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ENERGY OF FIBONACCI PRODUCT CORDIAL GRAPH

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Abstract: Let G be a Fibonacci product cordial graph with n vertices. Then the Fibonacci product cordial label energy of G is denoted by $E_{FPCL}[G]$ and is defined as $E_{FPCL}[G] = \sum_{i=1}^{n} |\rho_i|$, where ρ_i are the eigenvalues of the Fibonacci product cordial labeled matrix. In this paper, we introduce the Fibonacci product cordial labeled matrix, Fibonacci Product Cordial Labeled Laplacian matrix, Fibonacci product cordial label Laplacian energy, Fibonacci product cordial label equi-laplacian energic graphs respectively.

Keywords and Phrases: Fibonacci product cordial label energy, Fibonacci Product Cordial Labeled Laplacian matrix, Fibonacci product cordial label Laplacian energy, Fibonacci product cordial label equi-laplacian energic graphs.

2020 Mathematics Subject Classification: 05A, 05C31.

1. Introduction

Graph labeling is a strong communication between number theory and graph structure. For standard terminology and notations related to graph theory we refer to [8] while for graph labeling we refer to [5]. Rokad A. K. and G. V. Ghodasara created a new idea called Fibonacci cordial labeling by integrating the Fibonacci number and congruence notion in number theory with the cordial labeling idea

in Graph Labeling [9]. Tessymol Abraham and Shiny Jose have also introduced Fibonacci product cordial labeling [1].
In 1978, the energy of the graph G was first defined by Gutman [6] as the sum of G's absolute eigenvalues. It is a proper generalisation of a formula that may be used to compute the total π-electron energy of a conjugated hydrocarbon using the Huckel molecular orbital (HMO) approach in quantum chemistry.
In some cases, chemists prefer labeled graphs to graphs, such as when the vertices represent two distinct chemical species and the edges reflect a specific reaction between the two corresponding species. It's possible that the label energy we're looking at in this paper has uses in chemistry and other fields as well.

2. Preliminaries

Definition 2.1. *An injective function $\phi : V(G) \to \{F_1, F_2, ...F_n\}$ where F_j is the j^{th} Fibonacci number $(j = 1, 2, 3...n)$ is said to be Fibonacci product cordial labeling if the induced function $\chi^* : E(G) \to \{0, 1\}$ defined by $\phi^*(uv) = (\phi(u)\phi(v)) mod\ 2$ satisfies the condition $|e_{\phi^*}(0) - e_{\phi^*}(1)| \leq 1$. A graph which admits Fibonacci product cordial labeling is called Fibonacci product cordial graph* [1].

Definition 2.2. *The energy of a simple connected graph G is equal to the sum of the absolute value of eigenvalues of the graph G where the eigenvalue of a graph G is the eigenvalue of its adjacency matrix* [6].

Definition 2.3. *If G is graph with n vertices, and its Laplacian eigenvalues are $\mu_i, i = 1, 2, 3...n$ then the Laplacian energy of G, denoted $LE(G)$ and is equal to $LE(G) = \sum_{i=1}^{n} |\mu_i|$* [7].

3. Main Results

Definition 3.1. *Let G be a Fibonacci product cordial graph with vertex labeling ϕ. Then the Fibonacci product cordial labeled matrix of G (FPCL-Matrix) is denoted by $M_{FPCL}[G]$ and is defined as*

$$M_{FPCL}[G] = [m_{ij}] = \begin{cases} -1, & if \ \ i \neq j, u_i \ \ and \ \ u_j \ \ are \ \ non \ \ adjacent \\ 0, & if \ \ i \neq j, \phi(u_i)\phi(u_j) \ \ is \ \ odd \\ 1, & if \ \ i \neq j, \phi(u_i)\phi(u_j) \ \ is \ \ even \\ 0, & if \ \ i = j \end{cases}$$

Theorem 3.2. *Let G be a Fibonacci product cordial graph with vertex labeling ϕ. Then for the Fibonacci product cordial matrix $M_{FPCL}[G] = [m_{ij}]$ of G for distinct i, j, k (i) $m_{ij} = 0$, $m_{jk} = 1$ implies $m_{ik} \in \{1, -1\}$ (ii) $m_{ij} = 1$, $m_{jk} = 0$ implies $m_{ik} \in \{1, -1\}$ (iii) $m_{ij} = 0$, $m_{jk} = 0$ implies $m_{ik} \in \{0, -1\}$ and (iv) $m_{ij} = 1$,*

$m_{jk} = 1$ *implies* $m_{ik} \in \{0, 1, -1\}$

Proof. Let $M_{FPCL}[G] = [m_{ij}]$ be the FPCL-matrix of a graph G of order n. Suppose $m_{ij} = 0$, $m_{jk} = 1$. Then there exists three vertices u_i, u_j and u_k such that $\phi(u_i)\phi(u_j)$ is odd or $u_i = u_j$ and $\phi(u_j)\phi(u_k)$ is even. Therefore $\phi(u_i)$ and $\phi(u_j)$ are odd Fibonacci Numbers and $\phi(u_j)$ or $\phi(u_k)$ even Fibonacci Numbers. i.e $\phi(u_i)$ and $\phi(u_j)$ odd and $\phi(u_k)$ even. If there exist the edge u_iu_k then $\phi(u_i)\phi(u_k)$ is even and $m_{ik} = 1$ otherwise $m_{ik} = -1$

Suppose $m_{ij} = 1$, $m_{jk} = 0$. Then there exists three vertices u_i, u_j and u_k such that $\phi(u_i)\phi(u_j)$ is even and $\phi(u_j)\phi(u_k)$ is odd or $u_j = u_k$. Therefore $\phi(u_i)$ or $\phi(u_j)$ are even Fibonacci Numbers and $\phi(u_j)\phi(u_k)$ are odd Fibonacci Numbers. i.e $\phi(u_i)$ even and $\phi(u_j)$ and $\phi(u_k)$ are odd. If there exist the edge u_iu_k then $\phi(u_i)\phi(u_k)$ is even and $m_{ik} = 1$ otherwise $m_{ik} = -1$

Suppose $m_{ij} = 0$, $m_{jk} = 0$. Then there exists three vertices u_i, u_j and u_k such that $\phi(u_i)\phi(u_j)$ is odd or $u_i = u_j$ and $\phi(u_j)\phi(u_k)$ is odd or $u_j = u_k$ Therefore $\phi(u_i)$, $\phi(u_j)$ and $\phi(u_k)$ are odd Fibonacci Numbers. If there exist the edge u_iu_k then $\phi(u_i)\phi(u_k)$ is odd and $m_{i,k} = 0$ otherwise $m_{ik} = -1$

Suppose $m_{ij} = 1$,$m_{jk} = 1$. Then there exists three vertices u_i,u_j and $u_k$such that $\phi(u_i)\phi(u_j)$ is even and $\phi(u_j)\phi(u_k)$ is even. Therefore $\phi(u_i)$ or $\phi(u_j)$ are even Fibonacci Numbers and $\phi(u_j)$ or $\phi(u_k)$ are even Fibonacci Numbers. If $\phi(u_i)$ odd ,$\phi(u_j)$ even and $\phi(u_k)$ odd then $\phi(u_i)\phi(u_k)$ is odd. If there exist the edge u_iu_k then $m_{ik} = 0$ otherwise $m_{ij} = -1$. If $\chi(u_i)$ odd, $\chi(u_j)$ even and $\phi(u_k)$ even then $\phi(u_i)\phi(u_k)$ is even. If there exist the edge u_iu_k then $m_{ik} = 1$ otherwise $m_{ik} = -1$.

Definition 3.3. *Let G be a Fibonacci product cordial graph with n vertices. Then the Fibonacci product cordial label energy of G is denoted by $E_{FPCL}[G]$ and is defined as $E_{FPCL}[G]$ $=\sum_{i=1}^{n}|\rho_i|$, where ρ_i are the eigenvalues of the Fibonacci product cordial labeled matrix.*

Definition 3.4. *Let G be a Fibonacci product cordial graph with Fibonacci product cordial labeled matrix $M_{FPCL}[G] = [m_{ij}]$. The Fibonacci product cordial label degree of a vertex v_i denoted by M_i and is defined as M_i=$\sum_{j=1}^{n} m_{ij}$.*

Definition 3.5. *Let G be a Fibonacci product cordial graph with n vertices. The Fibonacci product cordial label degree matrix is denoted by $D_{FPCL}[G]$ and is defined as*

$$D_{FPCL}[G] = \begin{cases} M_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

where M_i is the Fibonacci product cordial label degree of the vertex v_i.

Definition 3.6. *Let G be a Fibonacci product cordial graph with vertex labeling*

ϕ. Then the Fibonacci Product Cordial Labeled Laplacian matrix of G (FPCL-L Matrix) is denoted by $L_{FPCL}[G]$ and is defined by $L_{FPCL}[G]$=$D_{FPCL}[G]$-$M_{FPCL}[G]$ where $D_{FPCL}[G]$ is the Fibonacci product cordial label degree matrix of G.

Definition 3.7. *Let G be a Fibonacci product cordial graph with n vertices. Then the Fibonacci product cordial label auxiliary eigenvalue μ_i is defined by μ_i=$\rho_i - \bar{M}$, $1 \leq i \leq n$ where $\bar{M} = \frac{1}{n}\sum_{j=1}^{n} M_j$, the average Fibonacci product cordial label degree of G and ρ_i are the Fibonacci product cordial label Laplacian eigenvalues of G.*

Definition 3.8. *Let G be a Fibonacci product cordial graph with n vertices. Then the Fibonacci product cordial label Laplacian energy of G is denoted by $LE_{FPCL}[G]$ and is defined as $LE_{FPCL}[G] = \sum_{i=1}^{n}|\mu_i|$ where μ_i are the Fibonacci product cordial label auxiliary eigenvalues.*

Definition 3.9. *Let G be a Fibonacci product cordial graph with n vertices. Then G is said to be r-regular Fibonacci product cordial graph if the Fibonacci product cordial label degree of all the vertex of G is r.*

Theorem 3.10. *Let G be a Fibonacci product cordial label graph of order n. If ρ_1, ρ_2, $\rho_3 \ldots \rho_n$ are the Fibonacci product cordial label eigenvalue of G. Then $\sum_{i=1}^{n} \rho_i = 0$.*

Proof. We know that Fibonacci product cordial label matrix $M_{FPCL}[G]$ is symmetric and hence its diagonal elements are zero. Therefore $\sum_{i=1}^{n} \rho_i = trace(M_{FPCL}[G]) = 0$.

Theorem 3.11. *Let G be a Fibonacci product cordial label graph of order n. If ρ_1, ρ_2, $\rho_3, \ldots, \rho_n$ are the Fibonacci product cordial label Laplacian eigenvalue of G. Then $\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} M_i$, where M_i is the Fibonacci product cordial label degree of the vertex v_i.*

Proof. We know that $\sum_{i=1}^{n} \rho_i = trace(L_{FPCL}[G])$ = sum of diagonal entries of the Fibonacci product cordial label degree matrix $D_{FPCL}[G] = \sum_{i=1}^{n} M_i$.

Theorem 3.12. *Let G be a Fibonacci product cordial label graph with ρ_1, ρ_2, $\rho_3 \ldots \rho_n$ are the Fibonacci product cordial label Laplacian eigenvalues. Then $\sum_{i=1}^{n} \rho_i^2 = n(n-1) + \sum_{i=1}^{n} M_i^2$, where M_i is the Fibonacci product cordial label degree of the vertex v_i.*

Proof. Let G be a Fibonacci product cordial label graph of order n and $L_{FPCL}[G]$ be the Fibonacci product cordial label Laplacian matrix with ρ_1, ρ_2, $\rho_3 \ldots \rho_n$ are the Fibonacci product cordial label Laplacian eigenvalues. Then

$$\sum_{i=1}^{n} \rho_i^2 = trace(L_{FPCL}[G])^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ij}$$

$$= \sum_{i=j} l_{ij}^2 + \sum_{i\neq j} l_{ij}^2$$
$$= \sum_{i=j} l_{ij}^2 + 2\sum_{i<j} l_{ij}^2$$
$$= \sum_{i=1}^{n} M_i^2 + \frac{2(n^2-n)}{2}(1)^2,$$

where M_i is the Fibonacci product cordial label degree of the vertex v_i.

$$\sum_{i=1}^{n} \rho_i^2 = n(n-1) + \sum_{i=1}^{n} M_i^2$$

Theorem 3.13. *Let G be a Fibonacci product cordial label graph of order n. If μ_1, μ_2, μ_3,...,μ_n are the Fibonacci product cordial label auxiliary eigenvalue of G. Then $\sum_{i=1}^{n}\mu_i = 0$.*

Proof. Consider
$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n}(\rho_i - \bar{M}) = \sum_{i=1}^{n} \rho_i - n\bar{M},$$

where $\bar{M} = \frac{1}{n}\sum_{j=1}^{n} M_j$, the average Fibonacci product cordial label degree of G. We know $\sum_{i=1}^{n}\rho_i = \sum_{j=1}^{n} M_j = n\bar{M}$. Therefore $\sum_{i=1}^{n}\mu_i = 0$

Theorem 3.14. *Let G be a Fibonacci product cordial label graph of order n. If μ_1, μ_2, μ_3,...,μ_n are the Fibonacci product cordial label auxiliary eigenvalue of G. Then*

$$\sum_{i=1}^{n}|(\mu_i)|^2 = n(n-1) + \sum_{i=1}^{n}\left(M_i - \frac{trace(L_{FPCL}[G])}{n}\right)^2.$$

Proof. Consider

$$\sum_{i=1}^{n}|(\mu_i)|^2 = \sum_{i=1}^{n}(\rho_i - \bar{M})(\rho_i - \bar{M})$$
$$= \sum_{i=1}^{n}\rho_i^2 + \sum_{i=1}^{n}\bar{M}^2 - 2\bar{M}\sum_{i=1}^{n}\rho_i$$
$$= n(n-1) + \sum_{i=1}^{n}\left(M_i - \frac{trace(L_{FPCL}[G])}{n}\right)^2.$$

Theorem 3.15. *Let G be a Fibonacci product cordial label graph of order n. If μ_1, μ_2, μ_3,...,μ_n are the Fibonacci product cordial label auxiliary eigenvalue of G. Then*

$$\sqrt{2K} \leq LE_{FPCL}[G] \leq \sqrt{nK},$$

where $K = \sum_{i=1}^{n} \mu_i^2$.

Proof. Consider

$$K = \sum_{i=1}^{n} \mu_i^2 = \left(\sum_{i=1}^{n} \mu_i\right)^2 - 2 \sum_{1\leq i\leq j\leq n} \mu_i\mu_j \leq 2|\sum_{1\leq i\leq j\leq n} \mu_i\mu_j| \leq 2 \sum_{1\leq i\leq j\leq n} |\mu_i||\mu_j|,$$

$$\textit{since} \quad \sum_{i=1}^{n} \mu_i = 0$$

$$(LE_{FPCL}[G])^2 = (\sum_{i=1}^{n}|\mu_i|)^2 = \sum_{i=1}^{n}|\mu_i|^2 + 2 \sum_{1\leq i\leq j\leq n} \mu_i\mu_j$$

$$\geq K + K$$

$$\sqrt{2K} \leq LE_{FPCL}[G]$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i| - |\mu_j|)^2 = \sum_{i=1}^{n}|\mu_i|^2 + \sum_{j=1}^{n}|\mu_j|^2 - 2\sum_{i=1}^{n}|\mu_i|\sum_{j=1}^{n}|\mu_j| \geq 0$$

$$2n\sum_{i=1}^{n}|\mu_i|^2 - 2\sum_{i=1}^{n}|\mu_i|\sum_{j=1}^{n}|\mu_j| \geq 0$$

$$2nK - 2(LE_{FPCL}[G])^2 \geq 0 \quad \textit{where} \quad K = \sum_{i=1}^{n}\mu_i^2$$

$$(LE_{FPCL}[G])^2 \leq nK$$

$$LE_{FPCL}[G] \leq \sqrt{nK}.$$

Definition 3.16. *Two FPCL graphs are said to be Fibonaci product cordial label equi-energic if their Fibonacci product cordial label energies are equal.*

Definition 3.17. *A Fibonnacci product cordial graph is called Fibonaci product cordial label equi-laplacian energic if their Fibonacci product cordial label energy and Fibonacci product cordial label Laplacian energy are equal.*

Theorem 3.18. *Let G be a r-regular Fibonnacci product cordial graph. Then the Fibonacci product cordial label Laplacian energy of G is same as Fibonacci product cordial label energy of G. ie. G is equi-laplacian energic graph.*

Proof. Let G be a r-regular Fibonnacci product cordial graph with Fibonaci product cordial label Laplacian characteristic polynomial $\phi_L(\rho)$ and Fibonaci product cordial label characteristic polynomial $\phi_M(\rho)$. Then $\phi_L(\rho) = det(\rho I – L_{FPCL}[G]) = det(\rho I\text{-}D_{FPCL}[G]+M_{FPCL}[G])=det(\rho I - rI + M_{FPCL}[G]) = (-1)^n det((r-\rho)I - M_{FPCL}[G]) = (-1)^n \phi_M(r-\rho)$, where $D_{FPCL}[G]$ is the Fibonacci product cordial label degree matrix of G, $L_{FPCL}[G]$ is the Fibonacci product cordial label Laplacian matrix and $M_{FPCL}[G]$ is the Fibonacci product cordial label matrix. Thus if ρ_1, ρ_2, ρ_3,...,ρ_n are the Fibonacci product cordial label eigenvalues of Fibonacci product cordial r-label regular graph then $r-\rho_1$, $r-\rho_2$, $r-\rho_3$, ...,$r-\rho_n$ are the Fibonacci product cordial Laplacian eigenvalues of Fibonacci Product Cordial r-label regular graph. Since G is Fibonacci Product Cordial r-label regular graph, the average Fibonacci product cordial label degree of G is $\bar{M} = \frac{1}{n}\sum_{j=1}^{n} M_j = r$. Hence Fibonacci Product cordial label Laplacian energy $L_{FPCL}[G] = \sum_{i=1}^{n}|\mu_i|= \sum_{i=1}^{n}|r-\rho_i-\bar{M}| = \sum_{i=1}^{n}|r-\rho_i-r| = \sum_{i=1}^{n}|\rho_i| = E_{FPCL}[G]$.

Corollary 3.19. *The Peterson graph is an equi-laplacian energic graph.*

Corollary 3.20. *The cycle is an equi-laplacian energic graph.*

4. Conclusion

The ideas of Fibonacci product cordial label matrix and Fibonacci product cordial label energy have been investigated in this work. The concept of Laplacian energy of Fibonacci product cordial graph has been proposed. Also discussed are some energy features of the Fibonacci product cordial graph.

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