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LINEAR CANONICAL TRANSFORM AND SCHWARTZ TYPE SPACES

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Abstract: In this paper we have defined the Schwartz type spaces $S_{\Delta,\alpha,A}$, $S^{\Delta,\beta,B}$, $S^{\Delta,\beta,B}_{\Delta,\alpha,A}$. We have studied the mapping properties of LCT between these spaces.

Keywords and Phrases: Schwartz type spaces, Linear canonical transform, Convolution.

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1. Introduction

The theory of Fourier transform has wide history and application in Engineering, Technology, Physics, Mathematics, etc. In recent past, linear canonical transformation was being studied by many mathematicians. Motivated by Pankaj Jain et al. [4], we define linear canonical transform, $\Delta_{x,a}$, $\Delta^*_{x,a}$ and obtain new results. The Fourier transform and the related convolution respectively defined by

$$\hat{f}(\lambda) = \mathcal{F}[f;\lambda] = \int_{\mathbb{R}} f(x)\, e^{-i\,x\,\lambda}\, dx \tag{1.1}$$

and

$$(f * g)\,(\lambda) = \int_{\mathbb{R}} f(\lambda - x)\, g(x)\, dx$$

are important tool for solving many practical problems. To give rise more general transforms and convolutions such as fractional Fourier transform [3], [5], [6], [12],

these notions have been generalized and extended due to their usefulness.
Linear canonical transform (LCT) is such generalization introduced by Quesne [10].
Let A be 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad with \quad ad - bc = 1.$$

The LCT is defined by

$$\mathcal{L}_A[f; \lambda] = \int_{\mathbb{R}} f(x)\, K_A(x, \lambda)\, dx\,,$$

where the kernal K_A is defined by

$$K_A(x, \lambda) = \begin{cases} \dfrac{1}{\sqrt{2\pi bi}}\, e^{\left[\frac{i}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\lambda + \frac{d}{b}\lambda^2\right)\right]} & , if\ b \neq 0 \\ \dfrac{1}{\sqrt{a}}\, e^{i\left(\frac{c}{a}\right)\lambda^2}\, \delta\left(x - \dfrac{\lambda}{a}\right) & , if\ b = 0 \end{cases}$$

and related convolution is defined by

$$(f *_A g)\,(x) = \int_{\mathbb{R}} f(\lambda)\, g(x - \lambda)\, e^{\left[i\,\frac{a}{b}\,\lambda(x-\lambda)\right]}\, d\lambda$$

The inverse LCT is defined by

$$\mathcal{L}_{A^{-1}}[f; x] = \int_{\mathbb{R}} f(\lambda)\, K_{A^{-1}}(\lambda, x)\, d\lambda\,,$$

where A^{-1} is the inverse of the matrix A, provided $|A| \neq 0$.
LCT has a range of applications in the study of wave propagation, electromagnetic, Harmonic oscillators and acoustic problems. There are some classical approaches already exist in [1], as LCT is a generalized form, it will be worthwhile to figure out and discover the relationship between LCT and those notions.

Remark. *Note that if we consider* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *then the kernel reduces to* $K_A(x, \lambda) = \frac{1}{\sqrt{2\pi i}}\, e^{-ix\lambda}$ *which is kernel of Fourier transform. Also, one can verify if* $A = \begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix}$ *then* $K_A(x, \lambda)$ *reduces to kernel of Fractional Fourier transform.*

2. Linear Canonical Transform based Convolution and Differential Operators

We recall one result from [4] as the following:

Theorem 2.1. *Let $1 \leq p < \infty, f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Then $(f *_A g) \in L^p(\mathbb{R})$ with*

$$\|f *_A g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \, \|g\|_{L^p(\mathbb{R})}. \tag{2.1}$$

Now, we prove the following theorem:

Theorem 2.2. *Let f be continuous and g be continuous with a compact support. Then $f *_A g$ is continuous.*

Proof. If $h \in \mathbb{R}$ then we have

$$\begin{aligned}
&\mid (f *_A g)\,(x+h) - (f *_A g)\,(x) \mid \\
&= \left| \int_{\mathbb{R}} f(y)\, g(x+h-y)\, e^{\left[i\, \frac{a}{b}\, y\, (x+h-y)\right]}\, dy \;-\; \int_{\mathbb{R}} f(y)\, g(x-y)\, e^{\left[i\, \frac{a}{b}\, y\, (x-y)\right]}\, dy \right| \\
&= \left| \int_{\mathbb{R}} f(y)\, \left(g(x+h-y)\, e^{\left[i\, \frac{a}{b}\, y\, h\right]} \;-\; g(x-y) \right)\, e^{\left[i\, \frac{a}{b}\, y\, (x-y)\right]}\, dy \right| \\
&\leq \int_{\mathbb{R}} \left| f(y)\, \left(g(x+h-y)\, e^{\left[i\, \frac{a}{b}\, y\, h\right]} \;-\; g(x-y) \right) \right|\, dy \\
&= \int_{\mathbb{R}} \left| f(y)\, \left(g(x+h-y)\, e^{\left[i\, \frac{a}{b}\, y\, h\right]} \;-\; g(x-y)\, e^{\left[i\, \frac{a}{b}\, y\, h\right]} + g(x-y)\, e^{\left[i\, \frac{a}{b}\, y\, h\right]} - g(x-y) \right) \right|\, dy \\
&\leq \int_{\mathbb{R}} |f(y)|\; |g(x+h-y) - g(x-y)|\, dy + \int_{\mathbb{R}} |f(y)|\; |g(x-y)|\; \left| e^{\left[i\, \frac{a}{b}\, y\, h\right]} - 1 \right| dy \\
&= I_1 + I_2,
\end{aligned}$$

where I_1 and I_2 are respectively the first and second integral in the above inequality. If $K = supp(g)$ is compact then for any fixed x, $x-K = \{x-y : y \in K\}$ is compact and thus f is uniformly continuous on $x-K$.
Therefore for each $\epsilon > 0$, there exist $\eta > 0$ such that if $\mid h \mid < \eta$ then $I_1 \to 0$ as $h \to 0$.
Note that as f, g are bounded on $x-K$,

$$I_2 \leq \int_{\mathbb{R}} |f(y)|\; |g(x-y)|\; 2\; |\sin(a/2b)\, y\, h|\, dy \;\to 0 \quad as \quad h \to 0.$$

Therefore

$$\mid (f *_A g)\,(x+h) - (f *_A g)\,(x) \mid \to 0 \quad as \quad h \to 0$$

and hence proof is completed.

Now we state and prove the following strong version of Theorem 2.2.

Theorem 2.3. *If $f \in C^\infty(\mathbb{R})$ and g is continuous with a compact support, then $f *_A g$ is C^∞.*

Proof. Consider $(f *_A g)(x+h) - (f *_A g)(x)$.
then

$$\begin{aligned}
&\frac{1}{h}[(f *_A g)(x+h) - (f *_A g)(x)] \\
&\qquad = \frac{1}{h}\int_{\mathbb{R}} g(y)\ \left[f(x+h-y)\, e^{i\frac{a}{b}yh} \ - \ f(x-y)\right]\ e^{i\frac{a}{b}y(x-y)}\, dy \\
&\qquad = \frac{1}{h}\int_{\mathbb{R}} g(y)\ \left[f(x+h-y)\, e^{i\frac{a}{b}yh} \ - \ f(x-y)\, e^{i\frac{a}{b}yh} \right. \\
&\qquad\qquad \left. + f(x-y)\, e^{i\frac{a}{b}yh} - f(x-y)\right]\ e^{i\frac{a}{b}y(x-y)}\, dy \\
&\qquad = \frac{1}{h}\int_{\mathbb{R}} g(y)\ [f(x+h-y) - f(x-y)]\ e^{i\frac{a}{b}y(x+h-y)}\, dy \\
&\qquad\qquad + \frac{1}{h}\int_{\mathbb{R}} g(y)\, f(x-y)\ \left[e^{i\frac{a}{b}yh} - 1\right] e^{i\frac{a}{b}y(x-y)}\, dy \\
&\qquad \to (Df *_A g)(x) + \left(f *_A \left(i\frac{a}{b}\right)(\cdot) g\right)(x), \quad as \quad h \to 0.
\end{aligned}$$

Therefore if f is differentiable then it follows that $f *_A g$ is also differentiable. Thus by induction it is not very difficult to prove that

$$D_x^n (f *_A g)(x) = \sum_{r=0}^{n} A_{n,r}\left(D^{n-r} f *_A \left(i\frac{a}{b}(\cdot)\right)^r g\right)(x),$$

where $A_{n,r}$ are appropriate constants.
Thus $f *_A g \in C^\infty$. Hence proof is completed.

Remark. *If we consider value of A as shown in earlier remark , $f *_A g$ will be ordinary convolution and the result is proved in [10, chapter 3]. Now we define the following generalized differential operators based on LCT:*

$$\Delta_{x,a} = -\left(D_x - i\frac{a}{b}x\right)$$
$$\Delta^*_{x,a} = \left(D_x + i\frac{a}{b}x\right), \quad where \quad D_x \equiv \frac{d}{dx}.$$

Theorem 2.4. *The following results are true.*

(i) $\Delta_{x,a}\, K_A(x,\lambda) = \left(\frac{i}{b}\lambda\right)\ K_A(x,\lambda)$

(ii) $\Delta_{\lambda,d}\, K_A(x,\lambda) = \left(\frac{i}{b}x\right)\ K_A(x,\lambda)$

(iii) $\Delta^*_{x,a}\, K_{A^{-1}}(x,\lambda) = \left(\frac{i}{b}\,\lambda\right)\; K_{A^{-1}}(x,\lambda)$

(iv) $\Delta^*_{\lambda,d}\, K_{A^{-1}}(x,\lambda) = \left(\frac{i}{b}\,x\right)\; K_{A^{-1}}(x,\lambda)$

Proof. We shall prove result (i).
(i) We have

$$\begin{aligned}\Delta_{x,a}\, K_A(x,\lambda) &= -\left(D_x - i\frac{a}{b}x\right)\,\frac{1}{\sqrt{2\pi bi}}\,e^{\frac{i}{2}\left(\frac{a}{b}\,x^2-\frac{2}{b}\,x\,\lambda+\frac{d}{b}\,\lambda^2\right)}\\ &= -\frac{1}{\sqrt{2\pi bi}}\,\left[\frac{i}{2}\left(\frac{a}{b}\,2x-\frac{2}{b}\,\lambda\right)+i\frac{a}{b}x\right]\,e^{\frac{i}{2}\left(\frac{a}{b}\,x^2-\frac{2}{b}\,x\,\lambda+\frac{d}{b}\,\lambda^2\right)}\\ &= \frac{1}{\sqrt{2\pi bi}}\,\left[-\frac{ia}{b}x+\frac{i}{b}\,\lambda+i\frac{a}{b}x\right]\,e^{\frac{i}{2}\left(\frac{a}{b}\,x^2-\frac{2}{b}\,x\,\lambda+\frac{d}{b}\,\lambda^2\right)}\\ &= \left(\frac{i}{b}\,\lambda\right)\,K_A(x,\lambda).\end{aligned}$$

(ii), (iii) and (iv) can be proved similarly.

3. Schwartz type Spaces based on LCT

Let us recall the Schwartz space $S(\mathbb{R})$ that consist of all functions $\phi \in C^\infty$ such that

$$\sup_{x\in\mathbb{R}}\left|x^k\,\phi^{(q)}(x)\right| \le m_{kq}, \quad k,q = 0,1,2,3\ldots$$

Following [6, 8], we define the space S_Δ as the space of all $\phi \in C^\infty$ for which

$$\sup_{x\in\mathbb{R}}\left|x^k\,\Delta^q_{x,a}\phi(x)\right| < \infty, \quad k,q \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}.$$

When $\Delta_{x,a}$ is the differential operator $\frac{d}{dx}$, the space S_Δ coincides with the standard Schwartz space S. The sequence m_{kq} in the construction of Schwartz space S depends on both k and q. The Gelfand and Shilov type spaces are variants of the space S in which the sequence m_{kq} depends only on k, or only on q or on both. Such spaces are respectively denoted by S_α, S^β, and S^β_α. These spaces have further been generalized to give rise to the spaces $S_{\alpha,A}$, $S^{\beta,B}$, and $S^{\beta,B}_{\alpha,A}$.
Further we define and study the generalizations of the spaces $S_{\alpha,A}$, $S^{\beta,B}$, and $S^{\beta,B}_{\alpha,A}$ in which the derivative $\frac{d}{dx}$ is replaced by more general operator Δ and Δ^*.
In [2] various spaces of type S such as S_α, S^β, and S^β_α have been defined and studied.

Definition 3.1. *Let $\delta > 0$. We define the space $S_{\Delta,\alpha,A}$ that consist of all $\phi \in C^\infty$ such that*

$$\left|x^k\,\Delta^q_{x,a}\phi(x)\right| \le C_{q,\delta}\,(A+\delta)^k\,k^{k\,\alpha},$$

where $k, q \in \mathbb{N}_0$ and $C_{q,\delta}$ depends on ϕ.

Definition 3.2. *Let $\rho > 0$. We define the space $S^{\Delta,\beta,B}$ that consist of all $\phi \in C^\infty$ such that*

$$\left|x^k \, \Delta_{x,a}^q \phi(x)\right| \leq C_{k,\rho} \, (B+\rho)^q \, q^{q\,\beta},$$

where $k, q \in \mathbb{N}_0$ and $C_{k,\rho}$ depends on ϕ.

Definition 3.3. *Let $\delta, \rho > 0$. We define the space $S_{\Delta,\alpha,A}^{\Delta,\beta,B}$ that consist of all $\phi \in C^\infty$ such that*

$$\left|x^k \, \Delta_{x,a}^q \phi(x)\right| \leq C_k \, (A+\delta)^k \, (B+\rho)^q \, k^{k\,\alpha} \, q^{q\,\beta},$$

where $k, q \in \mathbb{N}_0$ and C_k depends on ϕ.

Similarly we can define the spaces $S_{\Delta^*,\alpha,A}$, $S^{\Delta^*,\beta,B}$ and $S_{\Delta^*,\alpha,A}^{\Delta^*,\beta,B}$ where Δ be replaced by Δ^* in Definition 3.1, 3.2, 3.3.

Theorem 3.4. *Let $\phi \in S_{\Delta^*,\alpha,A}$. Then $\mathcal{L}_A[\phi;\cdot] \in S^{\Delta,\alpha,B}$*

Proof. Consider

$$\begin{aligned}
\lambda^k \, \Delta_{\lambda,d}^q \, \mathcal{L}_A[\phi;\lambda] &= \lambda^k \, \Delta_{\lambda,d}^q \int_{\mathbb{R}} K_A(x,\lambda) \, \phi(x) \, dx \\
&= \lambda^k \int_{\mathbb{R}} \Delta_{\lambda,d}^q \, K_A(x,\lambda) \, \phi(x) \, dx \\
&= \lambda^k \int_{\mathbb{R}} \left(\frac{ix}{b}\right)^q K_A(x,\lambda) \, \phi(x) \, dx \\
&= \left(\frac{i}{b}\right)^{q-k} \int_{\mathbb{R}} \left(\frac{i\lambda}{b}\right)^k K_A(x,\lambda) \, x^q \, \phi(x) \, dx \\
&= \left(\frac{i}{b}\right)^{q-k} \int_{\mathbb{R}} \left(\Delta_{x,a}\right)^k K_A(x,\lambda) \, x^q \, \phi(x) \, dx \\
&= \left(\frac{i}{b}\right)^{q-k} \int_{\mathbb{R}} K_A(x,\lambda) \, \left(\Delta_{x,a}^*\right)^k \left(x^q \, \phi(x)\right) \, dx \\
&= \left(\frac{i}{b}\right)^{q-k} \int_{\mathbb{R}} K_A(x,\lambda) \left(\sum_{r=0}^{k} A_{k,r} \, D_x^r \, x^q \, \left(\Delta_{x,a}^*\right)^{k-r} \phi(x)\right) dx \\
&= \left(\frac{i}{b}\right)^{q-k} \left(\sum_{r=0}^{k} A_{k,r} \int_{\mathbb{R}} K_A(x,\lambda) \, D_x^r \, x^q \, \left(\Delta_{x,a}^*\right)^{k-r} \phi(x) \, dx\right)
\end{aligned}$$

so that

$$\left|\lambda^k \, \Delta_{\lambda,d}^q \, \mathcal{L}_A[\phi;\lambda]\right| = \left|\left(\frac{i}{b}\right)^{q-k} \left(\sum_{r=0}^{k} A_{k,r} \int_{\mathbb{R}} K_A(x,\lambda) \, \frac{q!}{(q-r)!} \, \psi(x)^{q-r} \, \left(\Delta_{x,a}^*\right)^{k-r} \phi(x) \, dx\right)\right|.$$

where

$$\psi(x) = \begin{cases} x & if\ q-r \geq 0 \\ 0 & otherwise. \end{cases}$$

Denote

$$|A_k| = \sup_r |A_{k,r}|.$$

Then

$$\begin{aligned}
&\left|\lambda^k\, \Delta^q_{\lambda,d}\, \mathcal{L}_A[\phi;\lambda]\right| \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} \left(\sum_{r=0}^{k} |A_{k,r}| \int_{\mathbb{R}} |K_A(x,\lambda)|\, \frac{q!}{(q-r)!}\, |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, dx\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_A(x,\lambda)|\, \frac{q!}{(q-r)!}\, |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_A(x,\lambda)|\, \frac{q!}{k!\,(q-r)!}\, |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, dx\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k! \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_A(x,\lambda)|\, \frac{q!}{r!\,(q-r)!}\, |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!} \int_{\mathbb{R}} |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, dx\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!} \int_{\mathbb{R}} (1+|x|^2)\, |\psi(x)|^{q-r} \right. \\
&\qquad\qquad \left. \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, \frac{dx}{(1+|x|^2)}\right) \\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!} \left[\int_{\mathbb{R}} |\psi(x)|^{q-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, \frac{dx}{(1+|x|^2)} \right.\right. \\
&\qquad\qquad \left.\left. + \int_{\mathbb{R}} |\psi(x)|^{q+2-r}\, \left|(\Delta^*_{x,a})^{k-r}\, \phi(x)\right|\, \frac{dx}{(1+|x|^2)}\right]\right) \\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!} \int_{\mathbb{R}} 2\, |\psi(x)|^{q+2-r}\, \left|\left(\Delta^*_{x,a}\right)^{k-r}\, \phi(x)\right|\, \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!}\, C_{k-r,\delta}\, (A+\delta)^{q+2-r}\, (q+2-r)^{(q+2-r)\alpha} \right. \\
&\qquad\qquad \left. \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right) \\
&\leq 2 \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!} \int_{\mathbb{R}} C_{k-r,\delta}\, (A+\delta)^{q+2-r}\, (q+2)^{(q+2)\alpha}\, \frac{dx}{(1+|x|^2)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)| \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!}\, C_{k,\delta}\, (A+\delta)^{q+2-r}\, (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)\\
&\leq 2\left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)|\, C_{k,\delta} \left(\sum_{r=0}^{q+2} \frac{q!}{r!\,(q-r)!}\, (A+\delta)^{q+2-r}\, (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)\\
&= 2\left(\frac{1}{|b|}\right)^{q-k} |A_k|\, k!\, |K_A(x,\lambda)|\, C_{k,\delta}\, (1+A+\delta)^{q+2}\, (q+2)^{(q+2)\alpha} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\\
&= 2\pi \left(\frac{1}{|b|}\right)^{-k-2} |A_k|\, k!\, |K_A(x,\lambda)|\, C_{k,\delta} \left(\frac{1+A}{|b|}+\frac{\delta}{|b|}\right)^{q+2} (q+2)^{(q+2)\alpha}\\
&= 2\pi\, |b|^{k+2}\, |A_k|\, k!\, |K_A(x,\lambda)|\, C_{k,\delta} \left(\frac{1+A}{|b|}+\frac{\delta}{|b|}\right)^{q+2} (q+2)^{(q+2)\alpha}\\
&= D_{k,\delta}\, (B+\rho)^{q+2}\, (q+2)^{(q+2)\alpha}\\
&= D_{k,\rho}\, (B+\rho)^{q+2}\, (q+2)^{(q+2)\alpha}\\
&= E_{k,\rho}\, (B+\rho)^{q}\, q^{q\alpha}. \qquad (3.1)
\end{aligned}
$$

Thus proof is completed.

Theorem 3.5. *Let $\phi \in S^{\Delta^*,\beta,B}$. Then $\mathcal{L}_A[\phi;\cdot] \in S_{\Delta,\beta,A}$.*

Proof. If $\phi \in S_{\Delta^*,\beta,A}$ and $\rho > 0$ is arbitrary then by using (3.1), we can infer that

$$
\begin{aligned}
&\left|\lambda^k\, \Delta^q_{\lambda,d}\, \mathcal{L}_A[\phi;\lambda]\right|\\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k| \left(\sum_{r=0}^{k} \int_{\mathbb{R}} |K_A(x,\lambda)|\, \frac{q!}{(q-r)!}\, |\psi(x)|^{q-r} \left|\left(\Delta^*_{x,a}\right)^{k-r} \phi(x)\right| dx\right)\\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, |K_A(x,\lambda)|\, q! \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} |\psi(x)|^{q-r} \left|\left(\Delta^*_{x,a}\right)^{k-r} \phi(x)\right| dx\right)\\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, |K_A(x,\lambda)|\, q! \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} (1+|x|^2)\, |\psi(x)|^{q-r} \left|\left(\Delta^*_{x,a}\right)^{k-r} \phi(x)\right| \frac{dx}{(1+|x|^2)}\right)\\
&= \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, |K_A(x,\lambda)|\, q! \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \left[\int_{\mathbb{R}} |\psi(x)|^{q-r} \left|(\Delta^*_{x,a})^{k-r} \phi(x)\right| \frac{dx}{(1+|x|^2)}\right.\right.\\
&\qquad\qquad \left.\left. + \int_{\mathbb{R}} |\psi(x)|^{q+2-r} \left|\left(\Delta^*_{x,a}\right)^{q-r} \phi(x)\right| \frac{dx}{(1+|x|^2)}\right]\right)\\
&\leq \left(\frac{1}{|b|}\right)^{q-k} |A_k|\, |K_A(x,\lambda)|\, q! \left(\sum_{r=0}^{q} \frac{q!}{r!(q-r)!} \int_{\mathbb{R}} 2\, |\psi(x)|^{q+2-r} \left|\left(\Delta^*_{x,a}\right)^{k-r} \phi(x)\right| \frac{dx}{(1+|x|^2)}\right)\\
&= 2\left(\frac{1}{|b|}\right)^{q-k} |A_k|\, |K_A(x,\lambda)|\, q! \left(\sum_{r=0}^{q} \frac{q!}{r!\,(q-r)!}\, C_{q+2-r,\rho}\, (B+\rho)^{k-r}\, (k-r)^{(k-r)\beta} \int_{\mathbb{R}} \frac{dx}{(1+|x|^2)}\right)
\end{aligned}
$$

$$\le 2\left(\frac{1}{|b|}\right)^{q-k}|A_k|\,|K_A(x,\lambda)|\,q!\left(\sum_{r=0}^{q}\frac{q!}{r!\,(q-r)!}\int_{\mathbb{R}}C_{q+2-r,\rho}\,(B+\rho)^k\,(k)^{k\beta}\frac{dx}{(1+|x|^2)}\right)$$
$$\le 2\left(\frac{1}{|b|}\right)^{q-k}|A_k|\,|K_A(x,\lambda)|\,q!\left(\sum_{r=0}^{q}\frac{q!}{r!\,(q-r)!}C_{q,\rho}\,(B+\rho)^k\,(k)^{k\beta}\int_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\right)$$
$$\le 2\left(\frac{1}{|b|}\right)^{q-k}|A_k|\,|K_A(x,\lambda)|\,C_{q,\rho}\left(2^q\,(B+\rho)^k\,(k)^{k\beta}\int_{\mathbb{R}}\frac{dx}{(1+|x|^2)}\right)$$
$$=2\pi\left(\frac{1}{|b|}\right)^{q-k}|A_k|\,|K_A(x,\lambda)|\,q!\,2^q\,C_{q,\rho}\,(B+\rho)^k\,(k)^{k\beta}$$
$$=2\pi\left(\frac{1}{|b|}\right)^{q}|A_k|\,|K_A(x,\lambda)|\,q!\,2^q\,C_{q,\rho}\,\left(|b|\,|A_k|^{1/k}(B+\rho)\right)^k\,(k)^{k\beta}$$
$$=D_{q,\rho}\,(A+\rho)^k\,(k)^{k\beta}.$$

Thus proof is completed.

Similarly by using the same technique as in Theorem 3.5, we can prove the following Theorem 3.6 :

Theorem 3.6. *Let* $\phi \in S_{\Delta^*,\alpha,A}^{\Delta^*,\beta,B}$. *Then* $\mathcal{L}_A[\phi;\cdot] \in S_{\Delta,\beta,B}^{\Delta,\alpha,A}$.

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