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ON *h*-RANDERS EXPONENTIAL CHANGE OF FINSLER METRIC

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Abstract: Studying an (α, β)-metrics is a central idea in Finsler geometry, which is a generalization of Randers metric. In this paper, we have derived the Cartan connection for the Finsler space whose metric is given by h-Randers exponential change and also obtained the condition under which the Finslerian hypersurface to be hyperplane of first, second and third kind.

Keywords and Phrases: Finsler space, hypersurface, Randers change, exponential change, h-vector.

2020 Mathematics Subject Classification: 53B40.

1. Introduction

Nearly four decades ago, C. Shibata [17] introduced the idea of β-change in Finsler geometry. Randers change, Matsumoto change, exponential change, and Kropina change are very important example of β-change. Among them, exponential change is one of the interesting examples with $F = Le^{\beta/\alpha}$, where $\beta = b_j(x)y^j$ is 1-form and $\alpha = (a_{jk}(x)y^j y^k)^{1/2}$ is a Riemannian metric in the manifold M^n. In 2006, a Finsler space with metric function determined by exponential change has been studied by Yu Yao-Yong and You Ying [20]. In 2013, G. Shankar *et al.* [14] discussed Randers change of exponential metric. The first approximation of exponential change has been studied by T. N. Pandey *et al.* [12]. In 2016 Gupta and Gupta [2] have discussed h-exponential change and also obtained hypersurface for the exponential change of Finsler metric with an h-vector, given by $\overline{L} = Le^{\beta/L}$

[3], where $\beta = b_j(x,y)y^j$ is one form and b_j is an h-vector. The notion of h-vector $b_j(x,y)$ was introduced by H. Izumi [8], which is v-covariant constant concerning the Cartan connection and satisfies $LC^h_{jk}b_h = \rho h_{jk}$, where ρ is a non-zero scalar function and C^h_{jk} is component of Cartan tensor. Thus from the above definition of h-vector, we have $L\dot{\partial}_k b_j = \rho h_{jk}$, which shows that b_j is a function of direction also. Many geometers [2-5, 18] have studied the property of h-vector in Finsler geometry.

The notion of hypersurface in the Finslerian manifold has been initiated by E. Cartan [1]. Further, A. Rapcsàk [13] defined three different kinds of hypersurfaces and M. Matsumoto [11] has categorised them. A hypersurface is an $(n-1)$-dimensional manifold, embedded in an ambient space of dimension n. In 2002 M. Kitayama [9] studied a Finslerian hypersurface given by β-change. Later on many geometers [6, 7, 15, 16, 19] discussed the geometric property of the hypersurface.

In this paper, we have introduced h-Randers exponentially change of Finsler metric defined by

$$\overline{L} = Le^{\beta/L} + \beta, \tag{1.1}$$

where $\beta = b_j(x,y)y^j$ and b_j is an h-vector.

This paper is structured as follows: We obtain the following in section 2:

(i) The fundamental geometric properties of Finsler space, with metric (1.1).

(ii) The relation between the Cartan connection coefficients for both spaces F^n and $\overline{F}^n$.

In section 3, we have obtained the condition under which the hypersurface is the hyperplane of the first, second, and third kind.

The terminologies and notations are referred to Matsumoto [10].

2. The Finsler Space $\overline{F}^n$ with h-Randers Exponential Change

Let $F^n = (M^n, L)$ is an n-dimensional Finsler space with the fundamental function $L(x,y)$. The normalized supporting element, angular metric tensor, metric tensor, and Cartan tensor are defined by $l_j = \dot{\partial}_j L$, $h_{jk} = L\dot{\partial}_j\dot{\partial}_k L$, $g_{jk} = \frac{1}{2}\dot{\partial}_j\dot{\partial}_k L^2$ and $C_{jsk} = \frac{1}{2}\dot{\partial}_k g_{js}$ respectively. The Cartan connection in F^n is defined as $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$.

The Finsler space $\overline{F}^n = (M^n, \overline{L})$ with the basic function $\overline{L}(x,y)$ is defined by equation (1.1), where $\beta = b_j(x,y)y^j$, b_j is h-vector defined as

$$(a)\ \ b_j|_s = 0, \qquad (b)\ \ LC^h_{js}b_h = \rho h_{js}, \quad \rho \neq 0\,. \tag{2.1}$$

As a result of the above definition, we have

$$L\dot{\partial}_k b_j = \rho h_{jk}. \tag{2.2}$$

A bar over the quantity in this paper designates the geometric objects that correspond to the function $\overline{F}^n$. We have derived the normalized supporting element as well as the angular metric tensor of $\overline{F}^n$ as

$$\bar{l}_i = (\tau + e^\tau) l_i + (1 + e^\tau) m_i, \tag{2.3}$$

$$\overline{h}_{ij} = (\tau + e^\tau)\Big\{ \left[e^\tau(1 + \rho - \tau)\right] h_{ij} + e^\tau m_i m_j \Big\}, \tag{2.4}$$

where $m_i = b_i - \tau l_i$ and $\tau = \frac{\beta}{L}$.

For the transformed space $\overline{F}^n$ the metric tensor $\overline{g}_{ij}$ and the Cartan tensor $\overline{C}_{ijk}$ are derived as follows.

$$\overline{g}_{ij} = q g_{ij} + q_1 l_i l_j + q_2(l_i m_j + l_j m_i) + q_3 m_i m_j \tag{2.5}$$

and

$$\overline{C}_{ijk} = q C_{ijk} + U_{ijk}, \tag{2.6}$$

where

$$U_{ijk} = U_1(h_{ij} m_k + h_{ik} m_j + h_{jk} m_i) + U_2 m_i m_j m_k,$$

$$q = (\tau + e^\tau)\left[(\rho + 1 - \tau)e^\tau + \rho\right], \quad q_1 = (e^\tau + \tau)(\tau - \rho)(1 + e^\tau),$$

$$q_2 = (\tau + e^\tau)(1 + e^\tau) \quad q_3 = (2e^{2\tau} + \tau e^\tau + 2e^\tau + 1), \quad U_1 = \frac{1}{2L}[q_2 + q_3(\rho - \tau)],$$

$$U_2 = \frac{e^\tau}{2L}(4e^\tau + \tau + 3).$$

In $\overline{F}^n$, the inverse metric tensor $\overline{g}^{ij}$ is computed as

$$\overline{g}^{ij} = p g^{ij} + p_1 l^i l^j + p_2(l^i m^j + l^j m^i) + p_3 m^i m^j, \tag{2.7}$$

where

$$p = \frac{1}{q}, \quad p_1 = -\frac{(q^2 q_1 - q_2^2 q_3)(q + q_3 m^2)\left[q^2(q - q_1) - q_2^2 q_3\right]^2 + q^4 q_2^2 q_3 (q^3 - 2q_2^2 q_3)}{q\left[q^2(q - q_1) - q_2^2 q_3\right]\Big\{(q + q_3 m^2)\left[q^2(q - q_1) - q_2^2 q_3\right]^2 + q^4 q_2^2 q_3\Big\}},$$

$$p_2 = \frac{-q q_2 q_3\left[q^2(q - q_1) - q_2^2 q_3\right]}{(q + q_3 m^2)\left[q^2(q - q_1) - q_2^2 q_3\right]^2 + q^4 q_2^2 q_3},$$

$$p_3 = \frac{-q_3\left[q^2(q - q_1) - q_2^2 q_3\right]^2}{q(q + q_3 m^2)\left[q^2(q - q_1) - q_2^2 q_3\right]^2 + q^4 q_2^2 q_3}.$$

The Christoffel symbol for the Finsler space $\overline{F}^n$ is defined as

$$\overline{\gamma}_{jsk} = \frac{1}{2}\left\{\partial_j \overline{g}_{sk} - \partial_s \overline{g}_{jk} + \partial_k \overline{g}_{js}\right\} \tag{2.8}$$

Differentiating equation (2.5) with respect to x^k, we obtain

$$\begin{aligned}\partial_k \overline{g}_{ij} =& \, q\,\partial_k g_{ij} + q_2 \rho_k h_{ij} + 2(U_1 h_{ij} + U_2 m_i m_j)(\beta_k + N^r_k m_r)\\ &+ q_2(l_i b_{j|k} + l_j b_{i|k} + m_r F^r_{jk} l_i + m_r F^r_{ik} l_j + m_i F^r_{jk} l_r + m_j F^r_{ik} l_r)\\ &+ q_1(l_i l_r F^r_{jk} + l_j l_r F^r_{ik}) + 2U_1(h_{jr} N^r_k m_i + h_{ir} N^r_k m_j)\\ &+ q_3(m_i b_{j|k} + m_j b_{i|k} + m_i m_r F^r_{jk} + m_j m_r F^r_{ik})\,,\end{aligned} \tag{2.9}$$

where we have used the following notations

$$\rho_k = \rho_{|k} = \partial_k \rho \quad \text{and} \quad \beta_k = \beta_{|k}.$$

The coefficient of the Christoffel symbol is derived by applying the Christoffel process to the indices $i\,, j\,, k$ in equation (2.9), and then putting these values in equation (2.8), we get

$$\begin{aligned}\overline{\gamma}_{ijk} =& \, q\gamma_{ijk} + \mathfrak{A}_{ijk}\left\{\frac{q_2}{2}\rho_k h_{ij} + (\beta_k + N^r_k m_r)\, B_{ij} + U_1(h_{jr} m_i + h_{ir} m_j) N^r_k\right\} + Q_i F_{jk}\\ &+ Q_k F_{ji} + Q_j E_{ik} + (\overline{g}_{rj} - q\, g_{rj})\Big\{\gamma^r_{ik} + g^{rt}(C_{ikm} N^m_t - C_{tkm} N^m_i - C_{itm} N^m_k)\Big\},\end{aligned} \tag{2.10}$$

where the symbol $\mathfrak{A}_{ijk}$ is defined as $\mathfrak{A}_{ijk} A_{ijk} = A_{ijk} - A_{jki} + A_{kij}$ and we have used the following notations

$$Q_i = q_2 l_i + q_3 m_i\,, \quad B_{ij} = U_1 h_{ij} + U_2 m_i m_j\,, \tag{2.11}$$

$$2F_{js} = b_{j|s} - b_{s|j}, \qquad 2E_{js} = b_{j|s} + b_{s|j}.$$

The transformed Christoffel symbol is defined as

$$\overline{\gamma}^i_{jk} = \frac{1}{2}\overline{g}^{ir}\left\{\partial_j \overline{g}_{kr} + \partial_k \overline{g}_{rj} - \partial_r \overline{g}_{jk}\right\}\,.$$

Transvecting equation (2.10) by (2.7) we obtain

$$\begin{aligned}\overline{\gamma}^i_{jk} =& \, q\gamma^i_{jk} + (\overline{g}_{rj} - q\, g_{rj})\Big\{\gamma^r_{ik} + g^{rt}(C_{ikm} N^m_t - C_{tkm} N^m_i - C_{itm} N^m_k)\Big\} + \overline{g}^{is}\Bigg\{\mathfrak{S}_{ijk}\\ &\left\{\frac{q_2}{2}\rho_k h_{ij} + (\beta_k + N^r_k m_r)\, B_{ij} + U_1(h_{jr} m_i + h_{ir} m_j) N^r_k\right\} + Q_i F_{jk} + Q_k F_{ji} + Q_j E_{ik}\Bigg\}.\end{aligned} \tag{2.12}$$

By using $\overline{G}^i = \frac{1}{2}\overline{\gamma}^i_{jk} y^j y^k$, we obtain the spray coefficient $\overline{G}^i$ as follows

$$\overline{G}^i = G^i + D^i, \tag{2.13}$$

where

$$D^i = \frac{1}{2}\,\overline{g}^{is}\left\{2Lq_2F_{s0} + (q_2l_s + q_3m_s)E_{00}\right\}. \tag{2.14}$$

Differentiating equation (2.13) with respect to y^j, we obtain the Nonlinear connection $\overline{N}^i_j$ as follows

$$\overline{N}^i_j = N^i_j + D^i_j, \tag{2.15}$$

where

$$D^i_j = \overline{g}^{ir}\left\{-2D^m(qC_{mrj} + U_{mrj}) + Q_rE_{0j} + Q_jF_{r0} + q_2LF_{rj} + \frac{1}{2}\rho_k y^k q_2 h_{rj} + E_{00}B_{jr}\right\}. \tag{2.16}$$

The Cartan connection coefficient of the transformed Finsler space $\overline{F}^n$ is defined as

$$\overline{F}^i_{jk} = \overline{\gamma}^i_{jk} + \overline{g}^{it}\big(\overline{C}_{jkr}\overline{N}^r_t - \overline{C}_{tkr}\overline{N}^r_j - \overline{C}_{jtr}\overline{N}^r_k\big). \tag{2.17}$$

After plugging the values of (2.6), (2.7), (2.12), (2.15) in the above equation and calculating, we obtain the relation between the Cartan connection coefficients for both the spaces F^n and $\overline{F}^n$ as follows

$$\overline{F}^i_{jk} = F^i_{jk} + D^i_{jk}, \tag{2.18}$$

where

$$\begin{aligned} D^i_{jk} = \overline{g}^{is}\Big\{&Q_jF_{sk} + Q_sE_{kj} + Q_kF_{js} + q\left(C_{jkm}D^m_s - C_{skm}D^m_j - C_{jsm}D^m_k\right) \\ &+ U_{jkm}D^m_s - U_{skm}D^m_j - U_{jsm}D^m_k + B_{js}\beta_k - B_{jk}\beta_s + B_{sk}\beta_j \\ &+ \frac{q_2}{2}\left(\rho_k h_{js} - \rho_s h_{jk} + \rho_j h_{sk}\right)\Big\}. \end{aligned} \tag{2.19}$$

The contraction by y^k is denoted as zero '0' in subscript, for instance, $F_{jk}y^k = F_{j0}$. Thus, we have

Theorem 2.1. *The relation between the Cartan connection coefficients for both the spaces F^n and $\overline{F}^n$, for the h-Randers exponential change of Finsler space, is given by (2.18).*

3. Hypersurface $\overline{F}^{n-1}$ of the Transformed Finsler Space $\overline{F}^{n}$

The hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the Finsler space $F^n = (M^n, L)$ is given by the equation $x^i = x^i(u^\alpha)$, where $\alpha = 1, 2 \ldots\ldots n-1$.
The supporting element y^i at a point $u = (u^\alpha)$ of M^{n-1} is assumed to be tangent to M^{n-1}, *i.e.*

$$y^i = B^i_\alpha(u) v^\alpha,$$

where $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is the matrix of projection factors of rank $n-1$ can be assumed as the components of linearly independent vectors that are tangent to F^{n-1}.
At every point u^α of F^{n-1}, a unit normal vector B^i is defined as [11],

$$g_{ij} B^i B^j = 1 \quad \text{and} \quad g_{ij} B^j B^i_\alpha = 0. \tag{3.1}$$

The induced metric tensor $g_{\alpha\beta}$ and induced Cartan tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given as follows [11]:

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \qquad \text{and} \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma.$$

Now we obtain the condition under which the hypersurface for the transformed Finsler space $\overline{F}^{n}$ to be the hyperplane of the first, second and third kind.
Let $\overline{F}^{n-1} = (M^{n-1}, \underline{L}(u,v))$ be a Finslerian hypersurface of the transformed Finsler space $\overline{F}^{n}$. The unit normal vector $\overline{B}^i(u,v)$ of $\overline{F}^{n-1}$ is uniquely identified as

$$\overline{g}_{ij} B^i_\alpha \overline{B}^j = 0, \quad \overline{g}_{ij} \overline{B}^i \overline{B}^j = 1. \tag{3.2}$$

$\overline{B}^\alpha_i$ is the inverse projection factor of $\overline{B}^i_\alpha$, is uniquely defined by

$$\overline{B}^\alpha_i = g_{ij} \overline{g}^{\alpha\beta} B^j_\beta, \tag{3.3}$$

where $\overline{g}^{\alpha\beta}$ is the inverse metric tensor of the metric tensor $\overline{g}_{\alpha\beta}$ along $\overline{F}^{n-1}$. In view of the above equation and (3.2), it follow that

$$\overline{B}^i_\alpha \overline{B}^\beta_i = \delta^\beta_\alpha, \quad \overline{B}^i_\alpha \overline{B}_i = 0, \quad \overline{B}^i \overline{B}^\alpha_i = 0, \quad \overline{B}^i \overline{B}_i = 1. \tag{3.4}$$

Transvecting equation (3.1) by v^α and using $B^i_\alpha v^\alpha = y^i$, we obtain

$$y_j B^j = 0. \tag{3.5}$$

Contracting equation (2.5) by $B^i B^j$ and using (3.2) and (3.5), we get

$$\overline{g}_{ij} B^i B^j = q + q_3 (B^i m_i)^2, \tag{3.6}$$

which demonstrates that $B^i/\sqrt{q+q_3(m_iB^i)^2}$ is a unit normal vector. Equation (2.5) is again contracting by $B^i_\alpha B^j$ and using (3.2), (3.5), we obtain

$$\overline{g}_{ij}B^i_\alpha B^j = (q_2l_i + q_3m_i)B^i_\alpha\,(B^jm_j)\,, \tag{3.7}$$

which demonstrates that the vector B^j is normal to $\overline{F}^{n-1}$ if and only if

$$(q_2l_i + q_3m_i)B^i_\alpha(B^jm_j) = 0\,.$$

This implies that at least one of the following condition is correct.

$$(i)\quad B^i_\alpha(q_2l_i + q_3m_i) = 0 \qquad\qquad (ii)\quad B^jm_j = 0\,.$$

Transvecting the condition (i) by v^α gives $L = 0$, which is not possible. Therefore the condition (ii) holds, *i.e.*

$$B^jm_j = 0\,, \tag{3.8}$$

In view of (3.5), the above equation can be equivalently written as

$$B^jb_j = 0. \tag{3.9}$$

This proves that the vector B^j is normal to $\overline{F}^{n-1}$ if and only if the h-vector b_j is tangent to the Finsler space $\overline{F}^{n-1}$. According to equations (3.6), (3.7) and (3.9), $B^i/\sqrt{q}$ is a unit normal vector of $\overline{F}^{n-1}$ *i.e.*

$$\overline{B}^i = \frac{B^i}{\sqrt{q}}\,, \tag{3.10}$$

which gives

$$\overline{B}_i = \overline{g}_{ij}\overline{B}^j = \sqrt{q}\,B_i\,. \tag{3.11}$$

Thus, we have

Theorem 3.1. *Let $\overline{F}^n$ is obtained by the h-Randers exponential change (1.1) from F^n. If $\overline{F}^{n-1}$ is the hypersurface of the space $\overline{F}^n$ then the h-vector b_j is tangential to the hypersurface F^{n-1} if and only if each vector normal to F^{n-1} is also normal to $\overline{F}^{n-1}$.*

In view of the equation $g_{ij}B^iB^j = 1$, $g_{ij}B^jB^i_\alpha = 0$, (3.5), and the definition of the angular metric tensor $h_{ij} = g_{ij} - l_il_j$, we get

$$h_{ij}B^j_\alpha B^i = 0, \quad h_{ij}B^i = B_j. \tag{3.12}$$

The tensors B_{ij} and Q_i are given by (2.11) and satisfies the relations

$$B_{ij}B^iB^j_\alpha = 0, \quad B_{ij}B^i = B_j, \quad Q_jB^j = 0. \tag{3.13}$$

Transvecting (2.7) by B_i and using $l^iB_i = 0 = m^iB_i$, we get

$$\overline{g}^{is}B_i = pB^s. \tag{3.14}$$

For the hypersurface F^{n-1}, the normal curvature H_α is defined as [11], $H_\alpha = B_i(N^i_jB^j_\alpha + B^i_{0\alpha})$. In view of equation (2.15) and (3.11), we obtain the normal curvature $\overline{H}_\alpha$ for the hypersurface $\overline{F}^{n-1}$ as

$$\overline{H}_\alpha = \sqrt{q}(H_\alpha + B_iD^i_jB^j_\alpha).$$

Contracting the above equation by v^α and using $v^\alpha B^k_\alpha = y^k$, we get

$$\overline{H}_0 = \sqrt{q}(H_0 + B_iD^i). \tag{3.15}$$

Transvecting equation (2.14) and using $m_iB^i = 0$, $l_iB^i = 0$, we obtain

$$B_iD^i = pq_2LF_{i0}B^i. \tag{3.16}$$

Let the h-vector b_j be gradient, *i.e.* $b_{j|k} = b_{k|j}$, then

$$F_{jk} = 0. \tag{3.17}$$

In 2015, the following Lemma was demonstrated by M. K. Gupta and P. N. Pandey [4],

Lemma 1. *If the h-vector b_j is gradient then the scalar ρ is constant.*

In view of the above Lemma, we have

$$\rho_j = 0. \tag{3.18}$$

From equation (3.17), the equation (3.16) becomes

$$B_iD^i = 0. \tag{3.19}$$

and then equation (3.15) reduces to

$$\overline{H}_0 = \sqrt{q}H_0. \tag{3.20}$$

M. Matsumoto [11] has categorised the hypersurface to be hyperplane of first kind as *"A hypersurface F^{n-1} is a first kind hyperplane if and only if $H_\alpha = 0$ or equivalently*

$H_0 = 0$."

Thus, we have

Theorem 3.2. *Let the h-vector $b_i(x,y)$ be gradient and tangent to the hypersurface F^{n-1}, for the h-Randers exponential change. Then the hypersurface F^{n-1} is a first kind hyperplane if and only if the hypersurface $\overline{F}^{n-1}$ is a first kind hyperplane.*

The second fundamental h-tensor $H_{\alpha\beta}$ for the hypersurface F^{n-1} is defined as [11], $H_{\alpha\beta} = M_\alpha H_\beta + B_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta)$, which on contraction by v^β, gives

$$H_{\alpha 0} = H_{\alpha\beta}v^\beta = M_\alpha H_0 + H_\alpha, \qquad H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha. \tag{3.21}$$

where $M_\alpha = C_{ijk}B^i_\alpha B^j B^k$.

In view of equation (2.18) and (3.11), the second fundamental h-tensor $\overline{H}_{\alpha\beta}$ for hyperplane $\overline{F}^{n-1}$ is given as

$$\overline{H}_{\alpha\beta} - \overline{M}_\alpha \overline{H}_\beta = \sqrt{q}(H_{\alpha\beta} + D^i_{jk}B_iB^j_\alpha B^k_\beta) - \sqrt{q}M_\alpha H_\beta. \tag{3.22}$$

Using (3.17) and (3.18), equation (2.19) reduces to

$$\begin{aligned} D^i_{jk} = \overline{g}^{is}\Big\{ & Q_sE_{kj} + qC_{jkm}D^m_s + U_{jkm}D^m_s - qC_{skm}D^m_j - U_{skm}D^m_j \\ & - qC_{jsm}D^m_k - U_{jsm}D^m_k + B_{js}\beta_k + B_{sk}\beta_j - B_{jk}\beta_s\Big\}. \end{aligned} \tag{3.23}$$

Transvecting above equation by $B_iB^j_\alpha B^k_\beta$ and using $\overline{g}^{ij}B_j = pB^i$,
$B^sQ_s = 0\,,\ B_{sk}B^sB^k_\beta = 0\,,$
we get

$$\begin{aligned} D^i_{jk}B_iB^j_\alpha B^k_\beta = pB^sB^j_\alpha B^k_\beta\Big\{ & qC_{jkm}D^m_s + U_{jkm}D^m_s - qC_{skm}D^m_j - U_{skm}D^m_j \\ & - qC_{jsm}D^m_k - U_{jsm}D^m_k - B_{jk}\beta_s\Big\}. \end{aligned} \tag{3.24}$$

Equation (2.14) can be rewritten as

$$D^i = \frac{1}{2}\Big\{\big[(p+p_1)q_2+p_2q_3m^2\big]E_{00}+2p_2q_2LF_{\beta 0}\Big\}l^i+\frac{1}{2}\Big\{\mu E_{00}+2p_3q_2LF_{\beta 0}\Big\}m^i+pq_2F^i_0L, \tag{3.25}$$

where $\mu = (pq_3 + p_2q_2 + p_3q_3m^2)$ and $F_{\beta 0} = F_{s0}m^s$. In view of the above equation and applying the indicatory property of U_{jsk}, C_{jsk}, m_j, h_{js}, the equation (2.16) can be transformed as

$$D^m_s = \overline{g}^{mr}\Big\{\lambda h_{rs} + Q_rE_{s0} + \phi m_r m_s\Big\}, \tag{3.26}$$

where

$$\lambda = \left[-\mu\left(\frac{q\rho}{L} + m^2U_1\right) + U_1\right] E_{00} \quad \text{and} \qquad \phi = \left[-\mu\left(2U_1 + U_2m^2\right) + U_2\right] E_{00}. \tag{3.27}$$

Transvecting the equation (3.26) by C_{jkm}, we get

$$C_{jkm}D^m_s = C_{jkm}\Big\{pg^{mr}+p_1l^ml^r+p_2\left(l^mm^r + l^rm^m\right)+p_3m^mm^r\Big\}\Big\{\lambda h_{rs}+Q_rE_{s0}+\phi m_rm_s\Big\}, \tag{3.28}$$

which can be simplified as

$$C_{jkm}D^m_s = p\lambda C_{jsk} + \frac{\rho}{L}\mu h_{jk}E_{s0} + \Big[\left(p + p_3m^2\right)\phi + p_3\lambda\Big]\frac{\rho}{L}h_{jk}m_s. \tag{3.29}$$

Similarly we can write the expressions for $C_{skm}D^m_j$ and $C_{jsm}D^m_k$ as

$$C_{skm}D^m_j = p\lambda C_{jsk} + \frac{\rho}{L}\mu h_{sk}E_{j0} + \Big[\left(p + p_3m^2\right)\phi + p_3\lambda\Big]\frac{\rho}{L}h_{sk}m_j, \tag{3.30}$$

and

$$C_{jsm}D^m_k = p\lambda C_{jsk} + \frac{\rho}{L}\mu h_{sj}E_{k0} + \Big[\left(p + p_3m^2\right)\phi + p_3\lambda\Big]\frac{\rho}{L}h_{sj}m_k. \tag{3.31}$$

Transvecting equations (3.29), (3.30) and (3.31) by $B^sB^j_\alpha B^k_\beta$, and using (3.12), we get respectively

$$B^sC_{jkm}D^m_sB^j_\alpha B^k_\beta = p\lambda M_{\alpha\beta} + \frac{\rho}{L}\mu h_{jk}B^j_\alpha B^k_\beta B^sE_{s0}\,, \tag{3.32}$$

$$B^sC_{skm}D^m_jB^j_\alpha B^k_\beta = p\lambda M_{\alpha\beta}\,, \tag{3.33}$$

$$B^sC_{jsm}D^m_kB^j_\alpha B^k_\beta = p\lambda M_{\alpha\beta}, \tag{3.34}$$

where $M_{\alpha\beta}$ is the second fundamental v-tensor for the hypersurface F^{n-1} is defined as [11]

$$M_{\alpha\beta} = C_{ijk}B^j_\alpha B^k_\beta B^i. \tag{3.35}$$

Again, transvecting Equation (3.26) by U_{jkm}, we get

$$U_{jkm}D^m_s = U_{jkm}\Big\{pg^{mr} + \left(p_2l^r + p_3m^r\right)m^m\Big\}\Big\{\lambda h_{rs} + Q_rE_{s0} + \phi m_rm_s\Big\}. \tag{3.36}$$

U_{ijk} is an indicatory tensor and satisfies;

(i) $U_{ijk}m^i = (2U_1 + U_2m^2)m_jm_k + U_1m^2h_{jk}$,
(ii) $U_{ijk}g^{ir} = U_1\left(h_{jk}m^r + h^r_km_j + h^r_jm_k\right) + U_2m_jm_km^r$,

where $h_k^r = h_{ik} g^{ir}$.
By using the property of indicatory tensor U_{ijk}, equation (3.36) can be rewritten as

$$U_{jkm} D_s^m = \Big[p \left\{ U_1 [h_{jk} m^r + h_j^r m_k + h_k^r m_j] + U_2 m_j m_k m^r \right\} + (p_2 l^r + p_3 m^r) \\ \left[(2U_1 + U_2 m^2) m_j m_k + U_1 m^2 h_{jk} \right] \Big] \Big\{ \lambda h_{sr} + \phi m_s m_r + Q_r E_{s0} \Big\},$$

which can be simplified as

$$U_{jkm} D_s^m = \Big\{ \psi_1 U_1 h_{jk} + (\psi_1 U_2 + 2\psi_2 U_1) m_j m_k \Big\} m_s + q\lambda U_1 \left(h_{js} m_k + h_{sk} m_j \right) \\ + \Big\{ \mu \left[m^2 U_1 h_{jk} + (2U_1 + U_2 m^2) m_j m_k \right] \Big\} E_{s0}, \tag{3.37}$$

where

$$\psi_1 = \left(\lambda + \phi m^2 \right) \left(p + p_3 m^2 \right) \quad \text{and} \quad \psi_2 = \left(\lambda + \phi m^2 \right) p_3 + p\phi.$$

Similarly we can write the expression for $U_{skm} D_j^m$ and $U_{jsm} D_k^m$ as

$$U_{skm} D_j^m = \Big\{ \psi_1 U_1 h_{sk} + (\psi_1 U_2 + 2\psi_2 U_1) m_s m_k \Big\} m_j + q\lambda U_1 \left(h_{js} m_k + h_{jk} m_s \right) \\ + \Big\{ \mu \left[m^2 U_1 h_{sk} + (2U_1 + U_2 m^2) m_s m_k \right] \Big\} E_{j0}, \tag{3.38}$$

and

$$U_{jsm} D_k^m = \Big\{ \psi_1 U_1 h_{sj} + (\psi_1 U_2 + 2\psi_2 U_1) m_j m_s \Big\} m_k + q\lambda U_1 \left(h_{sk} m_j + h_{jk} m_s \right) \\ + \Big\{ \mu \left[m^2 U_1 h_{sj} + (2U_1 + U_2 m^2) m_s m_j \right] \Big\} E_{k0}. \tag{3.39}$$

Contracting equation (3.37), (3.38), (3.39) by $B^s B_\alpha^j B_\beta^k$ and using $B^i m_i = 0 = h_{ij} B^i B_\alpha^j$, we get respectively

$$B^s U_{jkm} D_s^m B_\alpha^j B_\beta^k = \mu \left\{ U_1 m^2 h_{jk} + (2U_1 + U_2 m^2) m_j m_k \right\} B_\alpha^j B_\beta^k B^s E_{s0}, \tag{3.40}$$

$$B^s U_{skm} D_j^m B_\alpha^j B_\beta^k = 0, \tag{3.41}$$

$$B^s U_{jsm} D_k^m B_\alpha^j B_\beta^k = 0. \tag{3.42}$$

Using equations (3.32), (3.33), (3.34), (3.40), (3.41) and (3.42) in equation (3.24), we get

$$D_{jk}^i B_i B_\alpha^j B_\beta^k = \left[\frac{\mu\rho}{L} - p \left\{ \left[(2U_1 + U_2 m^2) m_j m_k \right] + U_1 m^2 h_{jk} \right\} + B_{jk} \right] B_\alpha^j B_\beta^k B^s E_{s0} - p\lambda M_{\alpha\beta}. \tag{3.43}$$

The relative h- derivative of B^i_α and the normal vector B^i are given by [11]

$$B^i_{\alpha|\beta} = H_{\alpha\beta} B^i, \quad B^i{}_{|\beta} = -H_{\alpha\beta} B^\alpha_j g^{ij}. \tag{3.44}$$

Consider the h-covariant differentiation of $B^i b_i = 0$ with respect to the Cartan connection of the hypersurface F^{n-1}, we obtain

$$B^i b_{i|j} + b_i B^i_{|\beta} = 0\,. \tag{3.45}$$

In view of equation (3.44), equation (3.45) becomes

$$\left(b_{i|j} B^j H_\beta + b_{i|j} B^j_\beta \right) B^i - b_i H_{\alpha\beta} B^\alpha_j g^{ij} = 0,$$

Contracting the above equation by v^β and using (3.21), gives

$$B^i b_{i|0} = (H_\alpha + M_\alpha H_0) B^\alpha_j b^j - b_{i|j} H_0 B^i B^j.$$

If the hypersurface is first kind hyperplane then $H_0 = 0 = H_\alpha$. Thus the above equation reduces to $B^i b_{i|0} = 0$. The h-vector b_j is gradient, *i.e.* $b_{j|s} = b_{s|j}$, then we have

$$E_{s0} B^s = b_{s|0} B^s = 0. \tag{3.46}$$

Therefore equation (3.43) reduces to

$$D^i_{jk} B_i B^j_\alpha B^k_\beta = -p\lambda M_{\alpha\beta}. \tag{3.47}$$

In view of the above equation and (3.22), we get

$$\overline{H}_{\alpha\beta} - \overline{M}_\alpha \overline{H}_\beta = \sqrt{q}(H_{\alpha\beta} - p\lambda M_{\alpha\beta}) - \sqrt{q} M_\alpha H_\beta. \tag{3.48}$$

Now transvecting (2.6) by $B^i_\alpha B^j_\beta B^k$ and in view of equations (3.9) and (3.12), we obtain

$$\overline{C}_{ijk} B^i_\alpha B^j_\beta B^k = q C_{ijk} B^i_\alpha B^j_\beta B^k. \tag{3.49}$$

From equation (3.7) and (3.35), equation (3.49) may be written as

$$\overline{M}_{\alpha\beta} = \sqrt{q} M_{\alpha\beta}. \tag{3.50}$$

M. Matsumoto [11] has categorised the hypersurface to be hyperplane of second kind as *"A hypersurface F^{n-1} is a second kind hyperplane if and only if $H_{\alpha\beta} = 0$"* and the hypersurface to be hyperplane of third kind as *"A hypersurface F^{n-1} is a hyperplane of third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$."*
Thus, we have

Theorem 3.3. *Let the h-vector b_j is gradient and tangential to the hypersurface F^{n-1} for the h-Randers exponential change and satisfies the condition (3.46). Then* *(i) If F^{n-1} is a second kind hyperplane with $M_{\alpha\beta} = 0$ then $\overline{F}^{n-1}$ is also second kind hyperplane.*

(ii) If F^{n-1} is a third kind hyperplane then $\overline{F}^{n-1}$ is also third kind hyperplane.

References

[1] Cartan E., Les espaces de Finsler, Actualités Scientifiques et Industrielles, 79, Herman, Paris, 1934.

[2] Gupta M. K. and Gupta A. K., h-exponential change of Finsler metric, Facta Universitatis, Series: Mathematics and Informatics, 31(5) (2016), 1029–1039.

[3] Gupta M. K. and Gupta A. K., Hypersurface of a Finsler space subjected to an h-exponential change of metric, International Journal of Geometric Methods in Modern Physics, 13(10) (2016), 1650129.

[4] Gupta M. K. and Pandey P. N., Finsler space subjected to a Kropina change with an h-vector, Facta universitatis-series: Mathematics and Informatics, 30 (4) (2015), 513–525.

[5] Gupta M. K. and Pandey P. N., Hypersurface of a Finsler Space Subjected to a Kropina Change with an h-Vector, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 88(2) (2018), 241-246.

[6] Gupta M. K. and Pandey P. N., On hypersurface of a Finsler space with a special metric, Acta Math. Hungar., 120(1-2) (2008), 165-177.

[7] Gupta M. K. and Pandey P. N., Hypersurfaces of a Conformally and h-Conformally related Finsler spaces, Acta Math. Hungar., 123(3) (2009), 257-264.

[8] Izumi H., Conformal transformations of Finsler spaces II, Tensor, N.S., 33 (1980), 337–359.

[9] Kitayama M., On Finslerian hypersurfaces given by β-change, Balkan J. Geom. Appl., 7 (2002), 49-55.

[10] Matsumoto M., Foundations of Finsler geometry and special Finsler spaces, Kaiseisha, 1986.

[11] Matsumoto M., The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry, J. Math. Kyoto Univ, 25(3) (1985), 107–144.

[12] Pandey T. N. and Tripathi M. N., First Approximate Exponential Change of Finsler Metric, Mathematical Combinatorics, (4) (2013), 31-36.

[13] Rapcsàk A., Eine neue Charakterisierung Finslerscher Rume Skalarer und konstanter Krmmung und projektiv-ebene Rume, Acta Math. Acad. Sci. Hungar, 8(3) (1957), 1–8.

[14] Shankar G. and Ravindra, On Randers change of exponential metric, Journal of Applied Sciences, 15 (2013).

[15] Shankar G. and Yadav R., On the hypersurface of a Finsler space with special (α, β)-metric $\alpha + \beta + \frac{\beta^{n+1}}{\alpha^n}$, Journal of the Indian Mathematical Soc., 80(3-4) (2013), 329-339.

[16] Shankar G. and Yadav R., On the hypersurface of a Finsler space with special (α, β)-metric $\alpha + \frac{\beta^2}{\alpha - \beta}$, J. R. Acad. Phy. Sci., 12(1) (2013), 15-26.

[17] Shibata C., On invariant tensors of β-changes of Finsler metrics, Journal of Mathematics of Kyoto University, 24(1) (1984), 163-188.

[18] Shukla H. S., Pandey O. P. and Mishra A. K., Generalized h-Kropina Change of Finsler Metric, Mathematical Combinatorics, (4) (2015), 84-91.

[19] Singh U. P. and Bindu Kumari, On a Hypersurface of a Matsumoto space, Indian J. Pure Appl. Math., 32(4) (2001), 521-531.

[20] Yao-Yong Yu and Ying You, Projectively flat exponential Finsler metric, Journal of Zhejiang University-SCIENCE A, 7(6) (2006), 1068-1076.