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ON THE LANCZOS ORTHOGONAL DERIVATIVE

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Abstract: We use the Kempf et al. (2014 & 2015) process of integration by differentiation to obtain the Lanczos generalized derivative, and we give a simple deduction of the Rangarajan–Purushothaman's formula for the orthogonal derivative for higher orders. Besides, we show that the Lanczos derivative allows deduce an interesting algebraic expression for the first derivative of a function.

Keywords and Phrases: Differentiation by integration, Integration by differentiation, Lanczos derivative, Legendre polynomials, Orthogonal derivative.

2020 Mathematics Subject Classification: 26A24, 26A42, 33C45, 65D25.

1. Introduction

Kempf et al. [8, 9] show how to obtain a definite integral via differentiation, in fact, they find the interesting expression:

$$\int_a^b F(x)dx = \lim_{t\to 0} F\left(\frac{d}{dt}\right)\left[\frac{e^b t - e^a t}{t}\right] \tag{1}$$

Here we give an elementary proof of (1), and we use it to deduce the Cioranescu-(Haslam-Jones)- Lanczos generalized derivative [2, 3, 5-7, 10, 11]:

$$f'(x_0) = \lim_{\varepsilon\to 0} \frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} f(v+x_0)\, v\, dv, \tag{2}$$

which represents differentiation via integration [3, 5, 7, 11, 12]. This Lanczos derivative for higher orders was studied by the several authors [4, 13, 15] via Legendre polynomials, here we show a simple deduction of their corresponding formula. Besides, we exhibit that (2) gives an algebraic expression to determine $f'(x_0)$.

2. Expression of Kempf et al.

We have

$$\begin{aligned}\int_a^b x^n dx &= \frac{1}{(n+1)}(b^{n+1} - a^{n+1}) \\ &= \left[\frac{d^n}{dt^n}\sum_{r=0}^{\infty}\frac{b^{r+1}-a^{r+1}}{(r+1)!}t^r\right]_{t=0} \\ &= \lim_{\varepsilon\to 0}\left[\frac{d^n}{dt^n}\frac{1}{t}\sum_{k=1}^{\infty}\frac{b^k-a^k}{k!}t^k\right] \\ &= \lim_{\varepsilon\to 0}\frac{d^n}{dt^n}\frac{e^b t - e^a t}{t}.\end{aligned}$$

Then

$$\begin{aligned}\int_a^b F(x)dx &= \sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\int_a^b x^n dx \\ &= \lim_{t\to 0}\sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\frac{d^n}{dt^n}\frac{e^b t - e^a t}{t}.\end{aligned}$$

Hence (1) is immediate.

Now, we apply (1) for the case $F(x) = xf(x + x_0)$ with $a = -b = -\varepsilon$ and therefore, we have

$$\frac{e^{bt} - e^{at}}{t} = 2\varepsilon\left(1 + \frac{\varepsilon^2}{3!}t^2 + \frac{\varepsilon^4}{5!}t^4 + \cdots\right), \quad F(x) = f(x_0)x + f'(x_0)x^2 + \frac{1}{2!}f''(x_0)x^3 + \cdots .$$

Thus from (1), we get

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} xf(x + x_0)dx &= 2\varepsilon \lim_{t\to 0}\left[f(x_0)\frac{d}{dt} + f'(x_0)\frac{d^2}{dt^2} + \frac{1}{2!}f''(x_0)\frac{d^3}{dt^3} + \cdots\right] \\ &\quad \times \left(1 + \frac{\varepsilon^2}{3!}t^2 + \frac{\varepsilon^4}{5!}t^4 + \cdots\right), \\ &= 2\varepsilon^3\left[\frac{1}{3}f'(x_0) + \frac{\varepsilon^2}{5\cdot 3!}f'''(x_0) + \frac{\varepsilon^4}{7\cdot 5!}f^{v}(x_0) + \cdots\right].\end{aligned}$$

Then it is evident the expression (2):

$$f'(x_0) = \lim_{\varepsilon\to 0}\frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} xf(x + x_0)\,dx,$$

which coincides with the Lanczos orthogonal derivative [2, 3, 5-7, 10, 11]. The relation (1) represents integration by differentiation, but (2) expresses the inverse process, that is, differentiation by integration.

3. Generalized Derivative for Higher Orders

Now we consider the integral

$$\int_{-\varepsilon}^{\varepsilon} Q_n\left(\frac{t}{\varepsilon}\right) f(x + t)dt = \varepsilon\int_{-1}^{1} Q_n(u)f(x + \varepsilon u)du. \tag{3}$$

The Taylor expansion allows us to write the following:

$$f(x + \varepsilon u) = f(x) + \varepsilon f'(x)u + \cdots + \frac{\varepsilon^n}{n!}f^{(n)}(x)u^n + \frac{\varepsilon^{n+1}}{(n+1)!}f^{(n+1)}(x)u^{n+1} + \cdots ,$$

Then from (3), we have

$$\begin{aligned}\frac{1}{\varepsilon^{n+1}}\int_{-\varepsilon}^{\varepsilon} Q_n\left(\frac{t}{\varepsilon}\right) f(x + t)dt =& \sum_{k=0}^{n-1}\frac{\varepsilon^{k-n}}{k!}f^{(k)}(x)\int_{-1}^{1} u^k Q_n(u)du \\ &+ \frac{1}{n!}f^{(n)}(x)\int_{-1}^{1} u^n Q_n(u)du \\ &+ \sum_{j=n+1}^{\infty}\frac{\varepsilon^{j-n}}{j!}f^{(j)}(x)\int_{-1}^{1} u^j Q_n(u)du,\end{aligned} \tag{4}$$

which suggest selecting $Q_n(u)$ with the property $\int_{-1}^{1} u^k Q_n(u)du = 0$ for $k \leq n-1$, and it is evident that the Legendre polynomials satisfy the following identities [1, 13, 20]:

$$\int_{-1}^{1} u^k P_n(u)du = 0, \;\; k = 0, 1, 2, \cdots, n-1; \quad \int_{-1}^{1} u^n P_n(u)du = \frac{2(n!)}{(2n+1)!} \tag{5}$$

Thus, from (4) and (5), we obtain the celebrated formula of Rangarajan-Purushothaman [15] (See [4, 13]):

$$f^{(n)}(x) = \lim_{\varepsilon \to 0} \frac{(2n+1)!}{2\varepsilon^{n+1}} \int_{-\varepsilon}^{\varepsilon} P_n\left(\frac{t}{\varepsilon}\right) f(x+t)\,dt, \tag{6}$$

which reproduces (2) for $n = 1$ because $P_1\left(\frac{t}{\varepsilon}\right) = \frac{t}{\varepsilon}$.

4. Algebraic Calculation of $f'(x_0)$

Now we shall employ (2) to deduce an algebraic expression for $f'(x_0)$, where we shall accept that ε is very small. Thus,

$$f'(x_0) \sim \frac{3}{2\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t\, f(x_0+t)dt = \frac{3n}{2} \int_0^1 x \left[f\left(x_0 + \frac{x}{n}\right) - f\left(x_0 - \frac{x}{n}\right)\right] dx,$$
$$\varepsilon = \frac{1}{7}, \tag{7}$$

where n is very large. On the other hand, we know that (See [14, 16])

$$\int_0^1 F(x)dx \sim \frac{1}{\varphi(n)} \sum_{1 \leq k \leq n,\; (k,n)=1} F\left(\frac{k}{n}\right), \qquad n >> 1, \tag{8}$$

involving the Euler totient function $\varphi(n)$ [17-19], whose application in (7) implies the following interesting expression:

$$f'(x_0) \sim \frac{3}{2\varphi(n)} \sum_{1 \leq k \leq n,\; (k,n)=1} k \left[f\left(x_0 + \frac{k}{n^2}\right) - f\left(x_0 - \frac{k}{n^2}\right)\right] \tag{9}$$

If f is an odd function and $x_0 = 0$, then from (9) we have

$$f'(0) \sim \frac{3}{\varphi(n)} \sum_{1 \leq k \leq n,\; (k,n)=1} k\, f\left(\frac{k}{n^2}\right) \tag{10}$$

For example, if $f(x) = x\cos 2x$ and $f(x) = x\log(x^2+20)$, then $f'(0) = 1$ and $f'(0) = 2.9957$, respectively. Thus, from (10) we obtain that $f'(0) \sim 0.9672$ and $f'(0) \sim 2.9493$ for $n = 8$, respectively. If $f(x) = x\log(x+1)$, then $f'(1) = 1.1931$ and (9) gives $f'(1) \sim 1.1385$ for $n = 11$.
Finally, if $n = p$, a prime number, then (9) takes the form

$$f'(x_0) \sim \frac{3}{2(p-1)} \sum_{k=1}^{p-1} k \left[f\left(x_0 + \frac{k}{p^2}\right) - f\left(x_0 - \frac{k}{p^2}\right)\right] \tag{11}$$

For example, if $f(x) = x\sin x$, then $f'(2) = 0.0770$ and (11) gives $f'(2) \sim 0.0721$ for $p = 11$. Similarly, if $f(x) = x^2 e^x$, then $f'(0.5) = 2.0609$ and (11) implies $f'(0.5) \sim 2.0187$ for $p = 23$.

It is clear that for sufficiently large values of n, relation (9) will give excellent approximations for the first derivative of f at x_0.

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