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QUARTER SYMMETRIC NON-METRIC CONNECTION ON A $(k, \mu)-$CONTACT METRIC MANIFOLD

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Abstract: The object of the present paper is to introduce a new type of quarter symmetric non-metric connection on a $(k, \mu)-$contact metric manifold and study some properties of quarter symmetric non-metric connection on a $(k, \mu)-$contact metric manifold. Further, we obtain some properties of nearly Ricci recurrent on a $(k, \mu)-$contact metric manifold with respect to quarter symmetric non-metric connection. Finally, we present an example to verify our result.

Keywords and Phrases: $(k, \mu)-$contact metric manifold, quarter symmetric non-metric connection, Curvature tensor, symmetric and skew-symmetric and nearly Ricci recurrent.

2020 Mathematics Subject Classification: 53C25, 53D15..

1. Introduction

The notion of $(k, \mu)-$contact metric manifolds was introduced by Blair, Koufogiorgos and Papantoniou [2] where k and μ are real constants. A class of contact manifolds with contact metric structure (ϕ, ξ, η, g) in which the curvature tensor R satisfies the condition:

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

$\forall\ X, Y \in$TM, where k and μ are real constants, is called $(k, \mu)-$contact metric manifold. The class of $(k, \mu)-$contact metric manifolds contains both the class of

Sasakian ($k = 1$ *and* $h = 0$) and non-Sasakian ($k \neq 1$ *and* $h \neq 0$) manifolds. For example, the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure is a non-Sasakian $(k, \mu)-$contact metric manifold. The properties of $(k, \mu)-$contact metric manifold have been studied by many authors such as Koufogiorgos [14], Shaikh and Baishya [27], Shaikh and Jana [26], Sharma and Vranckew [28], Majhi and Ghosh [16], Ghosh and Sharma [10], De and Sarkar [5], Yildiz and De [34] etc.

Let $\overline{D}$ be a linear connection in a Riemannian manifold M. The torsion tensor $\overline{T}$ is given by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y].$$

The connection $\overline{D}$ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric. The connection $\overline{D}$ is a metric connection if there is a Riemannian metric g in M such that $\overline{D}_X g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

A. Friedmann and J.A. Schouten introduced the idea of a semi-symmetric linear connection [9]. A linear connection $\overline{D}$ is said to be a semi-symmetric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

$\forall\ X, Y \in$TM.

S. Golab introduced the idea of a quarter symmetric linear connection in a differentiable manifold [11]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a (1, 1) tensor field. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric metric connection reduces to the semi-symmetric metric connection [9]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Further in 1980, Mishra and Pandey [17] have studied quarter symmetric metric connection in Riemannian, Kähler and Sasakain manifolds.

A relation between the quarter-symmetric metric connection $\overline{D}$ and the Levi-Civita connection D in an n-dimensional SP-Sasakian manifold is given by [6]

$$\overline{D}_X Y = D_X Y + \eta(Y)\phi X - F(X,Y)\xi,$$

whose torsion and metric are $\overline{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$ and $(\overline{D}_X g)(Y,Z) = 0$. In this paper, the author proved that in an SP-Sasakian manifold, the Ricci tensor of the quarter symmetric metric connection is symmetric and found some interesting results.

Recently, in 2000, De and Sengupta [7] studied a quarter symmetric metric connection on a Sasakian manifold as

$$\overline{D}_X Y = D_X Y - \eta(X)\phi Y.$$

The quarter symmetric metric connection have been developed by several authors such as Srivastava, Sharma and Prasad [30], Prakash and Narain [23], Kumar, Bagewadi and Venkatesha [15], Haseeb [12], Prasad and Haseeb [25] etc. On the other hand, quarter symmetric non-metric connection have been studied by various authors such as Dwivedi [8], Mondal [18], Patra and Bhattacharyya [21], Somashekhara , Praveena and Venkatesha et al [29].

In recent paper, Shaikh and Jana [26] introduced and studied a new type of quarter symmetric metric connection on a $(k,\mu)-$contact metric manifold as

$$\overline{D}_X Y = D_X Y + \eta(Y)hX - g(hX,Y)\xi,$$

$\forall\ X, Y \in$TM, whose torsion tensor and metric are

$$\overline{T}(X,Y) = \eta(Y)hX - \eta(X)hY$$

and

$$(\overline{D}_X g)(Y,Z) = 0,$$

where h is a (1, 1) tensor field. They proved that the Ricci tensor of a non-Sasakian $(k,\mu)-$contact metric manifold (M^{2n+1}, g) with respect to the quarter-symmetric metric connection is symmetric if and only if the contact form η is closed and found many others results.

The motivation of the above ideas, we define a new type of quarter symmetric non-metric connection on a $(k,\mu)-$contact metric manifold as follows

$$\overline{D}_X Y = D_X Y + \eta(X)hY - \eta(Y)hX,$$

$\forall\ X, Y \in$TM, whose torsion tensor and metric are

$$\overline{T}(X,Y) = 2[\eta(X)hY - \eta(Y)hX]$$

and

$$(\overline{D}_X g)(Y,Z) = \eta(Y)g(hX,Z) + \eta(Z)g(hX,Y).$$

In 1952, Paatterson [19] introduced Ricci recurrent manifolds. According to him a manifold (M^n, g) was called "Ricci recurrent" if

$$(D_X S)(Y, Z) = A(X)S(Y, Z), \tag{1.1}$$

for some 1-form A where D and S denote the operator of covariant differentiation with respect to metric tensor g and Ricci tensor respectively. He denoted such a manifold by R_n. Ricci recurrent manifolds have been studied by several authors such as Roter [31], Chaki [4], Prakash [22], Venkatesha et al [32].

Very recently Prasad and Yadav [24] introduced a new type of non-flat Ricci recurrent manifold whose Ricci tensor S satisfies the condition:

$$(D_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + B(X)g(Y, Z), \tag{1.2}$$

$\forall\ X, Y, Z \in$TM, where A and B non-zero 1-forms, ρ_1 and ρ_2 be two vector fields such that

$$A(X) = g(\rho_1, X), \qquad B(X) = g(\rho_2, X). \tag{1.3}$$

Such a manifold called as a nearly Ricci recurrent manifold and 1-forms A and B be its associated 1-form. Nearly Ricci recurrent manifolds of this kind were denoted by him as a $N\{R(R_n)\}$. The name nearly Ricci recurrent Riemannian manifold was chosen because if $B = 0$ in (1.2) then the manifold reduces to a Ricci recurrent manifold which is very close to Ricci recurrent space. This justified the name "Nearly Ricci recurrent manifold" for a manifold defined by (1.2) and the use of the symbol $N\{R(R_n)\}$ for it.

2. Preliminaries

A (2n+1) dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if it admits a tensor ϕ of type (1,1), ξ is a contravariant vector fields of type (0,1) and 1-form η is a covariant tensor of the type (1,0) satisfying (Blair, [1], [3]):

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ \ trace\phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X) \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

$\forall\ X,\ Y \in$TM.

An almost contact metric structure becomes a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.5}$$

$\forall\ X,\ Y\ \in$TM. In a contact metric manifold we define a (1,1) tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\phi = -\phi h, \quad h\xi = 0, \quad trace(h) = trace(\phi h) = 0, \quad trace\phi = 0. \tag{2.6}$$

Also

$$D_X\xi = -\phi X - \phi hX, \tag{2.7}$$

$$(D_X\eta) = g(X + hX, \phi Y). \tag{2.8}$$

Blair, Koufogiorgos and Papantoniou [2] considered the $(k,\mu)-$nullity condition on a contact metric manifold. The $(k,\mu)-$nullity distribution $N(k,\mu)$ of a contact metric manifold M is defined by ([2], [20])

$$N(k,\mu) : p \rightarrow N_p(k,\mu) =$$
$$[Z \rightarrow T_pM : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} + \mu\{g(Y,Z)hX - g(X,Z)hY\}],$$

$\forall\ X,Y,Z\ \in$TM. A contact metric manifold M with $\xi \in N(k,\mu)$ is known as $(k,\mu)-$contact metric manifold if ([2], [20])

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.9}$$

Also in $(k,\mu)-$contact metric manifold, the following holds:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.10}$$

$$(D_X\phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.11}$$

$$\begin{aligned}(D_Xh)(Y) =&(1-k)[g(X,\phi Y)\xi - \eta(Y)\phi X]+ \\ &g(X, h\phi Y)\xi + \eta(Y)h\phi X - \mu\eta(X)\phi hX,\end{aligned} \tag{2.12}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.13}$$

$$\begin{aligned}S(X,Y) =&[2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &+ [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \ n \geq 1,\end{aligned} \tag{2.14}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.15}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \tag{2.16}$$

where Q is the Ricci operator, i.e. $g(QX,Y) = S(X,Y)$,

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y), \tag{2.17}$$

$\forall$ $X, Y \in$TM.

A Riemannian manifold is an Einstein manifold if

$$S(X,Y) = \lambda g(X,Y). \tag{2.18}$$

After introduction and preliminaries, we introduced a new type of quarter symmetric non-metric connection on a $(k,\mu)-$contact metric manifold with respect to the quarter symmetric non-metric connection $\overline{D}$ in section 3. In section 4, we find the curvature tensor of $(k,\mu)-$contact metric manifold with respect to the quarter symmetric non-metric connection $\overline{D}$ and its some proprieties. Section 5 is devoted to skew-symmetric and symmetric condition of Ricci tensor $\overline{S}$ of $\overline{D}$ on a $(k,\mu)-$contact metric manifold. Section 6 deals with nearly Ricci recurrent $(k,\mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$. Finally, the existence of nearly Ricci recurrent non-Sasakain $(k,\mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$ is ensured by a non-trivial example.

3. Quarter Symmetric Non-metric Connection $\overline{D}$ on a $(k,\mu)-$contact Metric Manifold

Let (M^{2n+1}, g) be a $(k,\mu)-$contact metric manifold with Levi-Civita connection D, we define a linear connection $\overline{D}$ on M^{2n+1} by

$$\overline{D}_X Y = D_X Y + \eta(X)hY - \eta(Y)hX \tag{3.1}$$

where η be 1-form associated with vector field ξ on M^{2n+1} given by

$$g(X,\xi) = \eta(X), \tag{3.2}$$

$\forall$ $X, Y \in$TM.

Using (3.1), the torsion tensor $\overline{T}$ on M^{2n+1} with respect to the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y],$$

which gives

$$\overline{T}(X,Y) = 2[\eta(X)hY - \eta(Y)hX]. \tag{3.3}$$

A linear connection satisfying (3.3) is called quarter symmetric connection. Again using (3.1), we have

$$(\overline{D}_X g)(Y,Z) = \eta(Y)g(hX,Z) + \eta(Z)g(hX,Y). \tag{3.4}$$

A linear connection $\overline{D}$ defined by (3.1) satisfying (3.3) and (3.4) is called quarter symmetric non-metric connection. Conversely, we will show that a linear connection $\overline{D}$ define on M^{2n+1} satisfying (3.3) and (3.4) is given by (3.1).

Let $\overline{D}$ is a linear connection M^{2n+1} given by

$$\overline{D}_X Y = D_X Y + H(X,Y). \tag{3.5}$$

Now, we shall determined the tensor field H such that $\overline{D}$ satisfies (3.3) and (3.4). In view of (3.5), we get

$$\overline{T}(X,Y) = H(X,Y) - H(Y,X). \tag{3.6}$$

We have

$$(\overline{D}_X g)(Y,Z) = \overline{D}_X g(Y,Z) - g(\overline{D}_X Y, Z) - g(Y, \overline{D}_X Z), \tag{3.7}$$

In view of (3.5) and (3.7), We get

$$g(H(X,Y),Z) + g(H(X,Z),Y) = -[\eta(Y)g(hX,Z) + \eta(Z)g(hX,Y)]. \tag{3.8}$$

From (3.5), (3.6) and (3.8), we get

$$\begin{aligned} & g(\overline{T}(X,Y),Z) + g(\overline{T}(Z,X),Y) + g(\overline{T}(Z,Y),X) = \\ & 2g(H(X,Y),Z) + 2\eta(Y)g(hX,Z) + 2\eta(X)g(hY,Z), \end{aligned}$$

which gives

$$H(X,Y) = \frac{1}{2}[\overline{T}(X,Y) +' \overline{T}(X,Y) +' \overline{T}(Y,X)] - \eta(Y)hX - \eta(X)hY, \tag{3.9}$$

where $'\overline{T}$ be a tensor field of type (1, 2) defined by

$$g('\overline{T}(X,Y),Z) = g(T(Z,X),Y) = 2[\eta(Z)g(hX,Y) - \eta(X)g(hZ,Y)]. \tag{3.10}$$

In view of (3.9) and (3.10), we get

$$H(X,Y) = \eta(X)hY - \eta(Y)hX.$$

This implies that

$$\overline{D}_X Y = D_X Y + \eta(X)hY - \eta(Y)hX.$$

Hence we have the following theorem:

Theorem 3.1. *Let (M^{2n+1}, g) be a $(k,\mu)-$contact metric manifold with $(k,\mu)-$contact* *structure* (ϕ, ξ, η, g) *admitting a quarter symmetric non-metric connection* $\overline{D}$ *which satisfies* (3.3) *and* (3.4). *Then the quarter symmetric non-metric connection is given by* $\overline{D}_X Y = D_X Y + \eta(X)hY - \eta(Y)hX$.

4. Curvature Tensor of $(k, \mu)-$contact Metric Manifold with respect to the Quarter Symmetric Non-metric Connection $\overline{D}$

Let R and $\overline{R}$ be the curvature tensor of the connection D and $\overline{D}$ respectively, then

$$\overline{R}(X,Y)Z = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z. \tag{4.1}$$

In view of (3.1) and (4.1), we have

$$\begin{aligned}\overline{R}(X,Y)Z =& R(X,Y)Z + [(D_X\eta)(Y)hZ - (D_Y\eta)(X)hZ] + \\ & [\eta(Y)(D_X h)(Z) - \eta(X)(D_Y h)(Z)] - \\ & [(D_X\eta)(Z)hY - (D_Y\eta)(Z)hX] - \\ & \eta(Z)[(D_X h)(Y) - (D_Y h)(X)].\end{aligned} \tag{4.2}$$

Using (2.8) and (2.12) in (4.2), we get

$$\begin{aligned}\overline{R}(X,Y)Z =& R(X,Y)Z - [2g(\phi X,Y)hZ - g(\phi X,Z)hY + g(\phi Y,Z)hX + \\ & g(\phi hX,Y)hZ - g(\phi hY,X)hZ - g(\phi hX,Z)hY + g(\phi hY,Z)hX] \\ & + (1-k)[\{g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)\} - 2g(X,\phi Y)\eta(Z)]\xi \\ & + [g(X,h\phi Z)\eta(Y) - g(Y,h\phi Z)\eta(X) - g(X,h\phi Y)\eta(Z) + \\ & g(Y,h\phi X)\eta(Z)]\xi + \mu[\eta(X)\phi hY - \eta(Y)\phi hX]\eta(Z).\end{aligned} \tag{4.3}$$

Hence we have have the following theorem:

Theorem 4.1. *The curvature tensor* $\overline{R}(X,Y)Z$ *of* $(k,\mu)-$*contact metric manifold with respect to quarter symmetric non-metric connection* $\overline{D}$ *is given by* (4.3).

In view of (2.5) and (4.3), we get

$$'\overline{R}(X,Y,Z,W) + {}'\overline{R}(Y,X,Z,W) = 0, \tag{4.4}$$

where $'\overline{R}(X,Y,Z,W) = g(\overline{R}(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

We also have

$$\begin{aligned}&\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = -4[d\eta(Y,X)hZ + \\ & d\eta(Z,Y)hX + d\eta(X,Z)hY] + 4(1-k)[d\eta(X,Y)\eta(Z) + d\eta(X,Z)\eta(Y) \\ & + d\eta(Z,Y)\eta(X)]\xi - 2[\{d\eta(Z,hY) - d\eta(Y,hZ)\}\eta(X) + \\ & \{d\eta(X,hZ) + d\eta(Z,hX)\}\eta(Y) + \{d\eta(Y,hX) - d\eta(X,hY)\}\eta(Z)]\xi.\end{aligned} \tag{4.5}$$

Hence we have the following theorem:

Theorem 4.2. *The curvature tensor* $\overline{R}(X,Y)Z$ *of* $(k,\mu)-$*contact metric manifold with respect to quarter symmetric non-metric connection satisfies*

$$'\overline{R}(X,Y,Z,W) + \ '\overline{R}(Y,X,Z,W) = 0,$$

and

$$\begin{aligned}&\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = -4[d\eta(Y,X)hZ+\\&d\eta(Z,Y)hX + d\eta(X,Z)hY] + 4(1-k)[d\eta(X,Y)\eta(Z) + d\eta(X,Z)\eta(Y)\\&+ d\eta(Z,Y)\eta(X)]\xi - 2[\{d\eta(Z,hY) - d\eta(Y,hZ)\}\,\eta(X)+\\&\{d\eta(X,hZ) + d\eta(Z,hX)\}\,\eta(Y) + \{d\eta(Y,hX) - d\eta(X,hY)\}\,\eta(Z)]\xi,\end{aligned}$$

$\forall\ X,Y,Z,W \in TM.$

Again if the 1-form η is closed i.e. if $d\eta(X,Y) = 0\ \forall\ X,Y \in$TM; then (4.5) implies that

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0. \tag{4.6}$$

Hence we have the following theorem:

Theorem 4.3. *The curvature tensor* $\overline{R}(X,Y)Z$ *of* $(k,\mu)-$*contact metric manifold with respect to quarter symmetric non-metric connection* $\overline{D}$ *satisfies the Bianchi identity if and only if the 1-form* η *is closed.*

Contracting (4.2), we have

$$\begin{aligned}\overline{S}(Y,Z) =& S(Y,Z) + 3g(Y,\phi hZ) + g(\phi hY,Z)+\\&g(\phi hY,hZ) - 3(1-k)g(Y,\phi Z),\end{aligned} \tag{4.7}$$

In view of (2.5) and (4.7), we get

$$\overline{S}(Y,Z) = S(Y,Z) + 3d\eta(Y,hZ) + d\eta(Z,hY) + d\eta(hZ,hY) - 3(1-k)d\eta(Y,Z), \tag{4.8}$$

$$\overline{S}(Y,\xi) = 2nk\eta, \tag{4.9}$$

and

$$\bar{r} = r. \tag{4.10}$$

From (4.7) and (4.8), we have have the following theorem:

Theorem 4.4. *The Ricci tensor* $\overline{S}$ *of* $(k,\mu)-$*contact metric manifold with respect* *to $\overline{D}$ is equal to Ricci tensor S of $(k,\mu)-$contact metric manifold with respect to D if and only if the 1-form η is closed.*

Theorem 4.5. *The scalar curvature $\overline{r}$ of $(k,\mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$ is equal to scalar curvature of manifold with respect to D .*

5. Skew-symmetric and Symmetric Condition of Ricci Tensor $\overline{S}$ of $\overline{D}$ on a $(k,\mu)-$contact Metric Manifold

From (4.7), we get

$$\begin{aligned}\overline{S}(Z,Y) =&S(Z,Y)+3g(Z,\phi hY)+g(\phi hZ,Y)+\\ &g(\phi hZ,hY)-3(1-k)g(Z,\phi Y),\end{aligned} \tag{5.1}$$

In view of (4.7) and (5.1), we have

$$\overline{S}(Y,Z)+\overline{S}(Z,Y) =2S(Y,Z)+4[g(Y,\phi hZ)+g(\phi hY,Z)]. \tag{5.2}$$

Using (2.5) in (5.2), we obtain

$$\overline{S}(Y,Z)+\overline{S}(Z,Y) =2S(Y,Z)+4[d\eta(Y,hZ)+d\eta(Z,hY)]. \tag{5.3}$$

If $\overline{S}(Y,Z)$ is skew-symmetric then left hand side of (5.3) vanishes and we have

$$S(Y,Z)=-2[d\eta(Y,hZ)+d\eta(Z,hY)]. \tag{5.4}$$

On the other hand if $S(Y,Z)$ is given by (5.4), then from (5.3), we get

$$\overline{S}(Y,Z)+\overline{S}(Z,Y)=0.$$

Hence we have the following theorem:

Theorem 5.1. *Let (M^{2n+1},g) be a $(k,\mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$. Then the Ricci tensor $\overline{S}$ of $\overline{D}$ is skew-symmetric if and only if the Ricci tensor S of the Levi-Civita connection D is given by* (5.4).

Again from (4.7) and (5.1), we have

$$\begin{aligned}\overline{S}(Y,Z)-\overline{S}(Z,Y) =&2[d\eta(Y,hZ)-d\eta(Z,hY)+\\ &d\eta(hZ,hY)-3(1-k)d\eta(Y,Z)].\end{aligned} \tag{5.5}$$

If the 1-form η is closed , then the equation (5.5) will be

$$\overline{S}(Y,Z)-\overline{S}(Z,Y)=0.$$

Hence we have the following theorem:

Theorem 5.2. *Let (M^{2n+1}, g) be a $(k, \mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$. Then the Ricci tensor $\overline{S}$ of $\overline{D}$ is symmetric if and only if the 1-form η is closed.*

6. Nearly Ricci Recurrent $(k, \mu)-$contact Metric Manifold with respect to Quarter Symmetric Non-metric Connection

Analogous to the definition of (1.3), we define nearly Ricci recurrent non-Sasakain $(k, \mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$ as follows

$$(\overline{D}_X\overline{S})(Y, Z) = [A(X) + B(X)]\overline{S}(Y, Z) + B(X)g(Y, Z). \tag{6.1}$$

Using (4.7) in (6.1), we get

$$\begin{aligned}(\overline{D}_X\overline{S})(Y, Z) =&[A(X) + B(X)][S(Y, Z) - 3g(Y, h\phi Z) + g(\phi hY, Z)+ \\ &g(\phi hY, hZ) - 3(1 - k)g(Y, \phi Z)] + B(X)g(Y, Z).\end{aligned} \tag{6.2}$$

Putting ξ for Z in (6.2) and using (2.1), (2.2) and (2.15), we get

$$(\overline{D}_X\overline{S})(Y, \xi) = [A(X) + (2nk + 1)B(X)]\eta(Y). \tag{6.3}$$

Now, we have

$$(\overline{D}_X\overline{S})(Y, \xi) = \overline{D}_X\overline{S}(Y, \xi) - \overline{S}(\overline{D}_XY, \xi) - \overline{S}(Y, \overline{D}_X\xi). \tag{6.4}$$

Using (2.3), (2.7), (3.1) and (4.7) in (6.4), we obtain

$$(\overline{D}_X\overline{S})(Y, \xi) = 2\overline{S}(Y, \phi X) - \overline{S}(Y, h\phi X) - 4nkg(Y, \phi X) + 2nkg(Y, h\phi X). \tag{6.5}$$

From (6.3) and (6.5), we have

$$\begin{aligned}[A(X) + (2nk + 1)B(X)]\eta(Y) =&2\overline{S}(Y, \phi X) - \overline{S}(Y, h\phi X)- \\ &4nkg(Y, \phi X) + 2nkg(Y, h\phi X).\end{aligned} \tag{6.6}$$

Further Y is replaced by ϕY in (6.6), we get

$$2\overline{S}(\phi Y, \phi X) - \overline{S}(\phi Y, h\phi X) = 4nkg(\phi Y, \phi X) - 2nkg(\phi Y, h\phi X). \tag{6.7}$$

Again X is replaced by hX in (6.7) and using (2.1), (2.6) and (2.10), we get

$$\begin{aligned}-2\overline{S}(\phi Y, h\phi X) + (1 - k)\overline{S}(\phi Y, \phi X) =& - 4nkg(\phi Y, h\phi X) \\ &- 2nk(1 - k)g(\phi Y, \phi X).\end{aligned} \tag{6.8}$$

In view of (6.7) and (6.8), we obtain

$$\overline{S}(\phi Y, \phi X) = 2nkg(\phi Y, \phi X),\ 3 + k \neq 0. \tag{6.9}$$

Using (2.3) and (4.7) in (6.9), we get

$$\begin{aligned} S(\phi Y, \phi X) =& 3(1-k)g(\phi Y, X) - 3g(\phi Y, hX) - g(hY, \phi X) \\ & - g(hY, h\phi X) + 2nk[g(Y, X) - \eta(Y)\eta(X)]. \end{aligned} \tag{6.10}$$

From (2.17) and (6.10), we have

$$\begin{aligned} S(Y, X) =& 2nkg(Y, X) + 3(1-k)g(\phi Y, X) - 3g(\phi Y, hX) \\ & - g(hY, \phi X) - g(hY, h\phi X) + 2(2n - 2 + \mu)kg(hY, X). \end{aligned} \tag{6.11}$$

Using (2.5) in (6.11), we have

$$\begin{aligned} S(X, Y) =& 2nkg(X, Y) + 3(1-k)d\eta(X, Y) - 3d\eta(hX, Y) \\ & - d\eta(hY, X) + d\eta g(hX, hY) + 2(2n - 2 + \mu)kg(hY, X). \end{aligned} \tag{6.12}$$

Hence we have the following theorem:

Theorem 6.1. *Let M^{2n+1} be a nearly Ricci recurrent $(k, \mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$. Then the Ricci tensor S of the Levi-Civita connection D is equal to Einstein manifold where $\lambda = 2nk$ if and only if $\mu = 2(1-n)$ and the 1-form η is closed, provided $k+3 \neq 0$.*

Again Putting $Y = \xi$ in (6.6), we get

$$B(X) = -\frac{1}{2nk+1}A(X). \tag{6.13}$$

Hence we have the following theorem:

Theorem 6.2. *Let M^{2n+1} be a nearly Ricci recurrent $(k, \mu)-$contact metric manifold with respect to quarter symmetric non-metric connection $\overline{D}$. Then $B(X) = -\frac{1}{2nk+1}A(X)$ hold on M^{2n+1}.*

7. Conclusion

1. For $(k, \mu)-$contact metric manifold (M^{2n+1}, g) admitting quarter symmetric non-metric connection $\overline{D}$,

   (i) The curvature tensor $\overline{R}$ of $\overline{D}$ is given by (4.3).

   (ii) $'\overline{R}(X, Y, Z, W) + {}'\overline{R}(Y, X, Z, W) = 0$.

(iii) $\overline{R}(X,Y)Z+\overline{R}(Y,Z)X+\overline{R}(Z,X)Y=0$ if and only if 1-form η is closed.

(iv) The Ricci tensor $\overline{S}$ of $\overline{D}$ is given by (4.7).

(v) The Ricci tensor $\overline{S}$ of $\overline{D}$ is equal to the Ricci tensor S of D if and only if 1-form η is closed.

(vi) The Ricci tensor $\overline{S}$ of $\overline{D}$ is skew-symmetric if and only if the Ricci tensor S of D is given by (5.4).

(iv) The Ricci tensor $\overline{S}$ of $\overline{D}$ is symmetric if and only if 1-form η is closed.

2. For nearly Ricci recurrent $(k,\mu)-$contact metric manifold (M^{2n+1},g) admitting quarter symmetric non-metric connection $\overline{D}$,

(i) The Ricci tensor S of D is equal to Einstein manifold where $\lambda=2nk$ if and only if $\mu=2(1-n)$ and the 1-form η is closed, provided $k+3\neq 0$.

(ii) $B(X)=-\frac{1}{2nk+1}A(X)$.

8. Example

Let us consider the 3-dimensional manifold $M=\{(x,y,z)\in \boldsymbol{R}^3, z\neq 0\}$, where (x,y,z) are standard co-ordinate of $\boldsymbol{R}^3$. We choose the vector fields

$$e_1=e^{-2z}\frac{\partial}{\partial x},\ e_2=e^{-2z}\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}, \tag{8.1}$$

which are linearly independent at each point of M.

Let g be the Riemannian metric denoted by

$$g(e_i,e_j)=\begin{cases}1, & i=j\\ 0, & i\neq j,\end{cases} \tag{8.2}$$

where $i,j=1,2,3$.

Let η be the 1-form defined by $\eta(U)=g(U,e_3)$ and $\eta(e_3)=1$ for any $U\in\chi(M^3)$. Let ϕ be tensor field of type (1,1) defined by

$$\phi e_1=e_2,\quad \phi e_2=-e_1,\quad \phi e_3=0. \tag{8.3}$$

From the properties of ϕ and η, we obtain

$$g(e_i,\phi e_i)=d\eta(e_i,e_i),\ i,j=1,2,3.$$

Then we have

$$\phi^2U=-U+\eta(U)e_3\quad g(\phi U,\phi W)=g(U,W)-\eta(U)\eta(W), \tag{8.4}$$

$\forall\ U, W \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defined an almost contact metric structure on M.

Let D be the Levi-Civita connection with respect to the Riemannian metric g. Then from equation (8.1), we have

$$[e_1, e_2] = 0 \quad [e_1, e_3] = 2e_1, \quad [e_2, e_3] = 2e_2. \tag{8.5}$$

The Riemannian connection D of the metric g is given by

$$\begin{aligned} 2g\left(D_X Y, Z\right) =& Xg\left(Y, Z\right) + Yg\left(X, Z\right) - Zg\left(X, Y\right) - g\left(X, [Y, Z]\right) \\ & - g\left(Y, [X, Z]\right) + g\left(Z, [X, Y]\right), \end{aligned} \tag{8.6}$$

which is known as Koszul's formula. Using (8.2) and (8.5) in (8.6), we get

$$\left.\begin{array}{lll} D_{e_1}e_3 = 2e_1, & D_{e_2}e_3 = 2e_2, & D_{e_3}e_3 = 0, \\ D_{e_1}e_2 = 0, & D_{e_2}e_2 = 2e_3, & D_{e_3}e_2 = 0, \\ D_{e_1}e_1 = 2e_3, & D_{e_2}e_1 = 0, & D_{e_3}e_1 = 0. \end{array}\right\} \tag{8.7}$$

Also we know

$$D_{e_1}e_2 = -\phi e_1 - \phi h e_1$$

Comparing two relations for $D_{e_1}e_2$ and using (8.3), we have

$$he_1 = -e_1.$$

Similarly

$$he_2 = -e_2 \ and\ he_3 = 0.$$

Also

$$\left.\begin{array}{lll} \overline{D}_{e_1}e_3 = 3e_1, & \overline{D}_{e_2}e_3 = 3e_2, & \overline{D}_{e_3}e_3 = 0, \\ \overline{D}_{e_1}e_2 = 0, & \overline{D}_{e_2}e_2 = 2e_3, & \overline{D}_{e_3}e_2 = -e_2, \\ \overline{D}_{e_1}e_1 = 2e_3, & \overline{D}_{e_2}e_1 = 0, & \overline{D}_{e_3}e_1 = -e_1. \end{array}\right\} \tag{8.8}$$

Use of (8.4) and (8.7), we can easily calculate the curvature tensor as follows:

$$\left.\begin{array}{lll} R\left(e_1, e_2\right)e_3 = 0, & R\left(e_2, e_3\right)e_3 = -4e_2, & R\left(e_1, e_3\right)e_3 = -4e_1, \\ R\left(e_1, e_2\right)e_2 = 4e_1, & R\left(e_2, e_3\right)e_2 = -4e_3, & R\left(e_1, e_3\right)e_2 = 0, \\ R\left(e_1, e_2\right)e_1 = -4e_2, & R\left(e_2, e_3\right)e_1 = 0, & R\left(e_1, e_3\right)e_1 = -4e_3. \end{array}\right\} \tag{8.9}$$

From (8.9), we have

$$\begin{aligned} R\left(e_1, e_2\right)e_3 &= 0 = 4[\eta(e_2)e_1 - \eta(e_1)e_2] + 8[\eta(e_2)he_1 - \eta(e_1)he_2], \\ R\left(e_2, e_3\right)e_3 &= -4e_2 = 4[\eta(e_3)e_2 - \eta(e_2)e_3] + 8[\eta(e_3)he_2 - \eta(e_2)he_3], \\ R\left(e_1, e_3\right)e_3 &= -4e_1 = 4[\eta(e_3)e_1 - \eta(e_1)e_3] + 8[\eta(e_3)he_1 - \eta(e_1)he_3]. \end{aligned}$$

In view of above the expression of the curvature tensor we can easily conclude that the manifold is a $(k,\mu)-$contact metric manifold with $k=4$ and $\mu=8$.

From (8.8), we have $\overline{T}(e_1,e_3)=2e_1$ and $2[\eta(e_1)he_3-\eta(e_3)he_1]=2e_1$. This show that the linear connection $\overline{D}$ defined as (3.1) is a quarter symmetric connection on (M^3,g). Also

$$\left(\overline{D}_{e_1}g\right)(e_1,e_3)=-1\neq 0.$$

Hence the above show that the quarter symmetric connection $\overline{D}$ is non-metric on (M^3,g). *This verifies Theorem* 3.1.

In view of (8.9), we obtain

$$S(e_1,e_1)=8,\qquad S(e_1,e_2)=8,\qquad S(e_2,e_2)=-8. \tag{8.10}$$

Similarly, we can obtain the non-vanishes components of the curvature tensor $\overline{R}$ and Ricci tensor $\overline{S}$ with respect to quarter symmetric non-metric connection $\overline{D}$ on a $(k,\mu)-$contact metric manifold as

$$\left.\begin{array}{lll}\overline{R}(e_1,e_2)e_3=0, & \overline{R}(e_2,e_3)e_3=-3e_2, & \overline{R}(e_1,e_3)e_3=-3e_1,\\ \overline{R}(e_1,e_2)e_2=6e_1, & \overline{R}(e_2,e_3)e_2=-6e_3, & \overline{R}(e_1,e_3)e_2=0,\\ \overline{R}(e_1,e_2)e_1=-6e_2, & \overline{R}(e_2,e_3)e_1=0, & \overline{R}(e_1,e_3)e_1=-6e_3,\end{array}\right\} \tag{8.11}$$

and

$$\overline{S}(e_1,e_1)=12,\qquad \overline{S}(e_1,e_2)=12,\qquad \overline{S}(e_2,e_2)=-6. \tag{8.12}$$

Since $\{e_1,e_2,e_3\}$ forms a basis of non-Sasakaian $(k,\mu)-$contact metric manifold any vector field $X,Y,Z\in\chi(M)$ can be written as

$$X=a_1e_1+b_1e_2+c_1e_3,\quad Y=a_2e_1+b_2e_2+c_2e_3,$$

where $a_i,b_i,c_i\in\boldsymbol{R}^+$ (the set of all positive real numbers), $i=1,2,3$.
This implies that

$$S(X,Y)=8(a_1a_2+b_1b_2-c_1c_2),\qquad g(X,Y)=a_1a_2+b_1b_2+c_1c_2, \tag{8.13}$$

and

$$\overline{S}(X,Y)=8(a_1a_2+b_1b_2-c_1c_2). \tag{8.14}$$

In view of (8.7), (8.12) and (8.14), we get

$$\left.\begin{array}{l}(\overline{D}_{e_1}\overline{S})=-24(a_1c_2+a_2c_1),\\ (\overline{D}_{e_2}\overline{S})=-24(b_1c_2+b_2c_1),\\ (\overline{D}_{e_3}\overline{S})=-24(a_1a_2+b_1b_2).\end{array}\right\} \tag{8.15}$$

Consequently, the manifold under consideration is neither Ricci symmetric nor Ricci recurrent. Let us now consider 1-forms A and B non vanishes

$$A(e_1) = \frac{-24(6k+1)(a_1c_2 + a_2c_1)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1}$$
$$B(e_1) = \frac{24(a_1c_2 + a_2c_1)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1}$$
$$A(e_2) = \frac{-24(6k+1)(b_1c_2 + b_2c_1)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1}$$
$$B(e_2) = \frac{24(b_1c_2 + b_2c_1)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1}$$
$$A(e_3) = \frac{-24(6k+1)(a_1a_2 + b_1b_2)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1}$$
$$B(e_3) = \frac{24(a_1a_2 + b_1b_2)}{6(k+1)(2a_1a_2 + 2b_1b_2 - c_1c_2) - 1},$$

where $k = 4$, at any point $x \in M$. From (1.3), we have

$$\left(\overline{D}_{e_j}\overline{S}\right)(X,Y) = [A(e_j) + B(e_j)]\overline{S}(X,Y) + B(e_j)g(X,Y), \quad j = 1,2,3. \tag{8.16}$$

It can be easily seen that the manifold with 1-forms satisfies relation (8.16).
Hence the manifold under consideration is a nearly Ricci recurrent $(k,\mu)-$contact metric manifold (M^3, g) with respect to quarter symmetric non-metric connection, which is neither Ricci recurrent nor Ricci symmetric. Thus we have the following theorem:

Theorem 8.1. *There exist a nearly Ricci recurrent $(k,\mu)-$contact metric manifold (M^3, g) with respect to quarter symmetric non-metric connection $\overline{D}$, which is neither Ricci recurrent nor Ricci symmetric.*

9. Application

We will continue the research on this quarter symmetric non-metric connection on the generalized $(k,\mu)-$contact metric manifold, (k,μ)-paracontact metric manifold, almost Kenmotsu manifold with nullity distribution and deal with the application of the results obtained in this paper.

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