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FIRST ZAGREB MATRIX AND ENERGY OF A T_2 HYPERGRAPH

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Abstract: Let H be a T_2 hypergraph of order $n \geq 4$. The first Zagreb matrix of H, denoted by $Z(H)$ is defined as the square matrix of order n, whose $(i, j)^{th}$ entry is $d_i + d_j$ if x_i and x_j are adjacent and zero for other cases. The first Zagreb energy $ZE(H)$ of H is the sum of the absolute values of the eigenvalues of $Z(H)$. It is shown that, for a T_2 hypergraph $ZE(H) \leq \left\lceil \frac{\sqrt{2}(n^2+3n+1)}{\sqrt{3}} \right\rceil$.

Keywords and Phrases: T_2 hypergraph, first Zagreb matrix, first Zagreb energy.

2020 Mathematics Subject Classification: 05C65, 05C50.

1. Introduction

The basic definitions and terminologies of a hypergraph are not given here and we refer to it [2] and [11]. The concept of hypergraph was introduced by Berge in 1967. In 2017, Seena V and Raji Pilakkat introduced Hausdorff hypergraph, T_0 hypergraph and T_1 hypergraph [7], [8] and [9]. Based on [8] and [9] S. Sujitha and D. Sharmila introduced T_2 hypergraph and studied Randic matrix and the corresponding energy in [10]. In 1977, Gutman [3] defind graph energy. In 2007, Nikiforov [6] extended the concept of graph energy to matrices. The first Zagreb energy was introduced by Nader Jafari Rad, Akbar Jahanbani and Ivan Gutman in [4] and later the same was studied by many authors. In this article, we study the first Zagreb matrix and first Zagreb energy of a T_2 hypergraph. Throughout this article, H is a simple connected T_2 hypergraph with order n and size m, where the order and size are the minimum number of vertices and edges need to define a T_2 hypergraph. The number of edges of a hpergraph H that are incident to a given vertex is called the degree of the vertex [1]. The maximum degree is denoted by Δ and the minimum degree is denoted by δ . The following definitions and theorems are used in sequel.

Definition 1.1. [10] *A hypergraph $H = (X, D)$ is said to be a T_2 hypergraph, if for any three distinct vertices u, v and w in X, there exist a hyperedge containing u and v but not w, and another hyperedge containing w, but not u and v.*

Example 1.2.

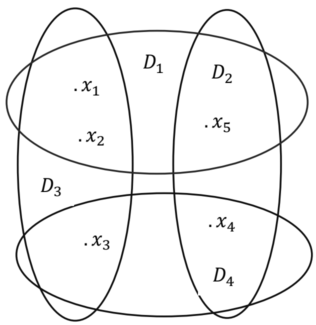

Figure 1: T_2 Hypergraph

Figure 1 is a T_2 Hypergraph with vertices x_1, x_2, x_3, x_4, x_5 and hyperedges D_1, D_2, D_3, D_4. It is easily seen that, for every three vertices x_i, x_j and x_k there exist a hyperedge containing x_i and x_j but not x_k and a hyperedge containing x_k but not x_i, x_j.

Result 1.3. [10]

(i) The minimum number of edges need to define a T_2 hypergraph is $m = \left[\frac{2n+5}{4}\right]$ where n is the number of vertices.

(ii) For a T_2 hypergraph H, the minimum degree $\delta(H) = 2$.

(iii) For a T_2 hypergraph H, rank $r = \left[\frac{2n+1}{4}\right]$ where $n \geq 5$.

Definition 1.4. [1] *A hypergraph H is said to be a k-uniform hypergraph (or a k-graph) for $k \geq 2$, if all edges have the same cardinality k.*

2. First Zagreb matrix and energy of a T_2 Hypergraph

The First Zagreb matrix and energy of a graph was introduced by Nader Jafari Rad, Akbar Jahanbani and Ivan Gutman. The first Zagreb matrix of a T_2 hypergraph is defined as follows,

Definition 2.1. [5] *The first Zagreb matrix is defined by*

$$Z(H)=\begin{cases} d_i + d_j \ if \ x_i x_j \in D \\ 0 \quad otherwise \end{cases}$$

Definition 2.2. [5] *The first Zagreb energy is defined by* $ZE(H)= \sum_{i=1}^{n} |\lambda_i|$

In this section, we find the energy of a T_2 hypergraph using first Zagreb matrix. Consider a T_2 hypergraph given in Figure 2 with 10 vertices and 6 edges.

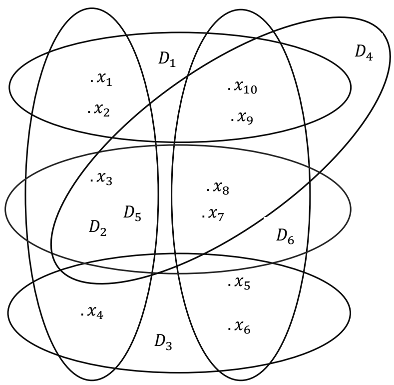

Figure 2: T_2 Hypergraph

The first Zagreb matrix of $H = T_2$ is given by

$$Z(H)=\begin{pmatrix} 0 & 4 & 5 & 5 & 0 & 0 & 0 & 0 & 5 & 5 \\ 4 & 0 & 5 & 5 & 0 & 0 & 0 & 0 & 5 & 5 \\ 5 & 5 & 0 & 6 & 0 & 0 & 6 & 6 & 6 & 6 \\ 5 & 5 & 6 & 0 & 5 & 5 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 5 & 0 & 4 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 5 & 4 & 0 & 5 & 5 & 5 & 5 \\ 0 & 0 & 6 & 6 & 5 & 5 & 0 & 6 & 6 & 6 \\ 0 & 0 & 6 & 6 & 5 & 5 & 6 & 0 & 6 & 6 \\ 5 & 5 & 6 & 6 & 5 & 5 & 6 & 6 & 0 & 6 \\ 5 & 5 & 6 & 6 & 5 & 5 & 6 & 6 & 6 & 0 \end{pmatrix}$$

The first Zagreb eigen values of $Z(H)$ are
$\lambda = 39.76, 8.92, -.67, -4, -4, -6, -6, -6, -10.45, -11.55$
Therefore, first Zagreb energy $ZE(H)= \sum_{i=1}^{n} |\lambda_i| = 97.35$

Result 2.3. For a T_2 hypergraph $ZE(T_2) \leq \left\lceil \frac{\sqrt{2}(n^2+3n+1)}{\sqrt{3}} \right\rceil$ where $4 \leq n \leq 20$. Equality holds only if $n = 4$ in a T_2 hypergraph.
Proof. The following table gives the first Zagreb energy of a T_2 hypergraph of order n, where $4 \leq n \leq 20$. It is clear that, $24 \leq ZE(H) \leq 323 \leq \left\lceil \frac{\sqrt{2}(n^2+3n+1)}{\sqrt{3}} \right\rceil$.
Therefore, the upperbound of ZE(H) with respect to the order of the T_2 hypergraph is $ZE(T_2) \leq \left\lceil \frac{\sqrt{2}(n^2+3n+1)}{\sqrt{3}} \right\rceil$

Vertices	Energy	$\left\lceil \frac{\sqrt{2}(n^2+3n+1)}{\sqrt{3}} \right\rceil$
4	24	24
5	25.41	34
6	34.97	45
7	48.14	58
8	63.97	73
9	79.82	90
10	97.35	107
11	115.39	128
12	128.12	148
13	148.05	171
14	143.97	196
15	178.46	222
16	195.79	250
17	239.57	279
18	236.94	310
19	268.41	343
20	323	377

Table 1: First Zagreb energy of a T_2 hypergraph

3. Main Results

In this section, we obtain the bounds of the first Zagreb energy of a T_2 hypergraph by using the graph parameters rank r, minimum degree δ, and maximum degree

Δ. Throughtout this section, we use A $= \sum_{i=1}^{n}\sum_{j=1}^{n}(d_i+d_j)$, B $= \sum_{i=1}^{n}\sum_{j=1}^{n}(d_i+d_j)^2$

Result 3.1. For a T_2 hypergraph $B \leq \delta r^4 + 10\Delta^4 - \delta$ where $n \geq 4$, equality holds only if $n = 6$ in T_2 hypergraph.
Proof. Let H be a T_2 hypergraph with order $4 \leq n \leq 20$.
The result can be easily proved by using the following table,

Vertices	B	$\delta r^4 + 10\Delta^4 - \delta$
4	192	670
5	224	320
6	320	320
7	488	1320
8	878	1320
9	1238	2058
10	2044	2058
11	2084	3400
12	2416	3400
13	3160	5610
14	3192	5610
15	5352	10750
16	4968	10750
17	7576	15680
18	8760	15680
19	8792	22558
20	18168	26248

Table 2: Values of B in a T_2 hypergraph

Theorem 3.2. *Let H be a T_2 hypergraph with n vertices, where $4 \leq n \leq 20$. Then* $ZE(H) < \frac{A}{n} + \sqrt{\frac{A}{n}} + \sqrt{(n-2)(\delta r^4 + 10\Delta^4 - \delta) - \frac{A}{n} - (\frac{A}{n})^2}$.
Proof. By Cauchy - Schwarz inequality
We have $(\sum_{i=2}^{n-1}|\lambda_i|)^2 \leq (n-2)\sum_{i=2}^{n-1}|\lambda_i|^2$

$$(ZE(H) - |\lambda_1| - |\lambda_n|)^2 \leq (n-2)[\sum_{i=1}^{n}|\lambda_i|^2 - |\lambda_1|^2 - |\lambda_n|^2]$$

$$ZE(H) \leq \ |\lambda_1| + |\lambda_n| + \sqrt{(n-2)[(\delta r^4 + 10\Delta^4 - \delta) - |\lambda_1|^2 - |\lambda_n|^2]}$$

Let $|\lambda_1| = x$ and $|\lambda_n| = y$

and $z(x,y) = x + y + \sqrt{(n-2)[(\delta r^4 + 10\Delta^4 - \delta) - x^2 - y^2]}$
Differentiating $z(x,y)$ partially with respect to x and y,

$$z_x = 1 - \frac{x(n-2)}{\sqrt{n-2}|[\delta r^4 + 10\Delta^4 - \delta - x^2 - y^2|]}$$

$$z_y = 1 - \frac{y(n-2)}{\sqrt{n-2}|[\delta r^4 + 10\Delta^4 - \delta - x^2 - y^2|]}$$

$$z_{xx} = -\frac{\sqrt{n-2}(\delta r^4 + 10\Delta^4 - \delta - y^2)}{(\delta r^4 + 10\Delta^4 - \delta - x^2 - y^2)^{\frac{3}{2}}}$$

$$z_{yy} = -\frac{\sqrt{n-2}(\delta r^4 + 10\Delta^4 - \delta - x^2)}{(\delta r^4 + 10\Delta^4 - \delta - x^2 - y^2)^{\frac{3}{2}}}$$

$$z_{xy} = -\frac{\sqrt{n-2}(xy)}{(\delta r^4 + 10^4 - \delta - x^2 - y^2)^{\frac{3}{2}}}$$

The stationary points are given by $z_x = 0$ and $z_y = 0$
$z_x = 0 \Rightarrow \delta r^4 + 10\Delta^4 - \delta - x^2(n-1) + y^2 = 0$
$z_y = 0 \Rightarrow \delta r^4 + 10\Delta^4 - \delta - y^2(n-1) + x^2 = 0$
Solving the above eqns,

$$x = \ y = \sqrt{\frac{\delta r^4 + 10\Delta^4 - \delta}{n}}$$

At x *and* y, z_{xx}, z_{yy}, z_{xy} are all negative and
$\Delta = (z_{xx})\ (z_{yy}) - (z_{xy})^2$ is positive.
If $x = y = \sqrt{\frac{\delta r^4 + 10\Delta^4 - \delta}{n}}$ then $z(x,y) = \sqrt{n(\delta r^4 + 10\Delta^4 - \delta)}$
Also, $z(x,y)$ decreases in the interval

$$\sqrt{\frac{\delta r^4 + 10\Delta^4 - \delta}{n}} < x < \sqrt{B} \text{ and}$$

$$0 < y < \sqrt{\tfrac{A}{n}} < \sqrt{B}$$

Therefore, $\sqrt{\frac{\delta r^4+10\Delta^4-\delta}{n}} < \frac{A}{n} < \lambda_1 < \sqrt{B}$

$0 < \lambda_n < \sqrt{\frac{A}{n}} < \sqrt{\frac{\delta r^4+10\Delta^4-\delta}{n}} < \sqrt{2B}$

Hence $z(x,y) < z(\frac{A}{n}, \sqrt{\frac{A}{n}}) < z(\frac{\delta r^4+10\Delta^4-\delta}{n}, \sqrt{\frac{\delta r^4+10\Delta^4-\delta}{n}})$

$ZE(H) < \frac{A}{n} + \sqrt{\frac{A}{n}} + \sqrt{(n-2)\left[(\delta r^4 + 10\Delta^4 - \delta) - \frac{A}{n} - (\frac{A}{n})^2\right]}$

where $B = \sum_{i=1}^{n}\sum_{j=1}^{n}(d_i + d_j)^2$

$$A = \sum_{i=1}^{n}\sum_{j=1}^{n}(d_i + d_j)$$

Theorem 3.3. *Let H be a T_2 hypergraph with $n \geq 4, n \neq 5$ and 6.*
Then $ZE(T_2) > \frac{Z(H)}{(detZ(H))^{\frac{1}{n}}}$.

Proof. From an arithmetic and a geometric mean inequality,

$$\frac{\sum\limits_{i=1}^{n} |\lambda_i|}{n} \geq (detZ(H))^{\frac{1}{n}}$$

we have $|\lambda_n| < (detZ(H))^{\frac{1}{n}}$

$$|\lambda_n| \sum_{i=1}^{n} |\lambda_i| < (detZ(H))^{\frac{1}{n}} \sum_{i=1}^{n} |\lambda_i| \ldots\ldots\ldots\ldots\ldots(1)$$

$$|\lambda_n| \sum_{i=1}^{n} |\lambda_i| = (|\lambda_n|)(|\lambda_1| + |\lambda_2| + ... + |\lambda_n|)$$
$$\geq |\lambda_n| (|\lambda_n| + |\lambda_n| + ... + |\lambda_n|)$$
$$= |\lambda_n|^2 [1 + 1 + ... + 1]$$
$$= n |\lambda_n|^2$$

$$n |\lambda_n|^2 \leq |\lambda_n| \sum_{i=1}^{n} |\lambda_i| \ldots\ldots\ldots\ldots\ldots(2)$$

From equation (1) and (2)

$$n |\lambda_n|^2 < (detZ(H))^{\frac{1}{n}} \sum_{i=1}^{n} |\lambda_i|$$
$$n |\lambda_n|^2 < (detZ(H))^{\frac{1}{n}} ZE(T_2)$$
$$\frac{n |\lambda_n|^2}{(detZ(H))^{\frac{1}{n}}} < ZE(T_2)$$

Since $|\lambda_n|^2 > \frac{A}{n}, \frac{A}{(detZ(H))^{\frac{1}{n}}} < ZE(T_2)$.

Theorem 3.4. *Let H be a T_2 hypergraph with $n \geq 4$ vertices. Then* $ZE(T_2) \leq \frac{A}{n} + \sqrt{(n-1)[B - (\frac{A}{n})^2]}$. *Equality holds only if H is a T_2 hypergraph with $n = 4$.*

Proof. We have $(\sum\limits_{i=2}^{n} |\lambda_i|)^2 \leq (n-1) \sum\limits_{i=2}^{n} |\lambda_i|^2$

$$(ZE(T_2) - |\lambda_1)^2| \leq (n-1)(\sum_{i=1}^{n} \lambda_i^2 - \lambda_1^2)$$

$$ZE(T_2) \leq \lambda_1 + \sqrt{(n-1)(\sum_{i=1}^{n} \lambda_i^2 - \lambda_1^2)}$$
$$\leq \lambda_1 + \sqrt{(n-1)[B - \lambda_1^2]}$$
$$Let f(x) = x + \sqrt{(n-1)[B - x^2]}$$
$$x^2 = \lambda_1^2 \leq B$$

$\Rightarrow x \leq \sqrt{B}$
$f'(x) = 0 \Rightarrow x = \sqrt{\frac{B}{n}}$
$\sqrt{\frac{B}{n}} \leq x \leq \sqrt{B}$
$\sqrt{\frac{B}{n}} \leq \frac{A}{n} \leq \lambda_1 \leq \sqrt{B}$
$f(\lambda_1) \leq f(\frac{A}{n})$
Thus $ZE(T_2) \leq \frac{A}{n} + \sqrt{(n-1)[B - (\frac{A}{n})]^2}$

Theorem 3.5. *Let H be a T_2 hypergraph with n vertices. Then* $\sqrt{B} < ZE(T_2) \leq \sqrt{nB}$
Proof. By Cauchy-schwarz inequality
$(\sum_{i=1}^{n} |\lambda_i|)^2 \leq n \sum_{i=1}^{n} |\lambda_i^2|$
$(ZE(T_2))^2 \leq nB$
$(ZE(T_2))^2 = (\sum_{i=1}^{n} |\lambda_i|)^2 > \sum_{i=1}^{n} |\lambda_i|^2 = B$
Thus $\sqrt{B} < ZE(T_2) \leq \sqrt{nB}$

Theorem 3.6. *Let H be a T_2 hypergraph with n vertices, rank r and maximum degree Δ. Then* $\lambda_1 < \sqrt{\frac{(n-1)(2r^4 + 10\Delta^4 - 2)}{n}}$.
Proof. We have $(\sum_{i=2}^{n} \lambda_i)^2 \leq (n-1) \sum_{i=2}^{n} \lambda_i^2$
$(-\lambda_1)^2 \leq (n-1)(\sum_{i=1}^{n} \lambda_i^2 - \lambda_1^2)$
$\leq (n-1)(B - \lambda_1^2)$
$n\lambda_1^2 < (n-1)(2r^4 + 10\Delta^4 - 2)$
$\lambda_1 < \sqrt{\frac{(n-1)(2r^4 + 10\Delta^4 - 2)}{n}}$

Result 3.7. In a T_2 hypergraph $\lambda_1 > \frac{8n - 7\sqrt{2} + r}{3}$ where $n \geq 4, n \neq 6$ *and* 7.

Result 3.8. In a T_2 hypergraph $\lceil \lambda_1 \rceil \geq \left\lceil \frac{8n - 7\sqrt{2} + r}{3} \right\rceil$ where $n \geq 4$, equality holds only if n=5,6 and 7.

Theorem 3.9. *Let H be a T_2 hypergraph with $n \geq 4$ and δ is the minimum degree,*

r is the rank. Then $\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil + \frac{(n-1)(\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil)^2}{(detZ(H))^{\frac{1}{n}}} > ZE(T_2)$

Proof. We have $\lceil \lambda_1 \rceil > \left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil > (detZ(H))^{\frac{1}{n}} \forall i$

$$\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil \sum_{i=2}^{n} \lambda_i > (detZ(H))^{\frac{1}{n}} \sum_{i=2}^{n} \lambda_i$$

$$\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil > |\lambda_i| \forall i = 2, 3, ...n$$

$$(n-1)(\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil)^2 > (detZ(H))^{\frac{1}{n}}(ZE(T_2) - \lambda_1)$$

$$\lambda_1 + \frac{(n-1)(\left\lceil \frac{8n-7\sqrt{2}+r}{3})^2 \right\rceil}{(detZ(H))^{\frac{1}{n}}} > ZE(T_2)$$

$$\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil + \frac{(n-1)(\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil)^2}{(detZ(H))^{\frac{1}{n}}} > ZE(T_2)$$

Remark 3.10. *For a T_2 hypergraph* $\left\lceil \frac{8n-7\sqrt{2}+r}{3} \right\rceil \leq \lambda_1 < \sqrt{\frac{(n-1)(2r^4+10\Delta^4-2)}{2}}$ *equality holds only if n=6 and 7.*

4. Conclusion

In this article, we studied the first Zagreb matrix and its energy for a T_2 hypergraph. Also, we found the bounds of the first Zagreb energy of a T_2 hypergraph with n vertices and m edges.

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