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Three Expansions for a Three Variable Hypergeometric Function

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Abstract: In this paper we record three summation results for a triple hypergeometric series X_2 and discuss various cases of reducibility.
Keywords: Hypergeometric function, Horn function, Appell function, Laguerre polynomial, Jacobi polynomial.

1. Introduction:

Exton [1] introduced a triple hypergeometric series whose representation is

$$X_2(a,b;c_1,c_2,c_3;x,y,z) = \sum_{0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \tag{1.1}$$

where

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \qquad a = 0, -1, -2, ...$$

The precise three dimensional region of convergence of (1.1) is given by, see [2],

$$2\sqrt{r} + 2\sqrt{s} + t \quad 1, |x| \quad r, |y| \quad s, \text{ and } \quad |z| \quad t$$

where the positive quantities r, s, and t are associated radii of convergence. For details of this function and other many related series refer to Exton [1] and Srivastava and Karlsson [3].

The Laplace type integral representation of (1.1.) due to Exton is

$$X_2(a,b;c_1,c_2,c_3;x,y,z)$$

$$= \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-s} s^{a-1} \ {}_0F_1(-;c_1;xs^2) \ {}_0F_1(-;c_2;ys^2) \ {}_1F_1(b;c_3;zs) ds \tag{1.2}$$

where $Re(a) > 0$.

2. In this section we derive the following,

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(\alpha+n)_m}{(1+\alpha)_m} X_2(a, m-n; c_1, c_2, 1+\alpha+m; x, y, z)$$

$$= \frac{(-z)^n}{(1+\alpha)_n} F_4\left(\frac{a}{2}, \frac{a+1}{2}; c_1, c_2; 2x, 2y\right). \tag{2.1}$$

where the function on the right is Appell function F_4 see [3].

$$\sum_{m=0}^{n} \frac{(-n)_m (1+\beta)_m}{m!(1+\alpha)_m} \, {}_2F_1(-n+m, \beta+m+1; \alpha+k+1; z) X_2(a, -m; c_1, c_2, 1+\beta; x, y, 1)$$

$$= X_2(a, -n; c_1, c_2, 1+\alpha; x, y, z). \tag{2.2}$$

where the parameter β is so restricted that the second number exists,

$$\sum_{m=0}^{n} X_2(a, -m; c_1, c_2, 1; x, y, z) = (n+1) X_2(a, -n; c_1, c_2, 2; x, y, z). \tag{2.3}$$

The following two special cases of (2.1) are worth mentioning.

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(\alpha+n)_m}{(1+\alpha)_m} X_2(a, m-n; c_1, c_2, 1+\alpha+m; x, y, z)$$

$$= \frac{(-z)^n}{(1+\alpha)_n} \, {}_4F_3\left(\frac{a}{2}, \frac{a+1}{2}, \frac{c_1+c_2}{2}, \frac{c_1+c_2-1}{2}; c_1, c_2, c_1+c_2-1; 4x\right). \tag{2.4}$$

which expressa special X_2 in terms of hypergeometric function.

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(\alpha+n)_m}{(1+\alpha)_m} X_2\left(2a, m-n; a+\frac{1}{2}, \frac{1}{2}, 1+\alpha+m; x, y, z\right)$$

$$= \frac{(-z)^n}{2(1+\alpha)_n} \left[(1+\sqrt{2y}-x)^{-a} + (1-\sqrt{2y}-x)^{-a}\right] \tag{2.5}$$

In particular, (for $\alpha = 0, x = y = z = 1/2$), (2.5) takes the elegant form,

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \Gamma(n+m) X_2\left(2, m-n, \frac{-1}{2}, \frac{1}{2}, 1+m; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{(-1)^n}{n 2^{2n+1}}$$

Proof: To prove (2.1), we use the well known result on the classical Laguerre polynomials viz., see [2, p. 1109],

$$\sum_{m=0}^{n} \frac{(-1)^n (\alpha+n)_m}{m!} L_{n-m}^{(\alpha+m)}(x) = \frac{(-x)^n}{n!} \tag{2.6}$$

where $L_n^\alpha(x) = \dfrac{(1+\alpha)_n}{n!}\ {}_1F_1(-n; 1+\alpha; x)$,
in the integral of (1.2), (for the ${}_1F_1$), change the order of integration and summation and itnterpret the resultatnt as Appell F_4.

In a similar manner (2.2) and (2.3) can be proved by employing the results, see [4, p. 152 and p. 166],

$$L_n^\alpha(xy) = \sum_{m=0}^{n} \frac{(1+\alpha)_n y^m}{(n-k)!(1+\alpha)_m}\ {}_2F_1(-n+m, \beta+m+1; \alpha+m+1; y) L_m^\beta(x),$$

and

$$\sum_{m=0}^{n} {}_1F_1(-m; 1; x) = (n+1)\ {}_1F_1(-n; 2; x)$$

in stead of (2.6).
Using the relation, see [3, p. 314]
$F_4(a, b; c, d; x, x) = {}_4F_3\left(a, b, \dfrac{c+d}{2}, \dfrac{c+d+1}{2}; c, d, c+d-1; 4x\right)$ in (2.1), we get (2.4). Further considering $a = 2, c_1 = a + \dfrac{1}{2}$ and $c_2 = \dfrac{1}{2}$, (2.4) gives (2.5).

References

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