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ON SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING q-DERIVATIVE OPERATOR WITH NEGATIVE COEFFICIENTS

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Abstract: The purpose of this work is to introduce and study new subclasses of analytic functions using a new q-derivative operator. This operator generalizes the operators introduced by Al-Oboudi, Catas, Cho and Kim, Cho and Srivastava, Maslina Darus and R W Ibrahim, Sălăgean, Uralegaddi and Somanatha. We investigate coefficient bounds, growth, distortion and closure theorems for the functions belonging to these classes. We also give a result which unifies radii of close-to-convexity, starlikeness and convexity.

Keywords and Phrases: q-derivative operator, coefficient bounds, growth, distortion and closure theorems.

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1. Introduction

We begin by denoting by $\mathcal{S}$ the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$.

First we briefly recall the notation of q-derivative operator which plays vital role in the theory of hypergeometric series, quantum physics and in the operator theory. Many known researchers and applied scientists have made valuable contributions in the modifications of the q-calculus and generalizations. In recent years extensions and generalization of q-calculus have witnessed a significant evolution, due to large range of application many researchers have explored q-calculus in depth. The first application and usage of the q-calculus was introduced by Jackson [12], [13].
For a function $f(z) \in \mathcal{S}$ the Jackson's q-derivative is defined as [11]

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0, 0 < q < 1). \tag{1.2}$$

From equation (1.2) it is clear that if $f(z)$ and $g(z)$ are two functions, then

$$\mathcal{D}_q(f(z) + g(z)) = \mathcal{D}_q f(z) + \mathcal{D}_q g(z). \tag{1.3}$$

$$\mathcal{D}_q(cf(z)) = c\mathcal{D}_q f(z). \tag{1.4}$$

We note that $\mathcal{D}_q f(z) \to f'(z)$ as $q \to 1^-$, where $f'(z)$ is the ordinary derivative of the function $f(z)$. Further by (1.2) the q-derivative of the function $h(z) = z^k$ is as follows:

$$\mathcal{D}_q h(z) = [k]_q z^{k-1} \tag{1.5}$$

where $[k]_q$ is given as:

$$[k]_q = \frac{1-q^k}{1-q} \qquad (0 < q < 1). \tag{1.6}$$

Note that $[k]_q \to k$ as $q \to 1^-$, therefore in view of equation (1.5), $\mathcal{D}_q h(z) = h'(z)$ as $q \to 1^-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z.
The q-derivative of the function $f(z)$, given by equation (1.1) is defined as

$$\mathcal{D}_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \qquad (0 < q < 1) \tag{1.7}$$

where $[k]_q$ is given by (1.6).
Motivated by the earlier investigations [1, 2, 14, 15, 21] and with the aid of the Jackson q-derivative, we define the new operator $\mathcal{D}^n_{\delta,\lambda,l,q} f(z)$. For $f(z) \in \mathcal{S}$, $0 < q < 1, \delta, \lambda, l \geq 0, n \in N_0 = N \cup \{0\}$.

$\mathcal{D}^0_{\delta,\lambda,l,q}f(z) = f(z).$
$\mathcal{D}^1_{\delta,\lambda,l,q}f(z) = (l+\delta-\lambda)f(z) + (1-\delta+\lambda)zD_qf(z) = \mathcal{D}_{\delta,\lambda,l,q}f(z).$
.
.
.
$\mathcal{D}^n_{\delta,\lambda,l,q}f(z) = \mathcal{D}_{\delta,\lambda,l,q}(\mathcal{D}^{n-1}_{\delta,\lambda,l,q}f(z)).$

$$\mathcal{D}^n_{\delta,\lambda,l,q}f(z) = z + \sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k z^k. \tag{1.8}$$

Note that as $q \to 1^-$ we obtain the differential operator studied by Latha and Shilpa in [16]. As $q \to 1^-$ and for suitable choices of parameters $\mathcal{D}^n_{\delta,\lambda,l,q}f(z)$ reduces to various operators studied by many authors Al-oboudi [3], Catas [6], Cho and Kim [7], Cho and Srivastava [8], Maslina Darus and Rabha W Ibrahim [9], S*ă*l*ă*gean [19], Uralegaddi and Somanatha [22].

Making use of the new q-derivative operator $\mathcal{D}^n_{\delta,\lambda,l,q}f(z)$ we introduce a new subclass of analytic functions as follows.

Definition 1.1. *A function $f(z) \in \mathcal{S}$ is said to be in the class $\mathcal{L}^n(\delta,\lambda,l,q,\alpha,\beta,\gamma)$ if it satisfies the following condition*

$$\Re\left\{\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))} - \alpha\right\} \tag{1.9}$$

$$> \beta\left|\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))} - 1\right|$$

where $0 < q < 1, -1 \le \alpha \le 1, \beta \ge 0,\ 0 \le \gamma < 1,\ z \in \mathcal{U}$.

Let $\mathcal{T} \subset S$ denote the family of functions defined in the open unit disc $\mathcal{U}$, introduced and studied by Silverman [20] which are of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \ge 0. \tag{1.10}$$

We further let $\mathcal{TL}^n(\delta,\lambda,l,q,\alpha,\beta,\gamma) = \mathcal{L}^n(\delta,\lambda,l,q,\alpha,\beta,\gamma) \cap \mathcal{T}$.
By specializing the parameters we can obtain several subclasses studied in [3], [4], [5], [10] and [18].

2. Main Results

We assume that $-1 \leq \alpha < 1, \quad \beta \geq 0, \quad 0 < q < 1, \quad n \in N_0, \quad 0 \leq \gamma < 1, \quad f(z) \in \mathcal{T}$ and $z \in \mathcal{U}$

Theorem 2.1. *A function* $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k \leq 1-\alpha. \tag{2.1}$$

Proof. First we assume that (2.1) holds. Then it is sufficient to show that

$$\beta\left|\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))}-1\right|$$

$$-\Re\left\{\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))}-1\right\} \leq 1-\alpha.$$

Note that

$$\beta\left|\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))}-1\right|$$

$$-\Re\left\{\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))}-1\right\}$$

$$\leq (1+\beta)\left|\frac{zD_q(D^n_{\delta,\lambda,l,q}f(z))}{(1-\gamma)D^n_{\delta,\lambda,l,q}f(z)+\gamma zD_q(D^n_{\delta,\lambda,l,q}f(z))}-1\right|$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}([k]_q-1)(1-\gamma)\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k}{1-\sum_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k}.$$

The last expression is bounded above by $(1-\alpha)$ since (2.1) holds.
Conversely if $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ and z is real, then

$$\frac{1-\sum_{k=2}^{\infty}[k]_q\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k z^{k-1}}-\alpha$$

$$\geq \beta \left| \frac{\sum_{k=2}^{\infty}([k]_q - 1)(1-\gamma)\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1+\gamma([k]_q-1)]\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n a_k z^{k-1}} \right|.$$

Letting $z \to 1^-$ along the real axis, we get the desired inequality (2.1). Which completes the proof.

Corollary 2.2. *For $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$, we have*

$$a_k \leq \frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}, (k \geq 2).$$

and

$$f(z) = z - \frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q-1)]\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n} z^k, (k \geq 2) \tag{2.2}$$

gives the sharpness.

3. Growth and Distortion Theorems

Theorem 3.1. *Let $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$. Then for $0 \leq i \leq n$,*

$$\left|D^i_{\delta,\lambda,l,q} f(z)\right| \geq |z| - \frac{1-\alpha}{\{[2]_q(1+\beta) - (\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}|z|^2 \tag{3.1}$$

and

$$\left|D^i_{\delta,\lambda,l,q} f(z)\right| \leq |z| + \frac{1-\alpha}{\{[2]_q(1+\beta) - (\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}|z|^2. \tag{3.2}$$

The equalities in (3.1) *and* (3.2) *are attained for*

$$f(z) = z - \frac{1-\alpha}{\{[2]_q(1+\beta) - (\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}} z^2. \tag{3.3}$$

Proof. We have $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ if and only if $D^i_{\delta,\lambda,l,q} f(z) \in \mathcal{TL}^{n-i}(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ where

$$D^i_{\delta,\lambda,l,q} f(z) = z - \sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^i a_k z^k \tag{3.4}$$

By Theorem 2.1, we have

$$\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\left[1+\left(\frac{1-\delta+\lambda}{l+1}\right)q\right]^{n-i}\sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^{i}a_k \tag{3.5}$$

$$\leq 1-\alpha$$

that is,

$$\sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^{i}a_k \tag{3.6}$$

$$\leq \frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}.$$

From (3.4) and (3.6) it follows that

$$|D^{i}_{\delta,\lambda,l,q}f(z)| \geq |z|-|z|^2\sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^{i}a_k \tag{3.7}$$

$$\geq |z|-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}|z|^2$$

and

$$|D^{i}_{\delta,\lambda,l,q}f(z)| \leq |z|+|z|^2\sum_{k=2}^{\infty}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^{i}a_k \tag{3.8}$$

$$\leq |z|+\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}|z|^2.$$

note that the bounds in (3.1) and (3.2) are attained for $f(z)$ defined by

$$D^{i}_{\delta,\lambda,l,q}f(z) = z-\frac{1-\alpha}{\{[2]_q(1+\beta)-(\alpha+\beta)(1+\gamma q)\}\left[1+(\frac{1-\delta+\lambda}{l+1})q\right]^{n-i}}z^2, (z\in\mathcal{U}). \tag{3.9}$$

Hence the proof is completed.

4. Closure Theorems

Let $f_i(z)$ be defined, for $i=1,2,3,..,m$, by

$$f_i(z) = z-\sum_{k=2}^{\infty}a_{k,i}z^k \quad (a_{k,i}\geq 0, z\in\mathcal{U}). \tag{4.1}$$

Theorem 4.1. *Let* $f_i(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ *for* $i = 1, 2, 3, ..., m$. *Then*

$$g(z) = \sum_{i=1}^{m} c_i f_i(z), \tag{4.2}$$

is also in the same class, where $c_i \geq 0$, $\sum_{i=1}^{m} c_i = 1$

Proof. Using (4.2), we have

$$g(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} c_i a_{k,i} \right) z^k. \tag{4.3}$$

Further, since $f_i(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$, we get

$$\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)(1+\gamma([k]_q - 1))\} \left[\frac{(l+\delta-\lambda) + (1-\delta+\lambda)[k]_q}{l+1} \right]^n a_{k,i} \leq 1 - \alpha. \tag{4.4}$$

Hence

$$\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)(1+\gamma([k]_q - 1))\} \left[\frac{(l+\delta-\lambda) + (1-\delta+\lambda)[k]_q}{l+1} \right]^n \left(\sum_{i=1}^{m} c_i a_{k,i} \right)$$

$$= \sum_{i=1}^{m} c_i \left[\sum_{k=2}^{\infty} \{[k]_q(1+\beta) - (\alpha+\beta)(1+\gamma(k_q - 1))\} \left[\frac{(l+\delta-\lambda) + (1-\delta+\lambda)[k]_q}{l+1} \right]^n a_{k,i} \right]$$

$$\leq \left(\sum_{i=1}^{m} c_i \right) (1-\alpha) = 1 - \alpha, \tag{4.5}$$

which implies that $g(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$. This completes the proof.

Theorem 4.2. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1} \right]^n} z^k, (k \geq 2) \tag{4.6}$$

then $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$ *if and only if*

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.7}$$

where $\mu_k \geq 0 (k \geq 1)$ *and* $\sum_{k=1}^{\infty} \mu_k = 1.$

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{4.8}$$

$$= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n} z^k$$

then it follows that

$$\sum_{k=2}^{\infty} \frac{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{1-\alpha}.$$

$$\frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n} \mu_k$$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \tag{4.9}$$

By Theorem 2.1, $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$.
Conversely, assume that $f(z) \in \mathcal{TL}^n(\delta, \lambda, l, q, \alpha, \beta, \gamma)$. Then

$$a_k \leq \frac{1-\alpha}{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n} (k \geq 2). \tag{4.10}$$

Setting

$$\mu_k = \frac{\{[k]_q(1+\beta) - (\alpha+\beta)[1+\gamma([k]_q - 1)]\} \left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{1-\alpha} (k \geq 2). \tag{4.11}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.12}$$

note that $f(z)$ can be expressed in the form (4.7). Hence the proof.

5. Some radii for the class $\mathcal{TL}^n(\delta,\lambda,l,q,\alpha,\beta,\gamma)$

Theorem 5.1. *Let $f(z)\in\mathcal{L}^n(\delta,\lambda,l,q,\alpha,\beta,\gamma)$. Then for $0\le\rho<1, k\ge 2$, $f(z)$ is (i). Close -to-convex of order ρ in $|z|<r_1$, where*

$$r_1=r_1(\delta,\lambda,l,q,\alpha,\beta,\gamma,\rho): \tag{5.1}$$

$$=inf_k\left[\frac{(1-\rho)\left\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\right\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{k(1-\alpha)}\right]^{\frac{1}{k-1}}.$$

(ii). Starlike of order ρ in $|z|<r_2$, where

$$r_2=r_2(\delta,\lambda,l,q,\alpha,\beta,\gamma,\rho): \tag{5.2}$$

$$=inf_k\left[\frac{(1-\rho)\left\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\right\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}.$$

(iii). Convex of order ρ in $|z|<r_3$, where

$$r_3=r_3(\delta,\lambda,l,q,\alpha,\beta,\gamma,\rho): \tag{5.3}$$

$$=inf_k\left[\frac{(1-\rho)\left\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\right\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}.$$

The result is sharp, for $f(z)$ given by (2.2).

Proof. To prove (i) we must show that

$$|f'(z)-1|\le 1-\rho$$

for $|z|<r_1(\delta,\lambda,l,q,\alpha,\beta,\gamma,\rho)$

From (1.10), we have $|f'(z)-1|\le\sum_{k=2}^{\infty}ka_k|z|^{k-1}$.

Thus $|f'(z)-1|\le 1-\rho$, if

$$\sum_{k=2}^{\infty}\frac{k}{1-\rho}a_k|z|^{k-1}\le 1. \tag{5.4}$$

But, by Theorem 2.1, (5.4) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{1-\alpha}$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left\{[k]_q(1+\beta)-(\alpha+\beta)[1+\gamma([k]_q-1)]\right\}\left[\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[k]_q}{l+1}\right]^n}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \tag{5.5}$$

which gives (5.1). To prove (ii) and (iii) we have to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \tag{5.6}$$

for $|z| < r_2$,

$$\left|\frac{zf''(z)}{f'(z)}-1\right| \leq 1-\rho$$

for $|z| < r_3$, respectively, by using arguments as in (i).

Remark 5.2. *By taking $l = 0, \quad \alpha = 1$ and as $q \to 1^-$ we obtain the results in* [17].

6. Conclusion

q-calculus has significant importance and applications in various fields of Science and Engineering. In this paper we introduce and study new subclasses of analytic functions using q-derivative operator. We find the coefficient bounds, growth, distortion and closure theorems for the functions belonging to these classes. Also we obtain a result which unifies radii of close-to-convexity, starlikeness and convexity. The results obtained from this study will enrich the theoretical foundation of this field and open new avenues for mathematical inquiry and applications. The results would generalize and improve the earlier results by several authors.

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