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NOTES ON EXTERNAL DIRECT PRODUCTS OF DUAL IUP-ALGEBRAS

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Abstract: The concept of the direct product of a finite family of B-algebras is introduced by Lingcong and Endam [15]. In this paper, we introduce the concept of the direct product of an infinite family of IUP-algebras and prove that it is a DIUP-algebra; we call the external direct product DIUP-algebra induced by IUP-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam. We find the result of the external direct product of special subsets of IUP-algebras. Also, we introduce the concept of the weak direct product

DIUP-algebras. Finally, we provide several fundamental theorems of (anti-)IUP-homomorphisms in view of the external direct product DIUP-algebras.

Keywords and Phrases: IUP-algebra, DIUP-algebra, external direct product, weak direct product, IUP-homomorphism, anti-IUP-homomorphism.

2020 Mathematics Subject Classification: Primary 03G25; Secondary 20K25.

1. Introduction and Preliminaries

Imai and Iséki [12] first commenced the study of BCK-algebras in 1966. In that same year, Iséki [13] introduced another class of algebras, called BCI-algebras, which are generalizations of BCK-algebras. Both types of abstract algebras have been actively studied by a large number of academics.

Recently, Iampan et al. [11] introduced a new algebraic structure called an independent UP-algebras (in short, IUP-algebras), which is independent of UP-algebras. Then, they introduced the concepts of IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals of IUP-algebras and investigated their properties and relationships. In addition, they also discussed the concept of homomorphisms between IUP-algebras and studied the direct and inverse images of four special subsets.

The concept of the direct product has been discussed and obtained some properties. Then, the notion of direct product are applied to B-algebras by Lingcong and Endam in 2016 [15, 16], BF-algebras by Endam and Teves in 2016 [10], BRK-algebras by Abebe in 2018 [1], BG-algebras by Widianto et al. in 2019 [18], BP-algebras by Setiani et al. in 2020 [17], GK-algebra by Kavitha and Gowri in 2021 [14]. Furthermore, in 2022, Chanmanee et al. introduced the concept of the direct product of an infinite family of B-algebras [5], BG-algebras [2], IUP-algebras [9], and BP-algebras [8], respectively, they called them the external direct product. Also, they introduced the notion of the direct product of an infinite family of UP-algebras and proved that it was a DUP-algebras (dual UP-algebra); they called the external direct product DUP-algebra induced by UP-algebras [6] and applied the concept of the internal direct product of a groupoid to a UP-algebra [7, 4]. In 2023, Chanmanee et al. [3] introduced the concept of the direct product of an infinite family of UP (BCC)-algebras; they called the external direct product and found the result of the external direct product of special subsets of UP (BCC)-algebras. Also, they introduced the concept of the weak direct product of UP (BCC)-algebras. Finally, they provided several fundamental theorems of (anti-)UP (BCC)-homomorphisms in view of the external direct product UP (BCC)-algebras.

In this paper, we introduce the concept of the direct product of an infinite family of IUP-algebras and prove that it is a DIUP-algebras; we call the external direct product DIUP-algebra induced by IUP-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam [15]. Moreover, we introduce the concept of the weak direct product DIUP-algebras. Finally, we discuss several (anti-)IUP-homomorphism theorems in view of the external direct product DIUP-algebras.

First of all, we start with the definitions and examples of IUP-algebras as well as other relevant definitions for the study in this paper, as follows:

Definition 1.1. [11] *An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $*$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 * x = x), \tag{IUP-1}$$
$$(\forall x \in X)(x * x = 0), \tag{IUP-2}$$
$$(\forall x, y, z \in X)((x * y) * (x * z) = y * z). \tag{IUP-3}$$

Example 1.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	3	1	5	2
2	2	5	0	4	3	1
3	5	4	1	0	2	3
4	1	3	5	2	0	4
5	3	2	4	5	1	0

Then $X = (X; *, 0)$ is an IUP-algebra.

Definition 1.3. *An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a dual IUP-algebra (DIUP-algebra) if it satisfies (IUP-2) and the following axioms:*

$$(\forall x \in X)(x * 0 = x), \tag{DIUP-1}$$
$$(\forall x, y, z \in X)((z * x) * (y * x)) = z * y). \tag{DIUP-2}$$

The binary relation $\leq$ on a DIUP-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Example 1.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	5	3	2	4	1
1	1	0	2	4	3	5
2	2	3	0	1	5	4
3	3	4	5	0	1	2
4	4	2	1	5	0	3
5	5	1	4	3	2	0

Then $X = (X; *, 0)$ is a DIUP-algebra.

In an IUP-algebra $X = (X; *, 0)$, the following assertions are valid (see [11]).

$$(\forall x, y \in X)((x * 0) * (x * y) = y), \tag{1.1}$$
$$(\forall x \in X)((x * 0) * (x * 0) = 0), \tag{1.2}$$
$$(\forall x, y \in X)((x * y) * 0 = y * x), \tag{1.3}$$
$$(\forall x \in X)((x * 0) * 0 = x), \tag{1.4}$$
$$(\forall x, y \in X)(x * ((x * 0) * y) = y), \tag{1.5}$$
$$(\forall x, y \in X)(((x * 0) * y) * x = y * 0), \tag{1.6}$$
$$(\forall x, y, z \in X)(x * y = x * z \Leftrightarrow y = z), \tag{1.7}$$
$$(\forall x, y \in X)(x * y = 0 \Leftrightarrow x = y), \tag{1.8}$$
$$(\forall x \in X)(x * 0 = 0 \Leftrightarrow x = 0), \tag{1.9}$$
$$(\forall x, y, z \in X)(y * x = z * x \Leftrightarrow y = z), \tag{1.10}$$
$$(\forall x, y \in X)(x * y = y \Rightarrow x = 0), \tag{1.11}$$
$$(\forall x, y, z \in X)((x * y) * 0 = (z * y) * (z * x)), \tag{1.12}$$
$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (z * x) * (z * y) = 0), \tag{1.13}$$
$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (x * z) * (y * z) = 0), \tag{1.14}$$

the right and the left cancellation laws hold. (1.15)

According to [11], the binary relation $\leq$ on an IUP-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Definition 1.5. [11] *A nonempty subset S of an IUP-algebra $X = (X; *, 0)$ is called*

(*i*) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x * y \in S), \tag{1.16}$$

(*ii*) *an IUP-filter of X if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.17}$$

$$(\forall x, y \in S)(x * y \in S, x \in S \Rightarrow y \in S), \tag{1.18}$$

(*iii*) *an IUP-ideal of X if it satisfies the condition* (1.17) *and the following condition:*

$$(\forall x, y, z \in S)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{1.19}$$

(*iv*) *a strong IUP-ideal of X if it satisfies the following condition:*

$$(\forall x, y \in S)(y \in S \Rightarrow x * y \in S). \tag{1.20}$$

From [11], we know that the concept of IUP-filters is a generalization of IUP-ideals and IUP-subalgebras, and IUP-ideals and IUP-subalgebras are generalizations of strong IUP-ideals. In an IUP-algebra X, we have that strong IUP-ideals. We get the diagram of the special subsets of IUP-algebras, which is shown in Figure 1.

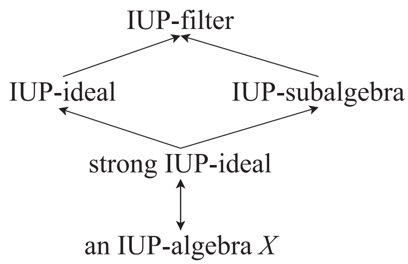

Figure 1: Special subsets of IUP-algebras

The concept of IUP-homomorphisms was introduced by Iampan et al. in [11]. Let $X_1 = (X_1; *_1, 0_1)$ and $X_2 = (X_2; *_2, 0_2)$ be IUP-algebras. A map $\psi : X_1 \to X_2$ is called an *IUP-homomorphism* if

$$(\forall x, y \in X_1)(\psi(x *_1 y) = \psi(x) *_2 \psi(y))$$

and an *anti-IUP-homomorphism* if

$$(\forall x, y \in X_1)(\psi(x *_1 y) = \psi(y) *_2 \psi(x)).$$

The *kernel* of ψ denoted by $\ker \psi$ is defined to be the set $\ker \psi = \{x \in X_1 \mid \psi(x) = 0_2\}$. An (anti-)IUP-homomorphism ψ is called an (anti-)IUP-monomorphism, (anti-)IUP-epimorphism, or (anti-)IUP-isomorphism if it is one-one, onto, or bijective, respectively. From [11] they prove that kernel of ψ is IUP-subalgebra, IUP-filter, and IUP-ideal.

Definition 1.6. *A nonempty subset S of a DIUP-algebra $X = (X; *, 0)$ is called*

(i) *a dual IUP-subalgebra (DIUPS) of X if it satisfies the condition (1.16).*

(ii) *a dual IUP-filter (DIUPF) of X if it satisfies the following conditions:*

$$\textit{the constant } 0 \textit{ of } X \textit{ is in } S, \tag{1.21}$$

$$(\forall x, y \in X)(x * y \in S, y \in S \Rightarrow x \in S), \tag{1.22}$$

(iii) *a dual IUP-ideal (DIUPI) of X if it satisfies the condition (1.21) and the following condition:*

$$(\forall x, y, z \in X)((x * y) * z \in S, y \in S \Rightarrow x * z \in S), \tag{1.23}$$

(iv) *a strong dual IUP-ideal (SDIUPI) of X if it satisfies the condition (1.21) and the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow y * x \in S). \tag{1.24}$$

2. External Direct Product DIUP-algebras

Lingcong and Endam [15] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated as follows:

Definition 2.1. [15] *Let $(X_i; *_i)$ be a groupoid for each $i \in \{1, 2, ..., k\}$. Define the direct product of algebras $X_1, X_2, ..., X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where*

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times ... \times X_k = \{(x_1, x_2, ..., x_k) \mid x_i \in X_i \ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, ..., x_k) \otimes (y_1, y_2, ..., y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, ..., x_k *_k y_k)$$

for all $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in \prod_{i=1}^{k} X_i$.

Definition 2.2. [5] *Let* X_i *be a nonempty set for each* $i \in I$. *Define the external direct product of sets* X_i *for all* $i \in I$ *to be the set* $\prod_{i\in I} X_i$, *where*

$$\prod_{i\in I} X_i = \{\wp : I \to \bigcup_{i\in I} X_i \mid \wp(i) \in X_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i\in I} X_i$ *with a function* $(x_i)_{i\in I} : I \to \bigcup_{i\in I} X_i$, *where* $i \mapsto x_i \in X_i$ *for all* $i \in I$.

Remark 2.3. [5] *Let* X_i *be a nonempty set and* S_i *a subset of* X_i *for all* $i \in I$. *Then* $\prod_{i\in I} S_i$ *is a nonempty subset of the external direct product* $\prod_{i\in I} X_i$ *if and only if* S_i *is a nonempty subset of* X_i *for all* $i \in I$.

Definition 2.4. [6] *Let* $X_i = (X_i; *_i)$ *be a groupoid for all* $i \in I$. *Define the binary operation* $\boxtimes$ *on the external direct product* $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes)$ *as follows:*

$$(\forall (x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i)((x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (y_i *_i x_i)_{i\in I}). \tag{2.1}$$

Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $\wp_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(\wp_{x_i}(j) = \begin{cases} x_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right). \tag{2.2}$$

Then $\wp_{x_i} \in \prod_{i\in I} X_i$.

Remark 2.5. *Let* $X_i = (X_i; *_i, 0_i)$ *be an IUP-algebra for all* $i \in I$. *For* $i \in I$, *we have* $\wp_{0_i} = (0_i)_{i\in I}$.

Lemma 2.6. *Let* $X_i = (X_i; *_i, 0_i)$ *be an IUP-algebra for all* $i \in I$. *For* $i \in I$, *let* $x_i, y_i \in X_i$. *Then* $\wp_{x_i} \boxtimes \wp_{y_i} = \wp_{y_i *_i x_i}$.

Proof. Now,

$$(\forall j \in I)\left((\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (IUP-2), we have

$$(\forall j \in I)\left((\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

By (2.2), we have $\wp_{x_i} \boxtimes \wp_{y_i} = \wp_{y_i *_i x_i}$.

The following theorem shows that the external direct product of IUP-algebras in terms of an infinite family of IUP-algebras is a DIUP-algebra.

Theorem 2.7. *$X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a DIUP-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$.

(DIUP-1) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-1), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \boxtimes (0_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(DIUP-2) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-3), we have $(x_i * y_i) *_i (x_i *_i z_i) = y_i * z_i$ for all $i \in I$. Thus

$$\begin{aligned}((z_i)_{i\in I} \boxtimes (x_i)_{i\in I}) \boxtimes ((y_i)_{i\in I} \boxtimes (x_i)_{i\in I}) &= (x_i *_i z_i)_{i\in I} \boxtimes (x_i *_i y_i)_{i\in I}\\ &= ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I}\\ &= (y_i *_i z_i)_{i\in I}\\ &= (z_i)_{i\in I} \boxtimes (y_i)_{i\in I}.\end{aligned}$$

(IUP-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-2), we have $x_i *_i x_i = 0_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a DIUP-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a DIUP-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4. Let $i \in I$.

(IUP-1) Let $x_i \in X_i$. Then $\wp_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (DIUP-1), we have $\wp_{x_i} \boxtimes (0_i)_{i\in I} = \wp_{x_i}$. By Remark 2.5, Lemma 2.6, and (2.2), we have $0_i *_i x_i = x_i$.

(IUP-2) Let $x_i \in X_i$. Then $\wp_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (IUP-2), we have $\wp_{x_i} \boxtimes \wp_{x_i} = (0_i)_{i\in I}$. By Lemma 2.6 and (2.2), we have $x_i *_i x_i = 0_i$.

(IUP-3) Let $x_i, y_i, z_i \in X_i$. Then $\wp_{x_i}, \wp_{y_i}, \wp_{z_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (DIUP-2), we have $(\wp_{z_i} \boxtimes \wp_{x_i}) \boxtimes (\wp_{y_i} \boxtimes \wp_{x_i}) = \wp_{z_i} \boxtimes \wp_{y_i}$. By Lemma 2.6 and (2.2), we have $(x_i *_i y_i) *_i (x_i *_i z_i) = y_i *_i z_i$.

Hence, $X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$.

We call the DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ in Theorem 2.7 the external direct product DIUP-algebra induced by an IUP-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Next, we introduce the concept of the weak direct product of infinite family of DIUP-algebras and obtain some of its properties as follows:

Definition 2.8. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Define the weak direct product DIUP-algebra induced by X_i for all $i \in I$ to be the structure $\prod^{\text{w}}_{i\in I} X_i = (\prod^{\text{w}}_{i\in I} X_i; \boxtimes)$, where*

$$\prod^{\text{w}}_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 2.9. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Then $\prod^{\text{w}}_{i\in I} X_i$ is a DIUP-subalgebra of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} 0_j *_j x_j & \text{if } j \in I_1 - I_2 \\ y_j *_j x_j & \text{if } j \in I_1 \cap I_2 \\ y_j *_j 0_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (IUP-1) and (IUP-2), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} x_j & \text{if } j \in I_1 - I_2 \\ y_j *_j x_j & \text{if } j \in I_1 \cap I_2 \\ y_j *_j 0_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i$. Hence, $\prod^{\text{w}}_{i\in I} X_i$ is a DIUP-subalgebra of $\prod_{i\in I} X_i$.

Theorem 2.10. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Then $\prod^{\text{w}}_{i\in I} X_i$ is a DIUP-ideal of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((x_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (z_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$ and $(y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid z_i *_i (y_i *_i x_i) \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. We shall show that $I_3 \subseteq I_1 \cup I_2$, where $I_3 = \{i \in I \mid z_i *_i x_i \neq 0_i\}$. Let $j \notin I_1 \cup I_2$. Then $j \notin I_1$ and $j \notin I_2$, so $z_j *_j (y_j *_j x_j) = 0_j$ and $y_j = 0_j$. By (IUP-1), we have $z_j *_j x_j = z_j *_j (0_j *_j x_j) = 0_j$. This implies that $j \notin I_3$, that is, $I_3 \subseteq I_1 \cup I_2$. Since $I_1 \cup I_2$ is finite, we have I_3 is finite. Therefore $(x_i)_{i\in I} \boxtimes (z_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is a DIUP-ideal of $\prod_{i\in I} X_i$.

Theorem 2.11. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is a DIUP-filter of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$ and $(y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid y_i *_i x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. We shall show that $I_3 \subseteq I_1 \cup I_2$, where $I_3 = \{i \in I \mid x_i \neq 0_i\}$. Let $j \notin I_1 \cup I_2$. Then $j \notin I_1$ and $j \notin I_2$, so $y_j *_j x_j = 0_j$ and $y_j = 0_j$. By (IUP-1), we have $x_j = 0_j *_j x_j = 0_j$. This implies that $j \notin I_3$, that is, $I_3 \subseteq I_1 \cup I_2$. Since $I_1 \cup I_2$ is finite, we have I_3 is finite. Therefore $(x_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is a DIUP-filter of $\prod_{i\in I} X_i$.

In a general case, $\prod^{\mathrm{w}}_{i\in I} X_i$ is not a strong dual IUP-ideal of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.

From Example 1.2, we let $X_i = X$ for all $i \in \mathbb{N}$ and let $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ defined by

$$(\forall j \in \mathbb{N}) \left((y_i)_{i\in\mathbb{N}}(j) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \right)$$

and $(\forall j \in \mathbb{N})((x_i)_{i\in\mathbb{N}}(j) = 4)$. Then $(y_i)_{i\in\mathbb{N}} \in \prod^{\mathrm{w}}_{i\in\mathbb{N}} X_i$ but

$$(\forall j \in \mathbb{N}) \left((x_i *_i y_i)_{i\in\mathbb{N}}(j) = \begin{cases} 3 & \text{if } j = 1 \\ 1 & \text{otherwise} \end{cases} \right).$$

This implies that $(y_i)_{i\in\mathbb{N}} \boxtimes (x_i)_{i\in\mathbb{N}} \notin \prod^{\mathrm{w}}_{i\in\mathbb{N}} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is not a strong dual IUP-ideal of $\prod_{i\in I} X_i$.

Theorem 2.12. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a DIUP-subalgebra of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an IUP-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (1.16), we have $x_i *_i y_i \in S_i$ for all $i \in I$, so $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a DIUP-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a DIUP-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $\wp_{x_i}, \wp_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By (1.16) and Lemma 2.6, we have $\wp_{x_i *_i y_i} = \wp_{y_i} \boxtimes \wp_{x_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i y_i \in S_i$. Hence, S_i is an IUP-subalgebra of X_i for all $i \in I$.

Theorem 2.13. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a DIUP-filter of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an IUP-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(y_i *_i x_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i *_i x_i \in S_i$ and $y_i \in S_i$, it follows from (1.18) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a DIUP-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a DIUP-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $\wp_{x_i}, \wp_{y_i} \in \prod_{i\in I} X_i$ and $\wp_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $\wp_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $\wp_{y_i} \boxtimes \wp_{x_i} = \wp_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.22), we have $\wp_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is an IUP-filter of X_i for all $i \in I$.

Theorem 2.14. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a DIUP-ideal of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an IUP-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((x_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (z_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(z_i *_i (y_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $z_i *_i (y_i *_i x_i) \in S_i$ and $y_i \in S_i$, it follows from (1.19) that $z_i *_i x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \boxtimes (z_i)_{i\in I} = (z_i *_i x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a DIUP-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a DIUP-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $\wp_{x_i}, \wp_{y_i}, \wp_{z_i} \in \prod_{i\in I} X_i$ and $\wp_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $\wp_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $(\wp_{z_i} \boxtimes \wp_{y_i}) \boxtimes \wp_{x_i} = \wp_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (1.23) and Lemma 2.6, we have $\wp_{x_i *_i z_i} = \wp_{z_i} \boxtimes \wp_{x_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i z_i \in S_i$. Hence, S_i is an IUP-ideal of X_i for all $i \in I$.

Theorem 2.15. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong IUP-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong DIUP-ideal of the external direct product DIUP-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a strong IUP-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i \in S_i$ for all $i \in I$, it follows from (1.20) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong DIUP-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong DIUP-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $\wp_{x_i}, \wp_{y_i} \in \prod_{i\in I} X_i$ and $\wp_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By (1.24) and Lemma 2.6, we have $\wp_{x_i *_i y_i} = \wp_{y_i} \boxtimes \wp_{x_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i y_i \in S_i$. Hence, S_i is a strong IUP-ideal of X_i for all $i \in I$.

Moreover, we discuss several IUP-homomorphism theorems in view of the external direct product of DIUP-algebras.

Definition 2.16. [5] *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be groupoids and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by*

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}). \tag{2.3}$$

Then $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function (see [5]).

Theorem 2.17. [5] *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be groupoids and $\psi_i : X_i \to S_i$ be a function for all $i \in I$.*

(i) ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 2.16,

(ii) ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,

(iii) ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.

Theorem 2.18. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be IUP-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then* *(i)* ψ_i *is an IUP-homomorphism for all* $i \in I$ *if and only if* ψ *is a DIUP-homomorphism which is defined in Definition* **??**,

(ii) ψ_i *is an IUP-monomorphism for all* $i \in I$ *if and only if* ψ *is a DIUP-monomorphism,*

(iii) ψ_i *is an IUP-epimorphism for all* $i \in I$ *if and only if* ψ *is a DIUP-epimorphism,*

(iv) ψ_i *is an IUP-isomorphism for all* $i \in I$ *if and only if* ψ *is a DIUP-isomorphism,*

(v) $\ker \psi = \prod_{i\in I} \ker \psi_i$ *and* $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.

Proof. (i) Assume that ψ_i is an IUP-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) &= \psi(x'_i *_i x_i)_{i\in I} \\
&= (\psi_i(x'_i *_i x_i))_{i\in I} \\
&= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I} \\
&= (\psi_i(x_i))_{i\in I} \boxtimes (\psi_i(x'_i))_{i\in I} \\
&= \psi(x_i)_{i\in I} \boxtimes \psi(x'_i)_{i\in I}.
\end{aligned}$$

Hence, ψ is a DIUP-homomorphism.

Conversely, assume that ψ is a DIUP-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $\wp_{x_i}, \wp_{y_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since ψ is a DIUP-homomorphism, we have $\psi(\wp_{x_i} \boxtimes \wp_{y_i}) = \psi(\wp_{x_i}) \boxtimes \psi(\wp_{y_i})$. Since

$$(\forall j \in I)\left((\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.4}$$

Since

$$(\forall j \in I)\left(\psi(\wp_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(\wp_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(\wp_{x_i}) \boxtimes \psi(\wp_{y_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.5}$$

By (2.4) and (2.5), we have $\psi_i(y_i *_i x_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an IUP-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.17 (i).

(iii) It is straightforward from (i) and Theorem 2.17 (ii).

(iv) It is straightforward from (i) and Theorem 2.17 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I \\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I \\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I}\ker\psi_i$. Now,

$$\begin{aligned}(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I} \\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I} \\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I \\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I}\psi_i(X_i)$.

Finally, we discuss several anti-IUP-homomorphism theorems in view of the external direct product of DIUP-algebras.

Theorem 2.19. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be IUP-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is an anti-IUP-homomorphism for all $i \in I$ if and only if ψ is an anti-DIUP-homomorphism which is defined in Definition 2.16,*

(ii) *ψ_i is an anti-IUP-monomorphism for all $i \in I$ if and only if ψ is an anti-DIUP-monomorphism,*

(iii) *ψ_i is an anti-IUP-epimorphism for all $i \in I$ if and only if ψ is an anti-DIUP-epimorphism,*

(iv) *ψ_i is an anti-IUP-isomorphism for all $i \in I$ if and only if ψ is an anti-DIUP-isomorphism.*

Proof. (i) Assume that ψ_i is an anti-IUP-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) &= \psi(x'_i *_i x_i)_{i\in I} \\ &= (\psi_i(x'_i *_i x_i))_{i\in I} \\ &= (\psi_i(x_i) *_i \psi_i(x'_i))_{i\in I} \\ &= (\psi_i(x'_i))_{i\in I} \boxtimes (\psi_i(x_i))_{i\in I} \\ &= \psi(x'_i)_{i\in I} \boxtimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-DIUP-homomorphism.

Conversely, assume that ψ is an anti-DIUP-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $\wp_{x_i}, \wp_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since ψ is an anti-DIUP-homomorphism, we have $\psi(\wp_{x_i} \boxtimes \wp_{y_i}) = \psi(\wp_{y_i}) \boxtimes \psi(\wp_{x_i})$. Since

$$(\forall j \in I)\left((\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(\wp_{x_i} \boxtimes \wp_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise}\end{cases}\right). \tag{2.6}$$

Since

$$(\forall j \in I)\left(\psi(\wp_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise}\end{cases}\right)$$

and

$$(\forall j \in I) \left(\psi(\wp_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right),$$

we have

$$(\forall j \in I) \left((\psi(\wp_{y_i}) \boxtimes \psi(\wp_{x_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases} \right). \tag{2.7}$$

By (2.6) and (2.7), we have $\psi_i(y_i *_i x_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is an anti-IUP-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.17 (i).

(iii) It is straightforward from (i) and Theorem 2.17 (ii).

(iv) It is straightforward from (i) and Theorem 2.17 (iii).

3. Conclusions and Future Work

In this paper, we have introduced the concept of the direct product of an infinite family of IUP-algebras and prove that it is a DIUP-algebra; we call the external direct product DIUP-algebra induced by IUP-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam [5]. We proved that the external direct product of IUP-algebras is DIUP-algebras. Also, we have introduced the concept of the weak direct product DIUP-algebras and proved that the weak direct product of IUP-algebras is DIUP-subalgebras, DIUP-ideals, and DIUP-filters, respectively. We have also shown that the external direct product of IUP-subalgebras (resp., IUP-filters, IUP-ideals, strong IUP-ideals) is a DIUP-subalgebra (resp., DIUP-filter, DIUP-ideal, strong DIUP-ideal) of the external direct product DIUP-algebras. Finally, we have provided several fundamental theorems of (anti-)IUP-homomorphisms in view of the external direct product DIUP-algebras.

Based on the concept of the external direct product DIUP-algebras in this article, we can apply it to the study of the external direct product in other algebraic systems. The external direct product dual IUP-algebras in this paper will be developed into new concepts for future studies: type 2 of internal direct products of IUP-algebras.

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