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THE NUMBER OF CC-DOMINATING SETS OF SOME GRAPHS

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Abstract: The aim of this paper is to study about the number of cc-dominating sets and to introduce the concept of *cc-domination polynomial* for simple finite undirected graphs. For a graph G on n vertices possessing $d_c(G,i)$ cc-dominating sets of cardinality i, the *cc-domination polynomial* is defined as $D_c[G;x]=\sum_{i=\gamma_{cc}(G)}^{n} d_c(G,i)x^i$, where $\gamma_{cc}(G)$ is the cc-domination number of G. We obtain some properties of $D_c[G;x]$ and compute the same for some special graphs. Moreover, the concept of cc-domination entropy is also introduced and studied.

Keywords and Phrases: CC-Domination Polynomial, CC-Domination Number, Closely-connected Vertices, CC-Degree of a Vertex, CC-Isolated Vertex, CC-Domination Entropy.

2020 Mathematics Subject Classification: 05C40, 05C69, 05A18.

1. Introduction and Preliminaries

Data communication networks are effective and efficient when they amalgamate performance, reliability, and security. The reliability of network infrastructure is crucial for the smooth functioning of the network and is strongly influenced by the network's capacity to handle topological changes. As a result, neither the entire network nor significant portions of it will fail as a result of such changes and that the remaining network will resume normal operation immediately. This feature can be achieved through several techniques and a number of algorithms are already available in the literature regarding this [12].

The main objective is to reduce network down time and keep communication flowing with the least amount of disruption. Hence while designing a distributed system paramount importance should be given for fault tolerance mechanisms so that the system can automatically recover from partial failures without seriously affecting its overall performance. Moreover in the modern era of cloud-computing, fault-tolerance mechanisms are indispensable to ensure high availability and authenticity to the users. The faults in the cloud environment may occur due to physical faults, network faults, processor faults, service expiry faults etc and so on [14]. Among these, the network faults arises mainly due to link failures. In order to minimize link failures, the nodes of the network must be associated "more closely" in such a way that unpredictable disruptions may not result in the failure of the whole network.

The concept of domination discussed in [7] entirely relies upon the "adjacency" property of vertices in a graph. But the property of "being adjacent" is not at all sufficient to characterize the vertex pairs in a graph as the deletion of the edges shared by adjacent vertices may or may not disconnect the graph. Moreover, there are numerous graphs in which the non-adjacent vertices are "so close" in the sense that the deletion of not all geodesics connecting them disconnects the graph. Consequently, these observations motivated the authors to introduce the more generalized concept of "closely-connected vertices" as they will ensure paths preserving the connectedness of a graph.

Let $G = (V, E)$ be a non-trivial simple undirected graph on n vertices. A path P in G is a cut path if there exists a graph $G' = (V, E \setminus \bar{E}')$ for some $\bar{E}' \subseteq \bar{E}(P)$ such that $\omega(G') > \omega(G)$. The vertices $u, v \in V$ are closely-connected if at least one of the geodesics connecting them is not a cut path and G is closely-connected if all its vertex pairs are closely-connected. The cc-number of u and v is defined as $\Gamma_{cc}(u, v) = \{p \in P(u, v) |\ p$ is not a cut path in $G\}$, where $P(u, v)$ is the set of all geodesics linking u and v in G. A subset C of V is a cc-dominating set if for every vertex $v \in V \setminus C$ there exists a vertex $u \in C$ such that $\Gamma_{cc}(u, v) \geq 1$. The minimum cardinality of a cc-dominating set is called the cc-domination number, denoted by $\gamma_{cc}(G)$. The open cc-neighborhood of a vertex $v \in V$ is the set $N_{cc}(v) = \{u \in V : \Gamma_{cc}(u, v) \geq 1\}$, whereas the closed cc-neighborhood of v is defined as $N_{cc}[v] = N_{cc}(v) \cup \{v\}$. The cardinality of $N_{cc}(v)$ is the cc-degree of v, denoted by $deg_{cc}(v)$. The vertex v is said to be cc-isolated if $N_{cc}(v) = \phi$. The maximum and minimum cc-degree of a vertex in G are given by

$$\Delta_{cc}(G) = \max_{v \in V} |N_{cc}(v)| \text{ and } \delta_{cc}(G) = \min_{v \in V} |N_{cc}(v)|.$$

The cc-complement of G is the graph $\bar{G_{cc}} = (V, E')$, where $uv \in E'$ iff $v \notin N_{cc}(u)$ in

G for $u, v \in V$. For more properties of closely-connected vertices [9] can be inferred. A special case of closely-connected vertices in which the cc-number coincides with the number of geodesics is studied in [10] and a corresponding index is formulated. A comprehensive treatment on graph domination is given in [8] and [7]. Further, for the domination polynomial introduced by Alikhani [2] and [3]. can be referred.

In this paper, the number of cc-dominating sets of graphs are studied and *cc-domination polynomial* $D_c[G; x]$ of a graph G is introduced. We investigate some special properties of $D_c[G; x]$ and obtain the same for some special graphs. Throughout this paper, G denotes a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$ and the graph theoretic terminologies and notations used are as in [6] unless specified otherwise.

2. Main Results

Definition 2.1. *Let $G = (V, E)$ be a graph of order n. For each $1 \le i \le n$, define*

$$D_c(G, i) = \left\{ (u_1, \ldots, u_i) \in \underbrace{V \times \ldots \times V}_{i \text{ times}} : \forall v \in V \setminus \{u_j\}_{j=1}^{i},\ \exists\ u_j \text{ with } N_{cc}(u_j) = v \right\}.$$

Definition 2.2. *Let G be a graph of order n. Then, the cc-domination polynomial $D_c[G; x]$ of G is defined as:*

$$D_c[G; x] = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G, i) x^i,$$

where $d_c(G, i) = |D_c(G, i)|$.

Theorem 2.3. *Let G be a graph of order n with $d_c(G, j) = \binom{n}{j}$ for some $j \le n$. Then,*

$$D_c[G; x] = (x + 1)^n + \sum_{i=0}^{j-1} \left[d_c(G, i) - \binom{n}{i} \right] x^i.$$

Proof. Since every subset of $V(G)$ of cardinality j is a cc-dominating set of G, it follows that $\gamma_{cc}(G) \le j$.

Claim: $d_c(G, i) = \binom{n}{i}\ \forall i \ge j$.

> **Proof of the claim :** Let H be a subgraph of G with $|V(H)| = i$, where $j < i \le n$. Then, there exists a subgraph H' of H with $|V(H')| = j$. Hence by our assumption it follows that $V(H')$ is a cc-dominating set of G. Since $V(H') \subset V(H)$, we get $V(H)$ is a cc-dominating set of G. Thus the vertex set of every subgraph of G of order exceeding j constitutes a cc-dominating set of G. Therefore, $d_c(G, i) = \binom{n}{i}\ \forall i \ge j$.

Thus,

$$\begin{aligned} D_c[G;x] &= \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i + \sum_{i=j}^{n} d_c(G,i)x^i \\ &= \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i + \sum_{i=j}^{n} \binom{n}{n-i} x^i \\ &= (x+1)^n - \sum_{i=0}^{j-1} \binom{n}{i} x^i + \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i. \end{aligned}$$

Since $d_c(G,i) = 0$ for $i = 0, 1, \ldots, \gamma_{cc}(G) - 1$, we get

$$\begin{aligned} D_c[G;x] &= (x+1)^n - \sum_{i=0}^{j-1} \binom{n}{i} x^i + \sum_{i=0}^{j-1} d_c(G,i)x^i \\ &= (x+1)^n + \sum_{i=0}^{j-1} \left[d_c(G,i) - \binom{n}{i} \right] x^i. \end{aligned}$$

This completes the proof.

Corollary 2.4. *Let G be a graph and let $j = inf\ \{i \in \mathbb{N} : \ d_c(G,i) = \binom{n}{i}\}$, where $|V(G)| = n$. Then,*

$$\left| \frac{D_c[G;x]}{f(x)} \right| < 1, \quad \text{where } f(x) = (x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1} \binom{n}{i} x^i.$$

Proof. Since $d_c(G,j) = \binom{n}{j}$, we get $d_c(G,i) < \binom{n}{i}$ $\forall i < j$. Therefore it follows from theorem 2.3 that

$$\begin{aligned} D_c[G;x] &< (x+1)^n - \sum_{i=0}^{j-1} \binom{n}{i} x^i + \sum_{i=\gamma_{cc}(G)}^{j-1} \binom{n}{i} x^i. \\ &= (x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1} \binom{n}{i} x^i - \sum_{i=\gamma_{cc}(G)}^{j-1} \binom{n}{i} x^i + \sum_{i=\gamma_{cc}(G)}^{j-1} \binom{n}{i} x^i. \\ &= f(x) \end{aligned}$$

Corollary 2.5. *A graph G of order n is closely-connected iff $D_c[G;x] = (x+1)^n - 1$. In this case, the bound of the number of cc-dominating sets of G is sharp and the corresponding alternating number is -1.*

Proof. Assume that G is closely-connected. Then, it follows from theorem 2.3 that $D_c[G;x] = (x+1)^n - 1$. Conversely, if $D_c[G;x] = (x+1)^n - 1$, then $d_c(G,i) = \binom{n}{i}\ \forall\ 1 \leq i \leq n$ so that every pair of vertices is closely-connected.

Moreover, for any graph G, the number of cc-dominating sets is given by

$$\begin{aligned} D_c[G;1] &= \sum_{i=\gamma_{cc}(G)}^{n} d_c(G,i) \\ &\leq \sum_{i=1}^{n} \binom{n}{i} \\ &= 2^n - 1 \end{aligned}$$

If G is closely-connected, then $D_c[G;x] = (x+1)^n - 1.$ so that $D_c[G;1] = 2^n - 1$. The alternating number of cc-dominating sets of G is the difference of cc-dominating sets of even cardinality and odd cardinality and is given by $D_c[G;-1] = -1$.

Remark 2.6. *For a closely-connected graph G of order n, it follows from theorem 2.3 that $D_c[G;x] = (x+1)^n - 1$. But, $\gamma_{cc}(G) = 1$ is not at all a sufficient condition to imply that $D_c[G;x] = (x+1)^n - 1$. For example, consider the complete bipartite graph $K_{2,n-2}$, where $n \geq 5$. Then $\gamma_{cc}(K_{2,n-2}) = 1$, but the vertices of degree $n-2$ in $K_{2,n-2}$ are not closely-connected so that*

$$d_c(K_{2,n-2}, i) = \begin{cases} \binom{n}{i} - 2, & \text{if } i = 1, \\ \binom{n}{i} \quad , & \text{if } i \neq 1. \end{cases}$$

Therefore, $D_c[K_{2,n-2};x] = (x+1)^n - 2x - 1$.

Theorem 2.7. *Let $G_1, \ldots, G_m$ be the components of a graph G. Then,*

$$D_c[G,x] = \prod_{i=1}^{m} D_c[G_i,x].$$

Proof. Without loss of generality, let $m = 2$. For $k \geq \gamma_{cc}(G)$, every cc-dominating set of cardinality k in G is obtained from a cc-dominating set of i vertices in G_1 and a cc-dominating set of $k-i$ vertices in G_2 for $i \in \{\gamma_{cc}(G_1), \ldots, |V(G_1)|\}$. Since the number of ways of doing this over all $i = \{\gamma_{cc}(G_1), \ldots, |V(G_1)|\}$ is exactly the coefficient of x^k in $D_c[G_1,x]D_c[G_2,x]$, both the sides of the above equation are identical.

Theorem 2.8. *Let G be a cut edge free graph such that $deg(v) \geq deg_{cc}(v)\ \forall v \in V(G)$. Then, $D_c[G;x] = D[G;x]$.*

Proof. Since G is free of cut edges, its adjacent vertices are closely-connected so that $deg_{cc}(v) \geq deg(v)\ \forall v \in V(G)$. That is, $\forall v \in V(G)$, $deg_{cc}(v) = deg(v)$. Thus the dominating sets and cc-dominating sets of G are the same and hence $D_c[G;x] = D[G;x]$.

Remark 2.9. *If $D_c[G;x]=D[G;x]$, then $\gamma_{cc}(G)=\gamma(G)$. But , the converse is not true. For example, consider the wheel graph W_5 shown in figure 1. Since $deg_{cc}(m_i)=4\ \forall i=1,\ldots,5$ and*

$$deg(m_i)=\begin{cases}4, & \text{if } i=5\\ 3, & \text{if } i\neq 5,\end{cases}$$

it follows that $\gamma_{cc}(W_5)=\gamma(W_5)=1$.
But, $d_c(W_5,1)=5$ whereas $d(W_5,1)=1$. Hence $D_c[W_5;x]\neq D[W_5;x]$.

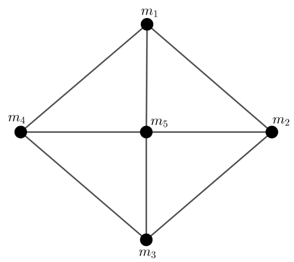

Figure 1: The Wheel graph W_5

Theorem 2.10. *Let G be a graph of order n with $d_c(G,\gamma_{cc}(G))=\binom{n}{\gamma_{cc}(G)}$ and G' be the graph obtained from G by adding k pendent vertices to $V(G)$. Then,*

$$D_c[G';x]=x^k\left[(x+1)^n-\sum_{i=0}^{\gamma_{cc}(G)-1}\binom{n}{n-i}x^i\right].$$

Proof. Let $V(G')=\{v_1,\ldots,v_n,u_1,\ldots,u_k\}$, where $v_i\in V(G)$ and u_j are pendent vertices for $i=1,\ldots,n$ and $j=1,\ldots,k$. Clearly, $\gamma_{cc}(G')=\gamma_{cc}(G)+k$ and $d_c(G',\gamma_{cc}(G)+k)=d_c(G,\gamma_{cc}(G)$. Also, it can be noted that $d_c(G',k+i)=d_c(G,i)$ for $i=\gamma_{cc}(G)+1,\ldots,n$. Therefore,

$$\begin{aligned}D_c[G';x]&=\sum_{i=k+\gamma_{cc}(G)}^{k+n}d_c(G,i-k)x^i\\&=x^k\sum_{i=\gamma_{cc}(G)}^{n}d_c(G,i)x^i\\&=x^kD_c[G;x].\end{aligned}$$

Now, it evidently follows from theorem 2.3 that

$$D_c[G';x]=x^k\left[(x+1)^n-\sum_{i=0}^{\gamma_{cc}(G)-1}\binom{n}{n-i}x^i\right].$$

Theorem 2.11. *Let $G = (V, E)$ be a graph of order n with $D_c[G; x] = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G, i)x^i$. If G has k cc-isolated vertices, then*

(i) $k = n - d_c(G, n-1)$.

(ii) $d_c(G, n-2) = \binom{n}{2} - k(n-1) + \binom{k}{2}$.

(iii) $d_c(G, 1) = |\{v \in V : deg_{cc}(v) = n-1\}|$.

Proof.

(i) Let $A \subseteq V$ be the set of all cc-isolated vertices in G. Then, for any vertex $v \in V \setminus A$, the set $V \setminus \{v\}$ is a cc-dominating set of G. Therefore, $d_c(G, n-1) = |V(G) \setminus A| = n - k$.

(ii) Let $D \subseteq V$ be a non cc-dominating set of cardinality $n-2$. Then, $D = V \setminus \{u, v\}$ for $u, v \in V$.

Case(i) Atleast one of the two vertices u or v are cc-isolated.
Let $u \in V$ be cc-isolated and $v \in V \setminus \{u\}$. Thus corresponding to every cc-isolated vertex u , there exists $n-1$ vertices in G such that $V \setminus \{u, v\}$ is not a cc-dominating set. Therefore, the total number of cc-dominating sets of cardinality $n-2$ in G is $d_c(G, n-2) = \binom{n}{2} - k(n-1) + \binom{k}{2}$.

Case(ii) Both u and v are not cc-isolated.
Since D is a non cc-dominating set, this is possible only if $\Gamma_{cc}(u, v) = 1$ and $\Gamma_{cc}(v, w) \geq 1$ for some $w \in D$. But this would eventually lead us to conclude that $\Gamma_{cc}(u, y) = 1$ for some $y \in D$, a contradiction to the fact that D is a non cc-dominating set.

(iii) For any vertex $v \in V$, $\{v\}$ is a cc-dominating set iff $\Gamma_{cc}(u, v) \geq 1$ $\forall u \in V \setminus \{v\}$.

Corollary 2.12. *Let G be a graph with $D_c[G; x] = [x(x+2)]^n$. Then, G is free of cc-isolated vertices.*

Proof. Let k be the number of cc-isolated vertices in G. Then from part (i) of theorem 2.11, $k = 2n - d_c(G, 2n-1) = 2n - 2n = 0$.

Theorem 2.13. *Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $D_c[G, x] = \sum_{i=0}^{n-\gamma_{cc}(G)} a_i x^{i+\gamma_{cc}(G)}$. Then, the coefficients $a_0, \ldots, a_{n-\gamma_{cc}(G)}$ are solutions of the following system, respectively for $i = 0, i = 1, \ldots, i = n - \gamma_{cc}(G)$.*

$$(C + I_n)V \geq 1_n \tag{2.1}$$

$$\sum_{i=0}^{n} v_i = \gamma_{cc}(G) + i \tag{2.2}$$

$$v_i \in \{0, 1\}, \ for \ i = 1, \ldots, n,$$

where I_n is the identity matrix of order n, $V = [v_1\ v_2 \cdots v_n]^t$ and C is the $n \times n$ matrix $[c_{ij}]$ with entries

$$c_{ij} = \begin{cases} 1, \text{ if } \ \Gamma_{cc}(v_i, v_j) \geq 1 \\ 0, \text{ otherwise.} \end{cases}$$

3. Special Cases

Theorem 3.1. *Let G be a closely-connected graph of order n and G' be the graph obtained by linking two copies of G through a bridge. Then*

$$D_c[G'; x] = \sum_{i=2}^{2n} d_c(G', i)x^i,$$

where

$$d_c(G', i) = \begin{cases} \binom{n}{\frac{i}{2}}^2 + 2\sum\limits_{k=1}^{\frac{i}{2}-1} \binom{n}{k} \times \binom{n}{i-k}, & \textit{if } i \leq n+1 \textit{ is even} \\ 2\sum\limits_{k=1}^{\lfloor \frac{i}{2} \rfloor} \binom{n}{k} \times \binom{n}{i-k} x^i, & \textit{if } 2 \leq i \leq n+1 \textit{ is odd} \end{cases}$$

and for $i > 1$,

$$d_c(G', n+i) = \begin{cases} \binom{n}{\frac{n+i}{2}}^2 + 2\sum\limits_{k=i}^{\lfloor \frac{n+i}{2} \rfloor - 1} \binom{n}{k} \times \binom{n}{n+i-k}, & \text{if both } n, i \text{ are odd} \\ \binom{n}{\frac{n+i}{2}}^2 + 2\sum\limits_{k=i}^{\lfloor \frac{n+i}{2} \rfloor - 2} \binom{n}{k} \times \binom{n}{n+i-k}, & \text{if both } n, i \text{ are even} \\ 2\sum\limits_{k=i}^{\lfloor \frac{n+i}{2} \rfloor} \binom{n}{k} \times \binom{n}{n+i-k}, & \text{otherwise.} \end{cases}$$

Proof. Since G is closely-connected, $\gamma_{cc}(G') = 2$. A cc-dominating set of G' of cardinality $i \geq 2$ is obtained by choosing $j > 1$ vertices from one copy of G and $i - j$ vertices from the other copy. This computation gives the desired result.

Theorem 3.2. *The cc-domination polynomial of the bipartite cocktail party graph B_n is given by*

$$D_c[B_u; x] = \begin{cases} x^4, \text{ if } n = 2, \\ x^6 + 6x^5 + 15x^4 + 14x^3 + 3x^2, \text{ if } n = 3, \\ (x+1)^n - 1, \text{ otherwise.} \end{cases}$$

Proof. Let B_u be the bipartite cocktail party graph with $V(B_u) = \{m_1, \dots, m_u\} \cup \{n_1, \dots, n_u\}$ as shown in figure 2.

Case(i) For $n = 2$, B_u is a disjoint union of two copies of K_2 so that all its vertices are cc-isolated.

Case(ii) For $n = 3$, B_u has no singleton cc-dominating sets. The only cc-dominating sets of size 2 are of the form $\{m_i, n_i\}$ for $i = 1, 2, 3$ so that $d_c(B_3, 2) = 3$. The cc-dominating sets of size 3 are obtained by taking the non-trivial subsets of $V(B_3)$ except those of the form $\{m_i, n_j, n_k\} \cup \{n_i, m_j, m_k\}$ for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k$. Thus $d_c(B_3, 3) = \binom{6}{3} - 6 = 14$. Now for $i > 3$, every subset of the vertex set is a cc-dominating so that $d_c(B_3, i) = \binom{6}{i}$.

Case(iii) It can be readily observed that B_u is closely-connected for $n > 3$. Hence the result follows immediately from corollary 2.5

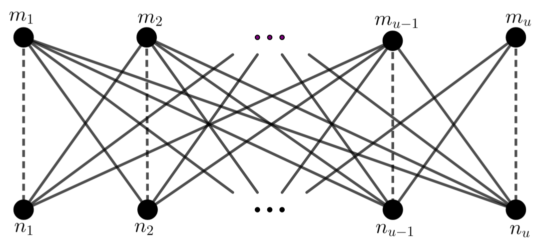

Figure 2: The bipartite cocktail party graph B_u

4. CC-Domination Entropy of Graphs

Since domination is an extremely sensitive graph invariant, the information content versions of some parameters have greater discrimination power than classical versions of these parameters. This resulted in the introduction of domination entropy of graphs in [4]. This has motivated the authors to define a new graph entropy measure to investigate the graph invariance properties of cc-dominating sets.

Definition 4.1. *Let G be a graph and $f : S \longrightarrow R^+$ be an information functional defined on $S = \{s_1, s_2, \ldots, s_k\}$ such that S is a set of elements of G. Then, the entropy is defined as follows:* [11]

$$I_f(G) = -\sum_{i=1}^{k} \frac{f(s_i)}{\sum_{j=1}^{k} f(s_j)} log\left(\frac{f(s_i)}{\sum_{j=1}^{k} f(s_j)}\right)$$

Definition 4.2. *Let G be a graph of order n without cc-isolates. The information functional f is defined as $f = |D_c(G, i)| = d_c(G, i)$. Then, the cc-domination entropy of G is defined as :*

$$I_{ccd}(G) = -\sum_{i=1}^{n} \frac{d_c(G,i)}{\gamma_{cc}^t(G)} log(\frac{d_c(G,i)}{\gamma_{cc}^t(G)}),$$

where

$$\gamma_{cc}^t(G) = \sum_{i=\gamma_{cc}(G)}^{|V(G)|} D_c(G, i).$$

Theorem 4.3. *Let G be a graph. Then, $I_{ccd}(G) = 0$ iff G is acyclic.*

Proof. Assume that G is acyclic. Then, $d_c(G, n) = 1$ and $d_c(G, i) = 0$ $\forall i < n$ so that $\gamma_{cc}^t(G) = 1$. Therefore, $I_{ccd}(G) = 0$.

Conversely, let $I_{ccd}(G) = 0$. Since $d_c(G, i) \geq 0$ $\forall i = 1, \ldots, n$ and $\gamma_{cct}(G) > 0$,

$$\begin{aligned} I_{ccd}(G) = 0 &\iff d_c(G, i) \cdot log(\frac{d_c(G, i)}{\gamma_{cc}^t(G)}) = 0 \\ &\iff d_c(G, i) = 0 \text{ or } log(\frac{d_c(G, i)}{\gamma_{cc}^t(G)}) = 0 \ \forall i = 1, \ldots, n. \\ &\iff log(\frac{d_c(G, i)}{\gamma_{cc}^t(G)}) = 0 \ \forall i \geq \gamma_{cc}(G) \\ &\iff \frac{d_c(G, i)}{\gamma_{cc}^t(G)} = 1 \ \forall i \geq \gamma_{cc}(G) \\ &\iff d_c(G, i) = \gamma_{cc}^t(G) \ \forall i \geq \gamma_{cc}(G) \\ &\iff d_c(G, i) = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G, i) \ \forall i \geq \gamma_{cc}(G) \\ &\iff \gamma_{cc}(G) = n. \end{aligned}$$

This completes the proof.

Theorem 4.4. *Let G be a closely-connected graph of order n. Then,*

$$I_{ccd}(G) = log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} log \binom{n}{i}.$$

Proof. Since $D_c[G; x] = (x+1)^n - 1$, it follows that $\gamma_{cc}^t(G) = \sum_{i=1}^{n} \binom{n}{i} = 2^n - 1$. Therefore,

$$\begin{aligned} I_{ccd}(G) &= -\sum_{i=1}^{n} \frac{\binom{n}{i}}{2^n - 1} log \left(\frac{\binom{n}{i}}{2^n - 1} \right) \\ &= log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} log \binom{n}{i}. \end{aligned}$$

Theorem 4.5. *For $n \geq 5$,*

$$I_{ccd}(K_{2,n-2}) = log(2^n - 3) - \frac{(n-2) log(n-2)}{2^n - 3} - \frac{1}{2^n - 3} \sum_{i=2}^{n} \binom{n}{i} log \binom{n}{i}.$$

Proof. Since $D_c[G;x] = (x+1)^n - 2x - 1$, it follows that $\gamma_{cc}^t(G) = \sum_{i=1}^{n} \binom{n}{i} = 2^n - 1$.

$$= log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} log \binom{n}{i}.$$

Theorem 4.6.

$$I_{ccd}(B_n) = 0, \qquad \text{if } n = 2,$$
$$= log39 - \frac{1}{39}\left[\sum_{i=1}^{2} \binom{2n}{i} log \binom{2n}{i} + 3log3 + 14log14\right], \qquad \text{if } n = 3,$$
$$= log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} log \binom{n}{i}, \qquad \text{otherwise.}$$

Proof. The result follows immediately from the expression of $D_c[B_n;x]$.

5. Conclusion

In this paper, the concept of cc-domination polynomial is studied using the number of cc-dominating sets of graphs. The introduction of closely-connected vertices facilitated to generalise the domination polynomial to a much more wider context of the cc-domination polynomial. Moreover, the concept of cc-domination entropy is introduced and evaluated for certain graphs.

6. Applications

1. **Network Model:** A common wired channel/network is used by all hosts for bidirectional communication. Every host has a fixed broadcast area and within this close-transmission range, link failures of the network does not disrupt the functioning of the entire system. A pair of hosts that are in communication with one another are referred to as closely-connected neighbors. This can be developed as a fault tolerant model to maximize the effectiveness and efficiency in communication networks.

2. **Military Communications:** In the military sphere, there always exists threats to the communication infrastructure as they are perceived as high-value targets. These threats makes them more vulnerable so that it is critically important to address network robustness. That is, the continued ability of the network to perform its function in the face of attack. Thus closely-connected neighbors serves as a new approach to design networks that sustain attacks [5].

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