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IDEALS OF FUNCTION SPACE IN THE LIGHT OF AN EXPONENTIAL ALGEBRA

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Abstract: Exponential algebra is a new algebraic structure consisting of a semigroup structure, a scalar multiplication, an internal multiplication and a partial order [introduced in [4]]. This structure is based on the structure 'exponential vector space' which is thoroughly developed by Priti Sharma et. al. in [11] [This structure was actually proposed by S. Ganguly et. al. in [1] with the name 'quasi-vector space'] Exponential algebra can be considered as an algebraic ordered extension of the concept of algebra. In the present paper we have shown that the function space $C^+(\mathbf{X})$ of all non-negative continuous functions on a topological space $\mathbf{X}$ is a topological exponential algebra under the compact open topology. Also we have discussed the ideals and maximal ideals of $C^+(\mathbf{X})$. We find an ideal of $C^+(\mathbf{X})$ which is not a maximal ideal in general; actually maximality of that ideal depends on the topology of $\mathbf{X}$. The concept of ideals of exponential algebra was introduced by us in [4].

Keywords and Phrases: Algebra, exponential algebra, function space, ideal, maximal ideal.

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1. Introduction

S. Ganguly et al. in [1] introduced a new algebraic structure called 'quasi vector space' or 'qvs' in short which consists of a semigroup structure, a partial order and a scalar multiplication. This structure has been topologised in [1] so that the topology becomes compatible with the operations and partial order. Although the study of this structure was initiated with hyperspace in [1], a large number of such algebraic structures have been found in [2], [3], [5] and [10] from various branches of mathematics. Later Priti Sharma and Sandip Jana further studied the same structure in [11] with a new nomenclature '*exponential vector space*' (in short 'evs'), since elements of this space behave exponentially due to the presence of the partial order, as explained in that paper. Let us first give the definition of exponential vector space.

Definition 1.1. [11] *Let $(X, \leq)$ be a partially ordered set, '$+$' be a binary operation on X [called addition] and '$\cdot$': $K \times X \longrightarrow X$ be another composition [called scalar multiplication, K being a field]. If the operations and partial order satisfy the following axioms then $(X, +, \cdot, \leq)$ is called an exponential vector space (in short evs) over K [This structure was initiated with the name quasi-vector space or qvs by S. Ganguly et al. in* [1]*].*

A_1 : $(X, +)$ *is a commutative semigroup with identity* θ

A_2 : $x \leq y \ (x, y \in X) \Rightarrow x + z \leq y + z$ *and* $\alpha \cdot x \leq \alpha \cdot y, \ \forall z \in X, \forall \alpha \in K$

A_3 : *(i)* $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$

(iv) $1 \cdot x = x$, *where '1' is the multiplicative identity in* K,

$\forall x, y \in X, \ \forall \alpha, \beta \in K$

A_4 : $\alpha \cdot x = \theta$ *iff* $\alpha = 0$ *or* $x = \theta$

A_5 : $x + (-1) \cdot x = \theta$ *iff* $x \in X_0 := \{z \in X : \ y \nleq z, \forall y \in X \smallsetminus \{z\}\}$

A_6 : *For each* $x \in X$, $\exists\, y \in X_0$ *such that* $y \leq x$.

In the above definition, X_0 is precisely the set of all minimal elements of the evs X with respect to the partial order on X and forms the maximum vector space (within X) over the same field as that of X [1]. In [11] the authors call this vector space X_0 as '*primitive space*' or '*zero space*' of X and the elements of X_0 as '*primitive elements*'.

After that in [4] we have introduced the concept of exponential algebra (or *ealg* in short) by defining an internal multiplication on an evs. Let us first write the definition of exponential algebra.

Definition 1.2. [4] *Let $(X,+,\cdot,\leq)$ be an exponential vector space over some field K and '$*$': $X\times X\longrightarrow X$ be a binary operation [called internal multiplication] satisfying the following axioms. Then $(X,+,\cdot,\leq,*)$ is called an exponential algebra (in short ealg) over K.*

$$EA_1 : (x*y)*z = x*(y*z),\ \forall\, x,y,z\in X$$
$$EA_2 : x\leq y \Longrightarrow z*x\leq z*y\ \&\ x*z\leq y*z,\ \forall\, z\in X$$
$$EA_3 : x*(y+z)\leq x*y+x*z\ \&\ (y+z)*x\leq y*x+z*x,$$
$$\text{eqality holds if } x\in X_0.$$
$$EA_4 : \alpha\cdot(x*y) = (\alpha\cdot x)*y = x*(\alpha\cdot y),\ \forall\,\alpha\in K,\ \forall\, x,y\in X$$
$$EA_5 : x*\theta=\theta*x=\theta,\ \forall\, x\in X,\ \theta \text{ being the additive identity of } X.$$

For convenient, henceforth we shall write 'xy' instead of '$x*y$' and 'αx' instead of '$\alpha\cdot x$', $\forall\, x,y\in X$ and $\forall\,\alpha\in K$.

In [4] it has been shown that X_0 is an algebra over the same field K. The algebra X_0 is called the '*primitive algebra*' of X. In [4] it has been shown that the converse of this is also true; given any algebra A over some field K, an exponential algebra X can be constructed such that X_0 is isomorphic with A as an algebra.

Definition 1.3. [4] *An exponential algebra X is said to be a unital exponential algebra if there exists an element $e\in X$, called unity in X, such that $xe=ex=x,\forall x\in X$.*

An exponential algebra X is said to be a commutative exponential algebra if $xy=yx,\forall\, x,y\in X$.

To define a topological exponential algebra we need the following concept.

Definition 1.4. [8] *Let '$\leq$' be a preorder in a topological space Z; the preorder is said to be closed if its graph $G_{\leq}(Z):=\big\{(x,y)\in Z\times Z : x\leq y\big\}$ is closed in $Z\times Z$ (endowed with the product topology).*

Theorem 1.5. [8] *A partial order '$\leq$' in a topological space Z will be a closed order iff for any $x,y\in Z$ with $x\not\leq y$, $\exists$ open nbds U,V of x,y respectively in Z such that $(\uparrow U)\cap(\downarrow V)=\emptyset$. $\big[$Here $\uparrow A:=\{x\in X : x\geq a$ for some $a\in A\}$ and $\downarrow A:=\{x\in X : x\leq a$ for some $a\in A\}$ for any $A\subseteq X\big]$.*

Definition 1.6. [4] *An exponential algebra X over the field $\mathbb{K}$ of real or complex numbers is said to be a topological exponential algebra if X has a topological struc-*

ture with respect to which the addition, scalar multiplication, internal multiplication are continuous and the partial order '$\leq$' is closed (Here $\mathbb{K}$ is equipped with the usual topology).

In view of Theorem 1.5 we can say that every topological exponential algebra is Hausdorff and hence X_0 is a Hausdorff topological algebra.

In [4] we have also introduced the concept of ideal and maximal ideal in an exponential algebra. In the present paper we consider the function space $C^+(\mathbf{X})$, the collection of all non-negative continuous functions on a topological space $\mathbf{X}$. We prove that it is an exponential algebra under suitably defined operations and partial order. We use compact open topology [6] to make $C^+(\mathbf{X})$ a topological ealg over the field $\mathbb{K}$. In the final section of this article we find some important ideals of $C^+(\mathbf{X})$. We have obtained the representation of a maximal ideal of $C^+(\mathbf{X})$ under some suitable condition.

2. The structure of topological ealg in the function space $C^+(\mathbf{X})$

Let $\mathbf{X}$ be a topological space and $C^+(\mathbf{X})$ be the set of all non-negative continuous functions on $\mathbf{X}$. The addition, scalar multiplication and partial order on $C^+(\mathbf{X})$ are defined as follows :
(i) $(f+g)(x) := f(x)+g(x)$, $\forall x \in \mathbf{X}$
(ii) $(\alpha f)(x) := |\alpha| f(x)$, $\forall x \in X$, $\forall \alpha \in \mathbb{K}$, the field of real or complex numbers.
(iii) $f \leq g \Leftrightarrow f(x) \leq g(x)$, $\forall x \in \mathbf{X}$

In [10] it has been shown that $C^+(\mathbf{X})$ forms an exponential vector space with respect to the operations and partial order defined as above.

We now define an internal multiplication '$*$' on $C^+(\mathbf{X})$ as follows: For any $f, g \in C^+(\mathbf{X})$, $(f*g)(x) := f(x)g(x)$, $\forall x \in \mathbf{X}$. Then by the following theorem we show that $C^+(\mathbf{X})$ forms an ealg with the additive identity being given by $\theta_{\mathbf{X}}(x) := 0, \forall\, x \in \mathbf{X}$ and the primitive algebra $[C^+(\mathbf{X})]_0 = \{\theta_{\mathbf{X}}\}$.

Theorem 2.1. $\Big(C^+(\mathbf{X}), +, \cdot, \leq, *\Big)$ *is a commutative unital exponential algebra over* $\mathbb{K}$.

Proof. Let $f, g \in C^+(\mathbf{X})$ and $a \in \mathbf{X}$. Then $|f(x)g(x) - f(a)g(a)| = |f(x)\{g(x) - g(a)\} + g(a)\{f(x) - f(a)\}| \leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)|$, $\forall\, x \in \mathbf{X} \cdots$ (1)
Since f, g are continuous at a, for any $\epsilon > 0, \exists$ open nbds. U, V of a in $\mathbf{X}$ such that $|f(x) - f(a)| < \epsilon, \forall\, x \in U$——(i) and $|g(x) - g(a)| < \epsilon, \forall\, x \in V$——(ii)
Again, f being continuous at a, $\exists$ an open nbd. W of a in $\mathbf{X}$ and $M > 0$ such that $|f(x)| \leq M, \forall\, x \in W$ [By nbd. property of continuous function]. Let $N := U \cap V \cap W$. Then N is an open nbd. of a in $\mathbf{X}$. Now from (1) using (i) and (ii) we have $|f(x)g(x) - f(a)g(a)| < M\epsilon + |g(a)|\epsilon, \forall\, x \in N$. This shows that $f * g$ is continuous at a. Arbitrariness of $a \in \mathbf{X}$ justifies that $f * g$ is continuous on $\mathbf{X}$

and hence $f * g \in C^+(\mathbf{X})$.
$\mathbf{EA_1}$: For $f, g, h \in C^+(\mathbf{X})$ we get $\big((f*g)*h\big)(x) = (f*g)(x)h(x) = f(x)g(x)h(x) = \big(f*(g*h)\big)(x), \forall\, x \in \mathbf{X}$.
$\mathbf{EA_2}$: Let $f \le g$. Then $f(x) \le g(x)$, $\forall x \in \mathbf{X}$.
Now $(h*f)(x) = h(x)f(x) \le h(x)g(x) = (h*g)(x)\,\&\,(f*h)(x) = f(x)h(x) \le g(x)h(x) = (g*h)(x)$, $\forall x \in \mathbf{X}$
$\mathbf{EA_3}$: For any $f, g, h \in C^+(\mathbf{X})$ we get $\big(f*(g+h)\big)(x) = f(x)(g+h)(x) = f(x)\big(g(x)+h(x)\big) = (f*g)(x)+(f*h)(x)$, $\forall x \in \mathbf{X}$. Therefore $f*(g+h) = f*g+f*h$. Similarly, $(g+h)*f = g*f+h*f$
$\mathbf{EA_4}$: $\big(\alpha\cdot(f*g)\big)(x) = |\alpha|(f*g)(x) = |\alpha|f(x)g(x) = (\alpha\cdot f)(x)g(x) = \big((\alpha\cdot f)*g\big)(x) = \big(f*(\alpha\cdot g)\big)(x)$, $\forall\, x \in \mathbf{X}, \forall\, \alpha \in \mathbb{K}$.
$\mathbf{EA_5}$: For any $f \in C^+(\mathbf{X})$, we get $(\theta_{\mathbf{X}} * f)(x) = \theta_{\mathbf{X}}(x)f(x) = 0 = f(x)\theta_{\mathbf{X}}(x) = (f*\theta_{\mathbf{X}})(x)$, $\forall x \in \mathbf{X} \implies \theta_{\mathbf{X}} * f = f * \theta_{\mathbf{X}} = \theta_{\mathbf{X}}$.
Thus according to definition 1.2, $\Big(C^+(\mathbf{X}), +, \cdot, \le, *\Big)$ is an exponential algebra over the field $\mathbb{K}$ (real or complex number field).

Clearly $C^+(\mathbf{X})$ is commutative and unital, the function given by $e(x) := 1$ $\forall\, x \in \mathbf{X}$ being the unity.

Let us consider the compact open topology on $C^+(\mathbf{X})$. For each subset K of $\mathbf{X}$ and each subset U of $\mathbb{R}$, we define
$$W(K, U) := \big\{f \in C^+(\mathbf{X}) : f(K) \subseteq U\big\}$$
The family of all sets of the form $W(K, U)$, where K is a compact subset of $\mathbf{X}$ and U is an open set in $\mathbb{R}$, is a subbase for the compact open topology for $C^+(\mathbf{X})$ [6]. Therefore for any $\epsilon > 0$ and any compact set K of $\mathbf{X}$, the family of sets of the form
$$V(f, K, \epsilon) := \big\{g \in C^+(\mathbf{X}) : |f(x) - g(x)| < \epsilon, \forall\, x \in K\big\}$$
is a subbase for the neighbourhood system of $f \in C^+(\mathbf{X})$ in the compact open topology. The following result is very much useful in handling the compact open topology.

Result 2.2. [10] *A net $(f_n)_{n\in D}$ in $C^+(\mathbf{X})$, D being a directed set, converges to f in the compact open topology of $C^+(\mathbf{X})$ iff $f_n \to f$ uniformly on every compact subset of $\mathbf{X}$.*

Theorem 2.3. [10] (1) *The addition '$+$' : $C^+(\mathbf{X}) \times C^+(\mathbf{X}) \longrightarrow C^+(\mathbf{X})$ is continuous.*
(2) *The scalar multiplication '$\cdot$' : $\mathbb{K} \times C^+(\mathbf{X}) \longrightarrow C^+(\mathbf{X})$ is continuous, where $\mathbb{K}$ is endowed with the usual topology.*
(3) *The partial order '$\le$' is closed.*

Theorem 2.4. *The internal multiplication '$*$' : $C^+(\mathbf{X}) \times C^+(\mathbf{X}) \longrightarrow C^+(\mathbf{X})$ is* *continuous and hence $C^+(\mathbf{X})$ is a topological ealg.*

Proof. Let $(f_n)_{n\in D}$ and $(g_n)_{n\in D}$ be two nets in $C^+(\mathbf{X})$, D being a directed set, such that $f_n \longrightarrow f$ and $g_n \longrightarrow g$ in $C^+(\mathbf{X})$ with compact open topology. Then by Result 2.2 we can say that $f_n \longrightarrow f$ and $g_n \longrightarrow g$ uniformly on every compact set in $\mathbf{X}$. Let K be a compact set in $\mathbf{X}$. So for any $\epsilon > 0$, $\exists\, p_1, p_2 \in D$ such that $|f_n(x) - f(x)| < \epsilon$, $\forall n \geq p_1 \cdots\cdots (i)$ and $|g_n(x) - g(x)| < \epsilon$, $\forall n \geq p_2 \cdots\cdots (ii)$. This is true for any $x \in K$. Now, D being a directed set, $\exists\, p \in D$ such that $p \geq p_1, p_2$. So

$$\begin{aligned}
\|f_n * g_n - f * g\|_K &:= \max_{x\in K} |(f_n * g_n)(x) - (f * g)(x)| \\
&= \max_{x\in K} |f_n(x)g_n(x) - f(x)g(x)| \\
&\leq \max_{x\in K} |f_n(x)||g_n(x) - g(x)| + \max_{x\in K} |g(x)||f_n(x) - f(x)| \\
&\leq \epsilon \cdot \max_{x\in K} |f_n(x)| + \epsilon \cdot \max_{x\in K} |g(x)|, \forall\, n \geq p \text{ [by (i) \& (ii)]} \\
&\leq \epsilon \cdot \max_{x\in K}\{|f_n(x) - f(x)| + |f(x)|\} + \epsilon \cdot \|g\|_K, \forall\, n \geq p \\
&= \epsilon \cdot \{\epsilon + \|f\|_K\} + \epsilon \cdot \|g\|_K, \forall\, n \geq p
\end{aligned}$$

Since f and g are continuous and K is compact in $\mathbf{X}$, $\|f\|_K$ and $\|g\|_K$ are finite. Thus it follows that $f_n * g_n \longrightarrow f * g$ uniformly on the compact set K in $\mathbf{X}$. So by 2.2 the internal multiplication $*$ is continuous. Therefore using the Theorem 2.3 we can say that $C^+(\mathbf{X})$ is a topological ealg.

Definition 2.5. [4] *A subset Y of an exponential algebra X is said to be a sub exponential algebra (subealg in short) if Y itself is an exponential algebra with all the compositions of X being restricted to Y.*

Note 2.6. Looking at the definition and necessary and sufficient condition of sub exponential vector space in [5] we have the following analogue for sub exponential algebra.

A subset Y of an exponential algebra X over the field K is a sub exponential algebra iff Y satisfies the following :

(i) $\alpha x + y, xy \in Y$, $\forall\, \alpha \in K$, $\forall\, x, y \in Y$.

(ii) $Y_0 \subseteq X_0 \bigcap Y$, where $Y_0 := \big\{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\big\}$

(iii) $\forall\, y \in Y$, $\exists\, p \in Y_0$ such that $p \leq y$.

If Y is a subealg of X then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$ we have $X_0 \cap Y \subseteq Y_0$.

Remark 2.7. *From above Note we can easily say that if Y is a subealg of an ealg X and Z is another subealg of X such that $Z \subseteq Y \subseteq X$ then Z is a subealg of Y*

also. In fact, $Z_0 = Z \cap X_0 = (Z \cap Y) \cap X_0 = Z \cap (Y \cap X_0) = Z \cap Y_0$. *This identity immediately suggests that if* Z *is a subealg of* Y *and* Y *is a subealg of an ealg* X *then* Z *is also a subealg of* X.

3. Ideals of $C^+(\mathbf{X})$

Definition 3.1. [4] *Let* X *be an ealg over the field* K. *Then a subealg* I *of* X *is said to be an ideal of* X *if* $XI \subseteq I$, $IX \subseteq I$ *and* $\downarrow I = I$.

In the above definition if we omit the condition $\downarrow I = I$ then I is called a *semiideal* of X. Clearly, every ideal is a semiideal, but converse may not be true.

Theorem 3.2. *Let* $C_b^+(\mathbf{X}) := \{f \in C^+(\mathbf{X}) : f$ *is bounded on* $\mathbf{X}\}$. *Then* $C_b^+(\mathbf{X})$ *is a subealg of* $C^+(\mathbf{X})$. *But not an ideal of* $C^+(\mathbf{X})$.
Proof. Let $f, g \in C_b^+(\mathbf{X})$ and $\alpha \in \mathbb{K}$. Then $\alpha f + g, f * g$ are bounded on $\mathbf{X}$ justifying that $\alpha f + g, f * g \in C_b^+(\mathbf{X})$.

Now $[C_b^+(\mathbf{X})]_0 = \{\theta_{\mathbf{X}}\} = C_b^+(\mathbf{X}) \cap [C^+(\mathbf{X})]_0$. Also for any $f \in C_b^+(\mathbf{X})$, $f \geq \theta_{\mathbf{X}}$. Thus in view of Note 2.6, $C_b^+(\mathbf{X})$ is a subealg of $C^+(\mathbf{X})$.

Remark 3.3. $C_b^+(\mathbf{X})$ *is not an ideal of* $C^+(\mathbf{X})$; *it is not even a semiideal of* $C^+(\mathbf{X})$. *In fact, if* $f \in C^+(\mathbf{X})$ *and* $g \in C_b^+(\mathbf{X})$ *then* $f * g$ *need not be bounded. For example, let* $\mathbf{X} = (0, \infty)$ *with usual subspace topology inherited from the real line* $\mathbb{R}$, $f(x) = x^2$, $g(x) = 1, \forall\, x \in \mathbf{X}$.

Definition 3.4. [9] *Let* $\mathbf{X}$ *be a topological space. A function* $f : \mathbf{X} \to \mathbb{R}$ *is said to be vanishing at infinity if for any* $\epsilon > 0, \exists$ *a compact set* K *in* $\mathbf{X}$ *such that* $|f(x)| < \epsilon, \forall\, x \notin K$.

Theorem 3.5. *Let* $C_0^+(\mathbf{X}) := \{f \in C^+(\mathbf{X}) : f$ *vanishes at infinity*$\}$. *Then* $C_0^+(\mathbf{X})$ *is a subealg of* $C^+(\mathbf{X})$.
Proof. Let $f, g \in C_0^+(\mathbf{X})$ and $\alpha \in \mathbb{K}$. Then $C^+(\mathbf{X})$ being an ealg, $\alpha f + g$, $f * g \in C^+(\mathbf{X})$. Let $\epsilon > 0$ be any number. Then $\exists$ compact sets K, F in $\mathbf{X}$ such that $|f(x)| < \epsilon, \forall\, x \notin K$ and $|g(x)| < \epsilon, \forall\, x \notin F$. Therefore $\big||\alpha| f(x) + g(x)\big| \leq |\alpha||f(x)| + |g(x)| < |\alpha|\epsilon + \epsilon, \forall\, x \notin K \cup F$, where $K \cup F$ is compact in $\mathbf{X}$. Also $|f(x)g(x)| < \epsilon^2, \forall\, x \notin K \cup F$. This shows that $\alpha f + g$, $f * g \in C_0^+(\mathbf{X})$.

Since for any $f \in C_0^+(\mathbf{X})$, we have $f \geq \theta_{\mathbf{X}}$, it follows that $[C_0^+(\mathbf{X})]_0 = \{\theta_{\mathbf{X}}\} = C_0^+(\mathbf{X}) \cap [C^+(\mathbf{X})]_0$. Therefore by Note 2.6, we can say that $C_0^+(\mathbf{X})$ is a subealg of $C^+(\mathbf{X})$.

Remark 3.6. *(i)* $C_0^+(\mathbf{X})$ *is neither an ideal nor a semiideal of* $C^+(\mathbf{X})$. *In fact, if* $f \in C^+(\mathbf{X})$ *and* $g \in C_0^+(\mathbf{X})$ *then* $f * g$ *need not vanish at infinity. For example, let* $\mathbf{X} = [0, \infty)$ *with usual subspace topology inherited from the real line* $\mathbb{R}$, $f(x) = x^2$,

$$g(x) = \begin{cases} \frac{1}{x}, & \text{if } x \geq 1 \\ x, & \text{if } 0 \leq x < 1 \end{cases}.$$

(ii) Let $f \in C_0^+(\mathbf{X})$. Then for $\epsilon = 1, \exists$ a compact set K in $\mathbf{X}$ such that $|f(x)| < 1, \forall x \notin K$. Again f being continuous and K being compact, $\|f\|_K := \max\limits_{x \in K} |f(x)|$ is finite. So $|f(x)| \leq \max\{1, \|f\|_K\}, \forall x \in \mathbf{X} \implies f \in C_b^+(\mathbf{X})$. Thus $C_0^+(\mathbf{X}) \subseteq C_b^+(\mathbf{X})$. So by Theorems 3.2, 3.5 and Remark 2.7 we can say that $C_0^+(\mathbf{X})$ is a subealg of $C_b^+(\mathbf{X})$.

We now show that $C_0^+(\mathbf{X})$ is an ideal of $C_b^+(\mathbf{X})$.

Theorem 3.7. *$C_0^+(\mathbf{X})$ is an ideal of $C_b^+(\mathbf{X})$.*

Proof. By Remark 3.6 (ii), $C_0^+(\mathbf{X})$ is a subealg of $C_b^+(\mathbf{X})$. Now let $f \in C_b^+(\mathbf{X})$ and $g \in C_0^+(\mathbf{X})$. Then $\|f\| := \max\limits_{x \in \mathbf{X}} |f(x)| < \infty$. Also for any $\epsilon > 0, \exists$ a compact set K in $\mathbf{X}$ such that $|g(x)| < \epsilon, \forall x \notin K$. Therefore $|f(x)g(x)| < |f(x)|\epsilon \leq \epsilon\|f\|, \forall x \notin K$ $\implies f * g \in C_0^+(\mathbf{X})$. Similarly we can show that $g * f \in C_0^+(\mathbf{X})$.

Now let $f \in\downarrow C_0^+(\mathbf{X})$. Then $\exists\, g \in C_0^+(\mathbf{X})$ such that $f \leq g \implies f(x) \leq g(x)$, $\forall x \in \mathbf{X}$. So for any $\epsilon > 0, \exists$ a compact set F in $\mathbf{X}$ such that $0 \leq g(x) < \epsilon, \forall x \notin F$ $\implies 0 \leq f(x) \leq g(x) < \epsilon, \forall x \notin F \implies f \in C_0^+(\mathbf{X})$. Consequently, $\downarrow C_0^+(\mathbf{X}) = C_0^+(\mathbf{X})$. Thus $C_0^+(\mathbf{X})$ is an ideal of $C_b^+(\mathbf{X})$.

Definition 3.8. [4] *A proper ideal M of an ealg X is said to be a maximal ideal if it is not contained in any other proper ideal of X.*

We now consider another subealg of $C^+(\mathbf{X})$ and show that it is an ideal. For this we need the following concept.

Definition 3.9. [9] *Let $\mathbf{X}$ be a topological space and $f : \mathbf{X} \to \mathbb{R}$ be a function. By support of f we define supp$(f) := \overline{\{x \in \mathbf{X} : f(x) \neq 0\}}$.*

Theorem 3.10. *Let $C_c^+(\mathbf{X}) := \{f \in C^+(\mathbf{X}) :$ supp(f) is compact in $\mathbf{X}\}$. Then $C_c^+(\mathbf{X})$ is an ideal of $C^+(\mathbf{X})$.*

Proof. We fist show that $C_c^+(\mathbf{X})$ is a subealg of $C^+(\mathbf{X})$. For this let $f, g \in C_c^+(\mathbf{X})$ and $\alpha \in \mathbb{K}$. Then $\alpha f + g, f * g \in C^+(\mathbf{X})$, since $C^+(\mathbf{X})$ is an ealg. Now supp$(\alpha f) = \overline{\{x \in \mathbf{X} : |\alpha| f(x) \neq 0\}}$. If $\alpha = 0$ then $(\alpha f)(x) = 0$, $\forall x \in \mathbf{X}$. So supp$(\alpha f) = \emptyset$, a compact subset in $\mathbf{X}$ trivially. If $\alpha \neq 0$, then supp$(f) = \overline{\{x \in \mathbf{X} : f(x) \neq 0\}} = \overline{\{x \in \mathbf{X} : |\alpha| f(x) \neq 0\}} =$ supp(αf). So $f \in C_c^+(\mathbf{X}) \Rightarrow \alpha f \in C_c^+(\mathbf{X})$.

We first note that $f(x) + g(x) \neq 0 \Rightarrow f(x) \neq 0$ or $g(x) \neq 0$ [$\because f, g$ both are non-negative] $\implies$ supp$(f + g) \subseteq$ supp$(f) \cup$ supp(g). Since supp(f) and supp(g) both are compact and supp$(f + g) = \overline{\{x \in \mathbf{X} : f(x) + g(x) \neq 0\}}$ is a closed subset of the compact set supp$(f) \cup$ supp(g), it follows that supp$(f + g)$ is compact in $\mathbf{X}$. Thus $f, g \in C_c^+(\mathbf{X}) \Rightarrow f + g \in C_c^+(\mathbf{X})$.

Now $f(x)g(x) \neq 0 \implies f(x) \neq 0$ and $g(x) \neq 0$. Therefore $\{x \in \mathbf{X} : f(x)g(x) \neq 0\} \subseteq \{x \in \mathbf{X} : f(x) \neq 0\} \cap \{x \in \mathbf{X} : g(x) \neq 0\}$. Consequently, $\operatorname{supp}(f * g) = \overline{\{x \in \mathbf{X} : f(x)g(x) \neq 0\}} \subseteq \overline{\{x \in \mathbf{X} : f(x) \neq 0\}} \cap \overline{\{x \in \mathbf{X} : g(x) \neq 0\}} = \operatorname{supp}(f) \cap \operatorname{supp}(g)$. Since $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are compact sets, it follows that $\operatorname{supp}(f * g)$ is a compact subset in $\mathbf{X}$. Thus $f, g \in C_c^+(\mathbf{X}) \Rightarrow f * g \in C_c^+(\mathbf{X})$.

Clearly, $\theta_{\mathbf{X}} \in C_c^+(\mathbf{X})$, since $\operatorname{supp}(\theta_{\mathbf{X}}) = \emptyset$, a trivial compact set in $\mathbf{X}$. Also $[C_c^+(\mathbf{X})]_0 = \{\theta_{\mathbf{X}}\} = [C^+(\mathbf{X})]_0 \cap C_c^+(\mathbf{X})$ and for any $f \in C_c^+(\mathbf{X})$, $\theta_{\mathbf{X}} \leq f$. Therefore by Note 2.6, $C_c^+(\mathbf{X})$ is a subealg of $C^+(\mathbf{X})$.

We now show that $C_c^+(\mathbf{X})$ is an ideal of $C^+(\mathbf{X})$. For this let $h \in C^+(\mathbf{X})$ and $f \in C_c^+(\mathbf{X})$. Since $C^+(\mathbf{X})$ is an ealg, we have $f * h, h * f \in C^+(\mathbf{X})$. Now by above discussion, $\operatorname{supp}(h * f) \subseteq \operatorname{supp}(h) \cap \operatorname{supp}(f) \subseteq \operatorname{supp}(f)$. So $\operatorname{supp}(h * f)$ being a closed subset of the compact set $\operatorname{supp}(f)$, is also a compact subset in $\mathbf{X}$. Thus $h * f \in C_c^+(\mathbf{X})$. Similarly we can show that $f * h \in C_c^+(\mathbf{X})$.

Now let $h \in \downarrow C_c^+(\mathbf{X})$. Then $\exists f \in C_c^+(\mathbf{X})$ such that $h \leq f \implies h(x) \leq f(x)$, $\forall x \in \mathbf{X}$. So $\operatorname{supp}(h) = \overline{\{x \in \mathbf{X} : h(x) \neq 0\}} \subseteq \overline{\{x \in \mathbf{X} : f(x) \neq 0\}} = \operatorname{supp}(f)$. Since $\operatorname{supp}(h)$ is a closed subset of the compact set $\operatorname{supp}(f)$ [$\because f \in C_c^+(\mathbf{X})$], we have $h \in C_c^+(\mathbf{X})$. Thus $\downarrow C_c^+(\mathbf{X}) = C_c^+(\mathbf{X})$.

Therefore according to the Definition 3.1, $C_c^+(\mathbf{X})$ is an ideal of $C^+(\mathbf{X})$.

Remark 3.11. *(i) Let $f \in C_c^+(\mathbf{X})$. Then $\operatorname{supp}(f) = K$ (say) is compact in $\mathbf{X}$. So $f(x) = 0$, $\forall\, x \notin K$. Therefore, for any $\epsilon > 0$ we have $|f(x)| = 0 < \epsilon$, $\forall\, x \notin K$. This justifies that $f \in C_0^+(\mathbf{X})$. Thus $C_c^+(\mathbf{X}) \subseteq C_0^+(\mathbf{X})$.*

(ii) From above discussion we have the following hierarchy for any topological space $\mathbf{X}$:

$$C_c^+(\mathbf{X}) \subseteq C_0^+(\mathbf{X}) \subseteq C_b^+(\mathbf{X}) \subseteq C^+(\mathbf{X})$$

Here each smaller space is a subealg of the larger one [by Remark 2.7].

(iii) If $\mathbf{X}$ is a compact topological space then clearly, $C_c^+(\mathbf{X}) = C_0^+(\mathbf{X}) = C_b^+(\mathbf{X}) = C^+(\mathbf{X})$.

(iv) Let $\mathbf{X} = [0, \infty)$ equipped with the subspace topology inherited from the real line $\mathbb{R}$. Let $f(x) := \begin{cases} \frac{1}{x}, & \text{if } x \geq 1 \\ x, & \text{if } 0 \leq x < 1 \end{cases}$

Then $f \in C_0^+(\mathbf{X}) \smallsetminus C_c^+(\mathbf{X})$. Next consider $g(x) := 1, \forall\, x \in \mathbf{X}$. Then $g \in C_b^+(\mathbf{X}) \smallsetminus C_0^+(\mathbf{X})$. If we consider the function $h(x) := x^2, \forall\, x \in \mathbf{X}$ then $h \in C^+(\mathbf{X}) \smallsetminus C_b^+(\mathbf{X})$.

(v) Since $C_c^+(\mathbf{X})$ is an ideal of $C^+(\mathbf{X})$ [by Theorem 3.10] and $C_c^+(\mathbf{X}) \subseteq C_b^+(\mathbf{X}) \subseteq C^+(\mathbf{X})$, we can say that $C_c^+(\mathbf{X})$ is an ideal of $C_b^+(\mathbf{X})$ also. Thus in view of the Theorem 3.7, if $C_c^+(\mathbf{X}) \subsetneqq C_0^+(\mathbf{X}) \subsetneqq C_b^+(\mathbf{X})$ then $C_c^+(\mathbf{X})$ is not a maximal ideal.

We now construct some more ideals of $C^+(\mathbf{X})$.

Theorem 3.12. *Let $\mathbf{X}$ be a topological space and $A \subseteq \mathbf{X}$. Then*

$I_A := \{f \in C^+(\mathbf{X}) : f(a) = 0, \forall\, a \in A\}$ is an ideal of $C^+(\mathbf{X})$. Moreover I_A is proper if $A \neq \emptyset$.

Proof. Let $f, g \in I_A$. Then $f(x) = 0 = g(x), \forall\, x \in A$. Now $(f+g)(x) = f(x) + g(x) = 0, \forall x \in A \Rightarrow f + g \in I_A$. For any $\alpha \in \mathbb{K}$ we have $(\alpha f)(x) = |\alpha| f(x) = 0$, $\forall x \in A \Rightarrow \alpha \cdot f \in I_A$. Again $(f * g)(x) = f(x)g(x) = 0$, $\forall x \in A$. Thus $f * g \in I_A$.

Clearly $\theta_{\mathbf{X}} \in I_A$. So $[I_A]_0 = \{\theta_{\mathbf{X}}\} = [C^+(\mathbf{X})]_0 \cap I_A$. Also for any $f \in I_A$ we have $f \geq \theta_{\mathbf{X}}$. Thus in view of Note 2.6, I_A is a subealg of $C^+(\mathbf{X})$.

Now let $f \in C^+(\mathbf{X})$ and $g \in I_A$. Since $(f * g)(a) = f(a)g(a) = 0, \forall\, a \in A$, we have $f * g, g * f \in I_A$.

Again for $\phi \in \downarrow I_A$, $\phi \leq f$ for some $f \in I_A \Rightarrow \phi(x) \leq f(x)$, $\forall x \in \mathbf{X}$. Since $f(x) = 0$, $\forall x \in A$ and $\phi(x) \geq 0, \forall\, x \in \mathbf{X}$ [$\because \phi \in C^+(\mathbf{X})$] we have $\phi(x) = 0$, for all $x \in A \implies \phi \in I_A$. Thus $\downarrow I_A = I_A$. Therefore according to the Definition 3.1, I_A is an ideal of $C^+(\mathbf{X})$.

Let $f(x) := 1, \forall\, x \in \mathbf{X}$. If $A \neq \emptyset$ then $f \in C^+(\mathbf{X}) \setminus I_A$. Clearly $I_\emptyset = C^+(\mathbf{X})$.

Proposition 3.13. *Let $A \subseteq B \subseteq \mathbf{X}$. Then $I_A \supseteq I_B$.*

Proof. Immediate from the construction of I_A.

Note 3.14. For any $c \in \mathbf{X}$ we define $I_c := I_{\{c\}} = \{f \in C^+(\mathbf{X}) : f(c) = 0\}$.

Theorem 3.15. *Let $\mathbf{X}$ be a compact topological space and $\mathscr{F}$ be a maximal ideal of $C^+(\mathbf{X})$ such that for any finite subset $\{f_1, \ldots, f_n\} \subseteq \mathscr{F}, \exists\, p \in \mathbf{X}$ such that $f_i(p) = 0, \forall\, i = 1, \ldots, n$. Then $\exists\, c \in \mathbf{X}$ such that $\mathscr{F} = I_c$.*

Proof. For $f \in \mathscr{F}$ define $Z(f) := \{x \in \mathbf{X} : f(x) = 0\}$. Then each $Z(f)$ is closed [$\because f$ is continuous]. By given hypothesis, for any finite subset $\{f_1, \ldots, f_n\} \subseteq \mathscr{F}, \exists\, p \in \mathbf{X}$ such that $f_i(p) = 0, \forall\, i = 1, \ldots, n$. This implies that $p \in \bigcap_{i=1}^{n} Z(f_i)$. This justifies that $\{Z(f) : f \in \mathscr{F}\}$ is a family of closed sets having finite intersection property. So $\mathbf{X}$ being compact, $\bigcap_{f \in \mathscr{F}} Z(f) \neq \emptyset$. Define $A := \bigcap_{f \in \mathscr{F}} Z(f)$

Let $f \in \mathscr{F}$. Then $A \subseteq Z(f) \implies f(A) = \{0\} \implies f \in I_A$. Thus $\mathscr{F} \subseteq I_A$. Again by Theorem 3.12, I_A is a proper ideal of $C^+(\mathbf{X})$. So maximality of $\mathscr{F}$ implies that $\mathscr{F} = I_A$.

Let $c \in A$ and if possible let A contains more points other than c. Then $\mathscr{F} = I_A \subsetneqq I_c$ [By Proposition 3.13]. This contradicts the maximality of $\mathscr{F}$, since I_c is a proper ideal of $C^+(\mathbf{X})$ [by Theorem 3.12 and Note 3.14]. So $A = \{c\}$. Therefore $\mathscr{F} = I_c$.

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