a_0 + ib_0|}{|a_0 + ib_0| + M_4} \leq |\omega_2| \leq 1 + \frac{M_3}{|a_q + ib_q|}\right\}$
where $M_1 = \max\limits_{0 \leq t \leq p-1}\{|a_t - ib_t|\},\ M_2 = \max\limits_{1 \leq t \leq p}\{|a_t - ib_t|\},\ M_3 = \max\limits_{0 \leq t \leq q-1}\{|a_t + ib_t|\},\ M_4 = \max\limits_{1 \leq t \leq q}\{|a_t + ib_t|\}$.

Proof. Taking $A_t = a_t - ib_t$ & $B_t = a_t + ib_t$, $P(z)$ can be written as

$$\begin{aligned}
P(z) &= \sum_{t=0}^{n}(A_t e_1 + B_t e_2)(\omega_1 e_1 + \omega_2 e_2)^t \\
&= \sum_{t=0}^{n}(A_t e_1 + B_t e_2)(\omega_1^t e_1 + \omega_2^t e_2) \\
&= \sum_{t=0}^{p} A_t \omega_1^t e_1 + \sum_{t=0}^{q} B_t \omega_2^t e_2 \\
&= \phi(\omega_1)e_1 + \psi(\omega_2)e_2 \\
&\text{where } \phi(\omega_1) = \sum_{t=0}^{p} A_t \omega_1^t \in \boldsymbol{A_1} \ \&\ \psi(\omega_2) = \sum_{t=0}^{q} B_t \omega_2^t \in \boldsymbol{A_2}.
\end{aligned}$$

Now, it follows for $|\omega_1| > 1$ that

$$\begin{aligned}
|\phi(\omega_1)| &\geq |A_p \omega_1^p| - |A_{p-1}\omega_1^{p-1} + A_{p-2}\omega_1^{p-2} + ... + A_0| \\
&\geq |A_p||\omega_1|^p - \left(|A_{p-1}||\omega_1|^{p-1} + |A_{p-2}||\omega_1|^{p-2} + ... + |A_0|\right) \\
&= |A_p||\omega_1|^p - |\omega_1|^{p-1}\left(|A_{p-1}| + \frac{|A_{p-2}|}{|\omega_1|} + ... + \frac{|A_0|}{|\omega_1|^{p-1}}\right) \\
&\geq |A_p||\omega_1|^p - |\omega_1|^{p-1}.M_1\left(1 + \frac{1}{|\omega_1|} + ... + \frac{1}{|\omega_1|^{p-1}}\right) \text{ where } M_1 = \max_{0 \leq t \leq p-1}\{|A_t|\} \\
&\geq |A_p||\omega_1|^p - |\omega_1|^{p-1}.M_1 \sum_{t=0}^{\infty}\frac{1}{|\omega_1|^t} \\
&= |A_p||\omega_1|^p - |\omega_1|^{p-1}.M_1.\frac{1}{1 - \frac{1}{|\omega_1|}} \\
&= |\omega_1|^p\left(|A_p| - \frac{M_1}{|\omega_1| - 1}\right)
\end{aligned}$$

Hence, we obtain for $|\omega_1| > 1$ that
$|\phi(\omega_1)| > 0$ if $|A_p| - \frac{M_1}{|\omega_1| - 1} > 0$

i.e, $|\phi(\omega_1)| > 0$ if $|\omega_1| > 1 + \frac{M_1}{|A_p|}$.
Therefore, no zeros of $\phi(\omega_1)$ in $|\omega_1| > 1$ in the auxiliary space $\boldsymbol{A_1}$ lie in

$$|\omega_1| > 1 + \frac{M_1}{|A_p|}.$$

Consequently, all the zeros of $\phi(\omega_1)$ in the auxiliary space $\boldsymbol{A_1}$ are contained in

$$|\omega_1| \leq 1 + \frac{M_1}{|A_p|}.$$

Again, let $\phi_1(\omega_1) = \omega_1^p \phi(\frac{1}{\omega_1}) = A_0\omega_1^p + A_1\omega_1^{p-1} + A_2\omega_1^{p-2} + ... + A_p$.
Then, we get for $|\omega_1| > 1$ that

$$\begin{aligned}
|\phi_1(\omega_1)| \geq & |A_0\omega_1^p| - |A_1\omega_1^{p-1} + A_2\omega_1^{p-2} + ... + A_p| \\
\geq & |A_0||\omega_1|^p - \left(|A_1||\omega_1|^{p-1} + |A_2||\omega_1|^{p-2} + ... + |A_p|\right) \\
= & |A_0||\omega_1|^p - |\omega_1|^{p-1}\left(|A_1| + \frac{|A_2|}{|\omega_1|} + ... + \frac{|A_p|}{|\omega_1|^{p-1}}\right) \\
\geq & |A_0||\omega_1|^p - |\omega_1|^{p-1}.M_2\left(1 + \frac{1}{|\omega_1|} + ... + \frac{1}{|\omega_1|^{p-1}}\right) \text{ where } M_2 = \max_{1\leq t\leq p}\{|A_t|\} \\
\geq & |A_0||\omega_1|^p - |\omega_1|^{p-1}.M_2 \sum_{t=0}^{\infty} \frac{1}{|\omega_1|^t} \\
= & |A_0||\omega_1|^p - |\omega_1|^{p-1}.M_2.\frac{1}{1 - \frac{1}{|\omega_1|}} \\
= & |\omega_1|^p\left(|A_0| - \frac{M_2}{|\omega_1| - 1}\right)
\end{aligned}$$

Hence, it follows for $|\omega_1| > 1$ that
$|\phi_1(\omega_1)| > 0$ if $|A_0| - \frac{M_2}{|\omega_1|-1} > 0$
i.e, $|\phi_1(\omega_1)| > 0$ if $|\omega_1| > 1 + \frac{M_2}{|A_0|}$
i.e, $|\phi(\frac{1}{\omega_1})| > 0$ if $|\omega_1| > 1 + \frac{M_2}{|A_0|}$.
Consequently, we obtain for $|\omega_1| < 1$ that $|\phi(\omega_1)| > 0$ if $|\omega_1| < \frac{1}{1+\frac{M_2}{|A_0|}}$.
Therefore, all the zeros of $\phi(\omega_1)$ in $|\omega_1| < 1$ in the auxiliary space $\boldsymbol{A_1}$ lie in

$$|\omega_1| \geq \frac{1}{1 + \frac{M_2}{|A_0|}}.$$

Finally, the annular region containing all the zeros of $\phi(\omega_1)$ in $\boldsymbol{A_1}$ is

$$\frac{1}{1+\frac{M_2}{|A_0|}} \leq |\omega_1| \leq 1+\frac{M_1}{|A_p|}.$$

Calculating in a similar fashion, it is observed that all the zeros of $\psi(\omega_2)$ in the auxiliary space $\boldsymbol{A_2}$ lie in

$$\frac{1}{1+\frac{M_4}{|B_0|}} \leq |\omega_2| \leq 1+\frac{M_3}{|B_q|}.$$

Thus the theorem is established.

Remark 3.1. *If all coefficients of the polynomial $P(z)$ are simultaneously multiples of e_1 or e_2, then either $\phi(\omega_1)$ or $\psi(\omega_2)$ vanishes identically. In this scenario, $P(z)$ has an uncountable number of zeros that are not isolated. Consequently, no bounded region containing all zeros of $P(z)$ can be identified.*

Remark 3.2. *The following example justifies the validity of Theorem 3.1.*

Example 3.1. Let $P(z) = 3(1+ij)z^3 + z^2 - 2(1-ij)z - 1,\ z = z_1 + jz_2$.
Then, we obtain by taking $z = \omega_1 e_1 + \omega_2 e_2$ that $\phi(\omega_1) = 6\omega_1^3 + \omega_1 - 1$ and $\psi(\omega_2) = \omega_2^2 - 4\omega_2 - 1$.
Here, $|a_0 - ib_0| = 1,\ |a_p - ib_p| = 6,\ |a_0 + ib_0| = 1,\ |a_q + ib_q| = 1,\ M_1 = 1,\ M_2 = 6,\ M_3 = 4\ \&\ M_4 = 4$.
Hence, by Theorem 3.1, the annular region containing all the zeros of $P(z)$ is

$$\left\{ z = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : \frac{1}{7} \leq |\omega_1| \leq 1 + \frac{1}{6},\ \frac{1}{5} \leq |\omega_2| \leq 5 \right\}.$$

Clearly, the zeros of $\phi(\omega_1)$ in $\boldsymbol{A_1}$ are $-\frac{1}{2},\ \pm\frac{1}{\sqrt{3}}$ and the zeros of $\psi(\omega_2)$ in $\boldsymbol{A_2}$ are $2 \pm \sqrt{5}$.
Hence, the six zeros of $P(z)$ in $\mathbb{C}_2$ are $-\frac{1}{2}e_1 + (2 \pm \sqrt{5})e_2,\ \pm\frac{1}{\sqrt{3}}e_1 + (2 \pm \sqrt{5})e_2$.

Remark 3.3. *For complex polynomials, Theorem 3.1 provides an improvement in the lower bound for the moduli of the zeros in Theorem A, as well as in both Theorem B, Theorem C, and Theorem D, even if $\beta = |a_{n-1}|$. Taking $P(z) = z^5 + 2z^4 - z^3 + 2z^2 - z + 1$, the sharpness of Theorem 3.1 is demonstrated in the following table.*

Sl. no.	*Theorem*	*Bounds for zeros of* $P(z)$
1	*Theorem 3.1*	$0.33 \leq \lvert z\rvert \leq 3$
2	*Theorem A*	$\lvert z\rvert \leq 3$
3	*Theorem B*	$\lvert z\rvert \leq 3$
4	*Theorem C*	$0.00056 \leq \lvert z\rvert \leq 2.99$
5	*Theorem D*	$0.303 \leq \lvert z\rvert \leq 2.73$

A bicomplex version of the Enström-Kakeya theorem is established in the next result.

Theorem 3.2. *If* $P(z) = \sum_{t=0}^{n} a_t z^t$ *is a bicomplex polynomial of degree* n *with real coefficients satisfying* $0 \leq a_1 \leq a_2 \leq ... \leq a_n$*, then all the zeros of* $P(z)$ *are contained in the closed discus* $\bar{D}(0;1,1)$.

Proof. Consider a bicomplex polynomial Q defined by the equation

$$\begin{aligned}(1-z)P(z) &= (1-z)\sum_{t=0}^{n} a_t z^t \\ &= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + ... + (a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= Q(z) - a_n z^{n+1}. \end{aligned} \tag{1}$$

Clearly, for $z = \omega_1 e_1 + \omega_2 e_2$,

$$\begin{aligned} Q(z) = &\{a_0 + (a_1 - a_0)\omega_1 + (a_2 - a_1)\omega_1^2 + ... + (a_n - a_{n-1})\omega_1^n\}e_1 + \\ &\{a_0 + (a_1 - a_0)\omega_2 + (a_2 - a_1)\omega_2^2 + ... + (a_n - a_{n-1})\omega_2^n\}e_2 \\ &= Q_1(\omega_1)e_1 + Q_2(\omega_2)e_2. \end{aligned} \tag{2}$$

Now, it follows for $|\omega_1| = 1$ that

$$\begin{aligned}\left|\omega_1^n Q_1\left(\frac{1}{\omega_1}\right)\right| &= |a_0\omega_1^n + (a_1 - a_0)\omega_1^{n-1} + ... + (a_{n-1} - a_{n-2})\omega_1 + (a_n - a_{n-1})| \\ &\leq |a_0||\omega_1|^n + |a_1 - a_0||\omega_1|^{n-1} + ... + |a_{n-1} - a_{n-2}||\omega_1| + |a_n - a_{n-1}| \\ &= a_0 + (a_1 - a_0) + ... + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\ &= a_n. \end{aligned}$$

Clearly, $\omega_1^n Q_1(\frac{1}{\omega_1})$ is holomorphic in $|\omega_1| \leq 1$; hence, considering the maximum modulus principle in $\mathbb{C}$,

$$\left|\omega_1^n Q_1\left(\frac{1}{\omega_1}\right)\right| \leq a_n \text{ for } |\omega_1| \leq 1$$

$$\text{i.e, } \left|Q_1\left(\frac{1}{\omega_1}\right)\right| \leq \frac{a_n}{|\omega_1|^n} \text{ for } |\omega_1| \leq 1$$

$$\text{i.e, } |Q_1(\omega_1)| \leq a_n|\omega_1|^n \text{ for } |\omega_1| \geq 1.$$

Similarly, we obtain for $|\omega_2| \geq 1$ that

$$|Q_2(\omega_2)| \leq a_n|\omega_2|^n.$$

From (1) & (2), we get that

$$\|(1-z)P(z)\| = \|\{Q_1(\omega_1) - a_n\omega_1^{n+1}\}e_1 + \{Q_2(\omega_2) - a_n\omega_2^{n+1}\}e_2\|.$$

Hence, in view of the property of the $\mathbb{C}_2$-norm, it follows for $|\omega_1| \geq 1$ that

$$\begin{aligned}\|(1-z)P(z)\| &\geq \frac{1}{\sqrt{2}}|Q_1(\omega_1) - a_n\omega_1^{n+1}| \\ &\geq \frac{1}{\sqrt{2}}\{a_n|\omega_1|^{n+1} - a_n|\omega_1|^n\} \\ &= \frac{1}{\sqrt{2}}a_n|\omega_1|^n(|\omega_1| - 1)\end{aligned}$$

and for $|\omega_2| \geq 1$,

$$\|(1-z)P(z)\| \geq \frac{1}{\sqrt{2}}a_n|\omega_2|^n(|\omega_2| - 1).$$

Thus, for $z = \omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2$ such that $|\omega_1| > 1$ & $|\omega_2| > 1$, $(1-z)P(z) \neq 0$.

Consequently, all the zeros of $P(z)$ lie in the closed discus $\bar{D}(0;1,1)$.

Remark 3.4. *The following example ensures the validity of Theorem 3.2.*

Example 3.2. Let $P(z) = 1 + 2z + 3z^2$, $z = z_1 + jz_2$.
All the zeros of $P(z)$ are $\left(\frac{-1\pm\sqrt{2}i}{3}\right)e_1 + \left(\frac{-1\pm\sqrt{2}i}{3}\right)e_2$.
Clearly, all the zeros of $P(z)$ are contained in the closed discus $\bar{D}(0;1,1)$.

3. Future prospect

In line with the work carried out in this paper, one may think of the extension of the results obtained by dealing with n-dimensional bicomplex numbers with the help of the idempotents $0, 1, \frac{1\pm i_1i_2}{2}, \frac{1\pm i_2i_3}{2}, \ldots, \frac{1\pm i_{n-1}i_n}{2}$ in $\mathbb{C}_n$. Consequently, the problem of taking the coefficients of the power series in $\mathbb{C}_n$ is still a virgin and may be considered an open problem for future researchers in this branch.

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