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DEGREE OF APPROXIMATION OF SIGNALS BELONGING TO WEIGHTED LIPSCHITZ CLASS BY ZWEIER-EULER PRODUCT MEANS OF ITS FOURIER SERIES

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Abstract: Many researchers have looked into various linear summation techniques for approximating periodic functions. In the current paper, an attempt is made to obtain the degree of approximation of signals belonging to weighted $(L^{\mu}, \varphi(v))$ class by Zweier-Euler product of its Fourier series.

Keywords and Phrases: Degree of approximation, Weighted Lipschitz class, Fourier series, Lebesgue integral.

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1. Introduction

Summability techniques and their applications are found to be important in approximation theory, and also can aid in the physical interpretation of a number of phenomena that occurs in science, and technology particularly in engineering. Fourier series is simply the representation of a function or signal which may be significant in a variety of contexts, both theoretical and practical. Nevertheless, infinite matrices and summability techniques have a direct bearing on a wide range of engineering and scientific fields. It is well known that over the past fourteen decades, the theory of approximation was found as on a well-known Weierstrass theorem, and was developed into an exciting interdisciplinary field of study. Due to their extensive use particularly in signal processing and in general signal analysis, these approximations have taken on significant new dimensions.

2. Definitions and Notations

Let $\sum a_\nu$ be an infinite series with the sequence (s_ν) of its partial sums. The transformation

$$t_\nu = \frac{1}{2}\sum_{\epsilon\ =\nu-1}^{\nu} s_\epsilon \tag{2.1}$$

defines the sequence (t_ν) of the Zweier means of the sequence (s_ν). If $t_\nu \to s$ as $\nu \to \infty$, where s is a finite number, we say that the series $\sum a_\nu$ is summable by Zweier mean or $Z_{\frac{1}{2}}$ mean to the sum s. It is known that Zweier method is regular. Let

$$\rho_\nu \ = \ \frac{1}{2^\nu}\sum_{\epsilon=0}^{\nu}\binom{\nu}{\epsilon} s_\epsilon. \tag{2.2}$$

Then we say that the sequence (ρ_ν) defines the Euler mean or simply $(E, 1)$ mean of the sequence (s_ν). If $\rho_\nu \to s$ as $\nu \to \infty$, where s is a finite number, we say that the series $\sum a_\nu$ is $(E, 1)$ summable to sum s. Clearly $(E, 1)$ method is regular ([6], [10]). The Zweier mean of the $(E, 1)$ transform of the sequence (s_ν) of partial sums of the series $\sum a_\nu$ is denoted by $Z_{\frac{1}{2}}(E, 1)$. Thus if Γ_ν is the ν^{th} $Z_{\frac{1}{2}}(E, 1)$ transform of $\sum a_\nu$, then

$$\Gamma_\nu = \ \frac{1}{\nu}\sum_{\epsilon\ =\nu-1}^{\nu}\rho_\epsilon \ = \ \frac{1}{\nu}\sum_{\epsilon\ =\nu-1}^{\nu}\frac{1}{2^\epsilon}\sum_{\eta=0}^{\epsilon}\binom{\epsilon}{\eta} s_\eta. \tag{2.3}$$

If $\Gamma_\nu \to s$ as $\nu \to \infty$, where s is a finite number, we say that the series $\sum a_\nu$ is summable $Z_{\frac{1}{2}}(E, 1)$ to sum s.

Let $\chi(v)$ be a periodic function with period 2π and integrable in the sense of Lebsegue over the interval $[0, 2\pi]$. Then the Fourier series of $\chi(v)$ is given by

$$\chi(v) \approx \frac{c_0}{2} + \ \sum_{\nu=1}^{\infty}(c_\nu \cos \nu t \ + \ d_\nu \sin \nu t) = \ \sum_{\nu=0}^{\infty} U_\nu(v). \tag{2.4}$$

The μ-norm $\|.\|_\mu$ of a function $\chi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|\chi\|_\mu \;=\; \Big(\int_0^{2\pi} |\chi(u)|^\mu \, du\Big)^{\frac{1}{\mu}}, \;\mu \geq 1 \tag{2.5}$$

and the degree of approximation $E_\nu(\chi)$ of a function $\chi \in L_\mu$ is given by [18]

$$E_\nu(\chi) \;=\; \min_{\mu_\nu} \|\chi \;-\; \mu_\nu\| \tag{2.6}$$

where μ_ν is a trigonometric polynomial of degree ν approximating the function χ.
For $0 < \gamma \leq 1$, a function $\chi \in Lip(\gamma)$, if [9]

$$|\chi(u+v) \;-\chi(u)| \;=\; O(|v|^\gamma), \tag{2.7}$$

and $\chi \in Lip(\gamma, \xi)$ for $0 \leq u \leq 2\pi, \;\xi \geq 1$, if [16]

$$\Big(\int_0^{2\pi} |\chi(u+v) \;-\chi(u)|^\mu \, du\Big)^{\frac{1}{\mu}} \;=\; O(|v|^\gamma), \;0 < \gamma \leq 1, \;\mu \geq 1. \tag{2.8}$$

For a given positive increasing function $\varphi(v)$ and an integer $\mu \geq 1$ a function $\chi \in Lip(\varphi(v), \mu)$, if [16]

$$\Big(\int_0^{2\pi} |\chi(u+v) \;-\chi(u)|^\mu \, du\Big)^{\frac{1}{\mu}} \;=\; O(\varphi(v)), \;\mu \geq 1 \tag{2.9}$$

and that $\chi \in W(L^\mu, \varphi(v))$, if [8]

$$\Big(\int_0^{2\pi} |\chi(u+v) \;-\chi(u)|^\mu \sin u\Big)^{\theta\mu} \, du)^{\frac{1}{\mu}} \;=\; O(\varphi(v)), \;\theta \geq 0, \;\mu \geq 1. \tag{2.10}$$

Clearly for $\theta = 0$, the class $W(L^\mu, \varphi(v))$ coincide with the class $Lip(\varphi(v), \mu)$.
We shall use the following notations throughout this paper :

$$\Psi(v) \;=\; \chi(u+v) + \;\chi(u-v) \;-\; 2\chi(u) \tag{2.11}$$

$$\sigma_\nu(\chi; u) \;=\; \nu^{th} \text{ partial sum of the Fourier series given by (2.4)} \tag{2.12}$$

and

$$\kappa_\nu(v) \;=\; \frac{1}{4\pi} \sum_{\epsilon \;=\nu-1}^{\nu} \frac{\cos^\epsilon(\frac{v}{2}) \;\sin(\epsilon+1)v}{\sin\frac{v}{2}} \tag{2.13}$$

3. Known Results

Alexits [1], Bernstein [2], Goel and Sahney [5], Chandra [3] and several others have determined the degree of approximation of the Fourier series of the functions of class $Lip\gamma$ by $|C,1|$, $|C,\gamma|$, $|N,p_\nu|$ and $|\bar{N},p_\nu|$ means. Subsequently working in the same direction Sahney and Rao [15] and Khan [7] have established results on the degree of approximation of the functions of the class $Lip(\gamma,\xi)$ by different summability methods. However dealing with product summabilty, Nigam [14], Misra et al.([11], [12], [13]) and Subrat [17] have established theorems on approximation of functions of class $Lip\gamma$, $Lip(\gamma,\xi), Lip(\varphi(v),\xi)$. Recently, Das et al. [4] have established the following theorems for Zweier-Euler product summability of Fourier series of functions of class $Lip(\gamma,\xi)$ and $Lip(\varphi(v),\xi)$:

Theorem 3.1. [4] *Let $Z_{\frac{1}{2}}$ be a Zweier method. Let $\chi : [0,2\pi] \to \mathbb{R}$ be 2π-periodic, integrable in the sense of Lebsegue and belonging to the class $Lip(\gamma,\xi), \xi \geq 1.$Then the degree of approximation of χ by $Z_{\frac{1}{2}}(E,1)$ transform of the fourier series satisfies* $||v_\nu^{Z_{\frac{1}{2}}E} - \chi||^{\xi}_{L_\gamma} = O\Big(\frac{1}{(\nu+1)^\gamma}\Big) + \ O\Big(\frac{\pi^\gamma}{\gamma}\Big(1-\frac{1}{(\nu+1)^\gamma}\Big)\Big)$

Theorem 3.2. [4] *Let $Z_{\frac{1}{2}}$ be a Zweier method. Let $\chi : [0,2\pi] \to \mathbb{R}$ be 2π-periodic, integrable in the sense of Lebsegue and belonging to the class $Lip(\varphi(v),\xi), \xi \geq 1.$Then the degree of approximation of χ by $Z_{\frac{1}{2}}(E,1)$ transform of the Fourier series satisfies* $||v_\nu^{Z_{\frac{1}{2}}E} - \chi||_{L^{\xi}_{\varphi}} = O\Big((\nu+1)\int_0^{\frac{\pi}{\nu+1}} \varphi(v)\ dv\Big) + O\Big(\int_{\frac{\pi}{\nu+1}}^{\pi} \frac{\varphi(v)}{v}\ dv\Big)$

4. Main Result

Extending the above results to the functions of weighted Lipchitz class, in the present paper we establish a theorem on degree of approximation of functions of $W(L^\mu,\varphi(v))$ class by $Z_{\frac{1}{2}}(E,1)$ mean of the Fourier series in the following form :

Theorem 4.1. *Let χ be a 2π-periodic and L-integrable function belonging to the weighted Lipschitz class $W(L^\mu,\varphi(v))$, where $\varphi(v)$ is a positive increasing function of v. Then the degree of approximation of the Fourier series of the function χ by the product $Z_{\frac{1}{2}}(E,1)$ summability mean is given by*

$$||\chi - \Gamma_\nu||_\mu \ = \ O\Big((\nu+1)^{\theta+\frac{1}{\mu}}\ \varphi\Big(\frac{1}{\nu+1}\Big)\Big),\ \mu \geq 1, \tag{4.1}$$

provided $\varphi(v)$ satisfies

$$\Big\{\int_0^{\frac{\pi}{\nu+1}} \bigg(\frac{v\ \Psi(v)\ \sin^\theta \frac{v}{2}}{\varphi(v)}\bigg)^\mu\ dv\Big\}^{\frac{1}{\mu}} \ = \ O\Big(\frac{1}{\nu+1}\Big) \tag{4.2}$$

and

$$\left\{ \int_{\frac{\pi}{\nu+1}}^{\pi} \left(\frac{v^{-\omega}\Psi(v)\ \sin^{\theta}\frac{v}{2}}{\varphi(v)} \right)^{\mu} dv \right\}^{\frac{1}{\mu}} = O\Big((\nu+1)^{\omega} \Big). \tag{4.3}$$

5. Required Lemmas

To establish theorem 4.1, following results are required :

Lemma 5.1. [4]

$$|\kappa_{\nu}(v)| = O(n+1),\ 0 < v \leq \frac{\pi}{\nu+1}$$

where $\kappa_{\nu}(v)$ *is as defined in (2.13).*

Lemma 5.2. [4]

$$\kappa_{\nu}(v) = O\Big(\frac{1}{v}\Big),\ \frac{\pi}{\nu+1} < v \leq \pi$$

where $\kappa_{\nu}(v)$ *is as defined in (2.13).*

6. Proof of Theorem 4.1

Using Riemnn-Lebsegue theorem for ν^{th} partial sum $\sigma_{\nu}(\chi; u)$ of the Fourier series represented by (2.4)of $\chi(u)$ and following Tichmarsh [18], we have

$$\sigma_{\nu}(\chi; u) - \chi(u) = \frac{1}{2\pi} \int_0^{\pi} \frac{\Psi(v)\sin(\nu+\frac{1}{2})v}{\sin\ (\frac{v}{2})}\ dv.$$

If ρ_{ν} denotes the ν^{th} $(E, 1)$ mean of $\sigma_{\nu}(\chi; u)$, then we have

$$\begin{aligned}
\rho_{\nu} - \chi(u) &= \frac{1}{2^{\nu}} \sum_{\epsilon=0}^{\nu} \binom{\nu}{\epsilon} \Big(\sigma_{\epsilon}(\chi; u) - \chi(u) \Big) \\
&= \frac{1}{2^{\nu+1}\pi} \int_0^{\pi} \frac{\Psi(v)}{\sin(\frac{v}{2})} \sum_{\epsilon=0}^{\nu} \binom{\nu}{\epsilon} \sin(\epsilon + \frac{1}{2})v\ dv \\
&= \frac{1}{2^{\nu+1}\pi} \int_0^{\pi} \frac{\Psi(v)}{\sin(\frac{v}{2})} Im\Big(\sum_{\epsilon=0}^{\nu} \binom{\nu}{\epsilon} e^{i(\epsilon+\frac{1}{2})v} \Big)\ dv \\
&= \frac{1}{2^{\nu+1}\pi} \int_0^{\pi} \frac{\Psi(v)}{\sin(\frac{v}{2})} Im\Big(e^{\frac{iv}{2}} \sum_{\epsilon=0}^{\nu} \binom{\nu}{\epsilon} e^{i\epsilon v} \Big)\ dv \\
&= \frac{1}{2^{\nu+1}\pi} \int_0^{\pi} \frac{\Psi(v)}{\sin(\frac{v}{2})} Im\Big(e^{\frac{iv}{2}} (1+e^{iv})^{\nu} \Big)\ dv
\end{aligned}$$

$$= \frac{1}{2^{\nu+1}\pi}\int_0^{\pi}\frac{\Psi(v)}{\sin(\frac{v}{2})} Im\Big(2^{\nu}\cos^{\nu}\Big(\frac{v}{2}\Big)e^{i(\nu+1)\frac{v}{2}}\Big)\, dv$$

$$= \frac{1}{2\pi}\int_0^{\pi}\Psi(v)\frac{\cos^{\nu}\Big(\frac{v}{2}\Big)\sin\frac{(\nu+1)v}{2}}{\sin\Big(\frac{v}{2}\Big)}\, dv$$

If Γ_ν denotes the ν^{th} Zweier mean of the sequence (ρ_ν), then we have

$$\frac{1}{2}\sum_{\epsilon=\nu-1}^{\nu}(\rho_\nu-\chi(u)) \;=\; \frac{1}{2}\sum_{\epsilon=\nu-1}^{\nu}\int_0^{\pi}\Psi(v)\frac{\cos^{\nu}\Big(\frac{v}{2}\Big)\sin\frac{(\nu+1)v}{2}}{\sin\Big(\frac{v}{2}\Big)}\, dv \tag{6.1}$$

That is

$$\Gamma_\nu-\chi(u) = \int_o^{\pi}\Psi(v)\;\kappa_\nu(v)\; dv \tag{6.2}$$

$$= \int_o^{\frac{\pi}{\nu+1}}\Psi(v)\;\kappa_\nu(v)\; dv + \int_{\frac{\pi}{\nu+1}}^{\pi}\Psi(t)\;\kappa_\nu(v)\; dv \tag{6.3}$$

$$= I_1 + I_2, (say). \tag{6.4}$$

Now assuming $\frac{1}{\mu}+\frac{1}{\lambda}=1$, using Holder's inequality and $(\sin\frac{v}{2})^{-1}\le\frac{\pi}{v}$, we get

$$|I_1| \;=\; \Big|\int_0^{\frac{\pi}{\nu+1}}\Psi(v)\kappa_\nu(v)\; dv\Big| \tag{6.5}$$

$$\le \Big(\int_0^{\frac{\pi}{\nu+1}}\Big|\frac{v\;\Psi(v)\sin^{\theta}\frac{v}{2}}{\varphi(v)}\Big|^{\mu}\; dv\Big)^{\frac{1}{\mu}}\Big(\int_0^{\frac{\pi}{\nu+1}}\Big|\frac{\varphi(v)\;\kappa_\nu(v)}{v\sin^{\theta}\frac{v}{2}}\Big|^{\lambda}\; dv\Big)^{\frac{1}{\lambda}} \tag{6.6}$$

$$= O(1)\Big\{\int_0^{\frac{\pi}{\nu+1}}\Big(\frac{\varphi(v)}{v^{1+\theta}}\Big)^{\lambda}\; dv\Big\}^{\frac{1}{\lambda}},\; using\ (4.2)\ and\ Lemma\ (5.1) \tag{6.7}$$

$$= O\Big(\varphi\Big(\frac{1}{\nu+1}\Big)\Big)\Big\{\int_0^{\frac{\pi}{\nu+1}}\Big(\frac{1}{v^{\lambda(1+\theta)}}\Big)^{\lambda}\; dv\Big\}^{\frac{1}{\lambda}} \tag{6.8}$$

$$= O\Big(\varphi\Big(\frac{1}{\nu+1}\Big)\Big)O((\nu+1)^{1+\theta-\frac{1}{\lambda}} \tag{6.9}$$

$$= O\Big(\Big\{\varphi\Big(\frac{1}{\nu+1}\Big)\Big\}O((\nu+1)^{\theta+\frac{1}{\mu}}\Big) \tag{6.10}$$

Similarly assuming $\frac{1}{\mu}+\frac{1}{\lambda}=1$, using Holder's inequality and $(\sin\frac{v}{2})^{-1}\le\frac{\pi}{v}$, we get

$$|I_2| \;=\; \Big|\int_{\frac{\pi}{\nu+1}}^{\pi}\Psi(v)\;\kappa_\nu(v)\; dv\Big| \tag{6.11}$$

$$\leq \Big(\int_{\frac{\pi}{\nu+1}}^{\pi} \Big|\frac{v^{-\omega}\Psi(v)\sin^{\theta}\frac{v}{2}}{\varphi(v)}\Big|^{\mu} dv\Big)^{\frac{1}{\mu}} \Big(\int_{0}^{\frac{\pi}{\nu+1}} \Big|\frac{\varphi(v)\kappa_{\nu}(v)}{v^{-\omega}\sin^{\theta}\frac{v}{2}}\Big|^{\lambda} dv\Big)^{\frac{1}{\lambda}} \tag{6.12}$$

$$= O((\nu+1)^{\omega})\Big\{\int_{\frac{\pi}{\nu+1}}^{\pi} \Big(\frac{\varphi(v)}{v^{1+\theta-\omega}}\Big)^{\lambda} dv\Big\}^{\frac{1}{\lambda}}, \text{ using (4.3) and Lemma (5.2)} \tag{6.13}$$

$$= O((\nu+1)^{\omega})\Big\{\int_{\frac{\pi}{\nu+1}}^{\pi} \Big(\frac{\varphi(\frac{1}{y})}{y^{\omega-\theta-1}}\Big)^{\lambda} \frac{dy}{y^{2}}\Big\}^{\frac{1}{\lambda}} \tag{6.14}$$

As $\varphi(v)$ is positive and increasing, $\Big(\frac{\varphi(\frac{1}{y})}{\frac{1}{y}}\Big)$ is also positive and increasing and hence by using second mean-value theorem for some ω with $\frac{1}{\pi} \leq \delta \leq \frac{\nu+1}{\pi}$, we have

$$|I_2| = O\Big((\nu+1)^{1+\omega}\varphi\Big(\frac{1}{\nu+1}\Big)\Big)\Big\{\int_{\delta}^{\frac{\nu+1}{\pi}} \frac{dy}{y^{2+\lambda(\omega-\theta-1)}}\Big\}^{\frac{1}{\lambda}} \tag{6.15}$$

$$= O\Big((\nu+1)^{1+\omega}\varphi\Big(\frac{1}{\nu+1}\Big)\Big)\, O\Big((\nu+1)^{\theta-\omega-\frac{1}{\lambda}}\Big) \tag{6.16}$$

$$= O\Big((\nu+1)^{\frac{1}{\mu}+\theta}\varphi\Big(\frac{1}{\nu+1}\Big)\Big) \tag{6.17}$$

Thus, from (6.4), (6.10) and (6.17), we get

$$|\Gamma_{\nu} - \chi(u)| = O\Big((\nu+1)^{\theta+\frac{1}{\mu}}\, \varphi\Big(\frac{1}{\nu+1}\Big)\Big) \tag{6.18}$$

Hence

$$E_{\nu}(\chi) = \|\Gamma_{\nu} - \chi(u)\|_{\mu} \tag{6.19}$$

$$= O\Big(\int_{0}^{2\pi} \Big\{\varphi\Big(\frac{1}{\nu+1}\Big)(\nu+1)^{\theta+\frac{1}{\mu}}\Big\}^{\mu} du\Big)^{\frac{1}{\mu}} \tag{6.20}$$

$$= O\Big(\varphi\Big(\frac{1}{\nu+1}\Big)(\nu+1)^{\theta+\frac{1}{\mu}}\Big)\Big(\int_{0}^{2\pi} du\Big)^{\frac{1}{\mu}} \tag{6.21}$$

$$= O\Big(\varphi\Big(\frac{1}{\nu+1}\Big)(\nu+1)^{\theta+\frac{1}{\mu}}\Big) \tag{6.22}$$

This establishes the theorem.

7. Conclusion

Summability techniques are regarded as helpful tools in fields like signal filtering and system stabilization. Infinite series, Fourier series, and wavelets are just a few examples of the different forms and types of series signals. The theory of

Summability can be very useful in determining the degree of approximation and error of such signals. Both pure and applied mathematics have long-established fields of study for summability theory and Fourier analysis. There are various types of results concerning the approximations of periodic - signals (functions) of different weighted $W(L^\mu, \varphi(v))$ classes. Present theorem in this paper is an attempt to establish the approximation of the signal or function of weighted $W(L^\mu, \varphi(v))$ of Fourier series which generalizes several known theorems. This result can be extended to the product summability of Zweier mean with Euler mean of different order.

References

[1] Alexits, G., Über die Annäherung einer stetigen Funktion durch die Cesàroschen Mittel ihrer Fourierreihe, Mathematische Annalen, 100 (1928), 264–277.

[2] Bernstein S., Sur l' orders de la Melleure approximation des function continue par des polynomes de degree' donne'e,, Memories Acad. Roy-Belyique, 4 (1912), 1-104.

[3] Chandra P., On degree of approximation of functions belonging to Lipschitz class, Nanta Math, 8 (1975), 88-91.

[4] Das S. and Dutta H., Degree of approximation of signals in certain Lipschitz classes by the Zweier–Euler product summability method of Fourier series, Math Meth Appl Sci., 44, No. 6, (2021), 4881-4887.

[5] Goel D. S., and Sahney B. N., On the degree of approximation of continuous functions, Ranchi University Mathematical Journal, 4 (1973), 50-53.

[6] Hardy G. H., Divergent Series, First Edition, Clarendon Press, Oxford University, 1949.

[7] Khan H., On Degree of approximation of function belonging to the class, Indian Journal of Pure and Applied Mathematics, 13, No. 6 (1982), 132-136.

[8] Khan H. H., A note on a theorem Izumi, Comm. Fac. Sci. Math. Ankara (TURKEY), 31 (1982), 123-127.

[9] McFadden L., Absolute Norlund summability, Duke Maths. Journal, 9 (1942), 168-207.

[10] Misra U. K., Current Topics in Summability Theory and Applications, chapter-I, Springer Sciences, Warsaw, 1 (2016).

[11] Misra U. K., Misra M., Padhy B. P. and Muduli M. K., On degree of approximation by product mean $(E, q)(N, p_n)$ of Fourier series, Gen. Math. Notes, 6, No. 2 (2011), 113-141.

[12] Misra U. K., Misra M., Padhy B. P. and Das P. K., Degree of approximation of Fourier series of a function of Lipchitz Class by Product Means, Frontier of Mathematics and its Application, 1, No. 3 (2014), 52-58.

[13] Misra U. K., Misra M., Padhy B. P., Palo P., and Samanta P., Approximation of the Fourier Series of a function of Lipchitz class by Product Means, Journal of Advanced in Mathematics, 9, No. 4 (2014), 2475-2484.

[14] Nigam H. K. and Sharma Ajay, On degree of Approximation by product means, Ultra Scientist of Physical Sciences, 22, No. 3 (2010), 889-894.

[15] Sahney B. N., and Rao G., Errors bound in the approximation function, Bulletin of Australian Mathematical Society, 6 (1972), 11-18.

[16] Shyam Lal, On degree of approximation of Fourier series of function belonging to weighted Lipchitz class by $(C, 1)(E, 1)$ means, Tamkang Journal of mathematics, 30 (1999), 47-52.

[17] Subrat Sahu, Acharya D., Nayak P. C. and Misra U. K., Approximation of Fourier series of a function of Lipchitz class by Product Means, Journal of Progressive Research in Mathematics, 4, No. 4 (2015), 399-407.

[18] Titchmarsh E. C., Theory of Functions, Second Edition, Oxford University Press, London, 1939.

[19] Zygmund A., Trigonometric Series, Second Edition, Cambridge University Press, 1959.

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