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# EDGE ITALIAN DOMINATION OF SOME GRAPH PRODUCTS

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**Abstract:** An edge Italian dominating function (EIDF) of a graph $G = (V, E)$ is a function $f : E(G) \rightarrow \{0, 1, 2\}$ such that every edge $x$ with $f(x) = 0$ is adjacent to some edge $y$ with $f(y) = 2$ or adjacent to at least two edges $z_1, z_2$ with$f(z_1) = f(z_2) = 1$. The weight of an edge Italian dominating function is the sum $\sum_{x \in E(G)} f(x)$ and the minimum weight of an edge Italian dominating function of $G$ is called the edge Italian domination number of $G$ and is denoted by $\gamma_I^{'}(G)$. In this paper, we determine the edge Italian domination number of some graph products.

**Keywords and Phrases:** Edge Italian Domination, Edge Italian dominating function, Edge Italian Domination number.

**2020 Mathematics Subject Classification:** 05C.

## 1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A graph $G$ with $p$ vertices and $q$ edges will be referred to as a $(p, q)$-graph. A subset $S$ of $V$ is called a dominating set of $G$ if every vertex not in $S$ is adjacent to some vertex in $S$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality taken over all dominating sets of $G$. The concept of edge domination in graphs was introduced by Mitchell and Hedetniemi [6]. A subset $F$ of edges of a graph $G$ is called an edge dominating set of $G$ if every edge not in

$F$ is adjacent to some edge in $F$. The edge domination number of $G$, denoted by $\gamma^{'}(G)$, is the minimum cardinality taken over all edge dominating sets of $G$.

Cockayne et al. [2] introduced Roman Dominating Function. A function $f : V(G) \to \{0,1,2\}$ such that every vertex $v$ with $f(v) = 0$ is adjacent to some vertex $u$ with $f(u) = 2$ is called a Roman dominating function. The weight of a Roman dominating function is the value $\sum_{v \in V(G)} f(v)$. The minimum weight of a Roman dominating function on $G$ is called the Roman domination number and is denoted by $\gamma_R(G)$. Roushini Leely Pushpam et al. [7] introduced edge version of Roman Domination. An edge Roman Dominating Function of a graph $G$ is a function, $f : E(G) \to \{0,1,2\}$ such that every edge $e$ with $f(e) = 0$ is adjacent to some edge $e_1$ with $f(e_1) = 2$. The edge Roman domination number of $G$, denoted by $\gamma^{'}_R(G)$, is the minimum weight of an edge Roman dominating function of $G$.

Italian domination was first introduced as Roman $\{2\}$- domination by Chellali et al. [1]. It was further researched and renamed as Italian domination by Henning and Klostermeyer [4]. An Italian dominating function of a graph $G$ is a function $f : V(G) \to \{0,1,2\}$ such that every vertex $v$ with $f(v) = 0$ is adjacent to some vertex $u$ with $f(u) = 2$ or is adjacent to at least two vertices $x, y$ with $f(x) = f(y) = 1$. The weight of an Italian dominating function is $\sum_{v \in V(G)} f(v)$. The minimum weight of such a function on $G$ is called the Italian domination number of $G$ and is denoted by $\gamma_I(G)$.

We have introduced the edge version of Italian domination in graphs in [5]. An Edge Italian dominating function (EIDF) of a graph $G = (V, E)$ is a function $f : E(G) \to \{0,1,2\}$ such that every edge $x$ with $f(x) = 0$ is adjacent to some edge $y$ with $f(y) = 2$ or adjacent to at least two edges $z_1$ and $z_2$ with $f(z_1) = f(z_2) = 1$. The weight of an Edge Italian dominating function is $\sum_{x \in E(G)} f(x)$. The Edge Italian Domination number of $G$, denoted by, $\gamma^{'}_I(G)$ is the minimum weight of all Edge Italian dominating functions of $G$. The following results will be used in this paper.

**Theorem 1.1.** [5] *For the path $P_n$, $\gamma^{'}_I(P_n) = \lceil \frac{n}{2} \rceil$, $n \geq 2$.*

**Theorem 1.2.** [5] *For the path $C_n$, $\gamma^{'}_I(C_n) = \lceil \frac{n}{2} \rceil$, $n \geq 3$.*

**Theorem 1.2.** [5] *For the Star $K_{1,n}$, $\gamma^{'}_I(K_{1,n}) = 2$, when $n \geq 2$.*

The corona $G_1 \odot G_2$ of an $(n_1, m_1)$-graph $G_1$ on an $(n_2, m_2)$-graph, $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$ for$1 \leq i \leq n_1$. Clearly, $G_1 \odot G_2$ has $n_1 + n_1 n_2$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.

Let $G_1$ and $G_2$ be two graphs with the vertex sets$V(G_1) = \{u_1, u_2, u_3, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, v_3, ..., v_n\}$ respectively. The Cartesian Product of the graphs

$G_1$ and $G_2$, denoted by $G_1 \Box G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$are adjacent in $G_1 \Box G_2$ if (i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or (ii) $v_1 = v_2$ and $u_1u_2 \in E(G_1)$. The Cartesian product of $P_n$ and $P_2$ is called the ladder graph, $L_n$. The Tensor Product $G_1 \times G_2$ of the graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G_1 \times G_2$ if and only if $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent to $v_2$ in $G_2$. Tensor product of graphs is also known as the categorical product or direct product. The Strong product $G_1 \boxtimes G_2$ of the graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in$G_1 \boxtimes G_2$, if (i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or (ii) $v_1 = v_2$ and $u_1u_2 \in E(G_1)$ or (iii) $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

For terms and definitions not explicitly defined here, reader may refer to Harary [3]. For graph products and the related terminology, reader may refer to Richard Hammack et. al [8].

In this paper, we investigate the edge Italian domination and the edge Italian domination number of some graph products like Corona, Cartesian product, Tensor product and Strong product of graphs.

## 2. Edge Italian Domination in some Graph Products

**Proposition 2.1.** *For any path $P_n$, $\gamma_I'(P_n \odot K_1) = n$.*

**Proof.** Consider an EIDF, $f$ on $P_n \odot K_1$ such that the $n$ edges connecting the $n$ copies of $K_1$ to $P_n$ get the weight 1 and all the edges of $P_n$ get the weight 0. Then, $\sum f(e) \leq n$. Since, $P_n \odot K_1$ has $2n$ vertices and $2n - 1$ edges, it follows from the definition of EIDF that in $\gamma_I'$ function on $P_n \odot K_1$, $\sum f(e) \geq \lceil \frac{2n-1}{2} \rceil = n$. Therefore, $\gamma_I'(P_n \odot K_1) = n$.

**Proposition 2.2.** *For any cycle $C_n$, $\gamma_I'(C_n \odot K_1) = n$.*

**Proof.** Consider the graph $C_n \odot K_1$. Let $f$ be an EIDF on $C_n \odot K_1$ such that the $n$ edges connecting the copies of $K_1$ to $C_n$ get the weight 1 and all the edges of $C_n$ get the weight 0 and then $\sum f(e) \leq n$. Since $C_n \odot K_1$ has $2n$ vertices and $2n$ edges, in order to satisfy the condition of EIDF, $\sum f(e) \geq n$. Hence, $\gamma_I'(C_n \odot K_1) = n$.

**Theorem 2.3.** *For any path $P_n$, $\gamma_I'(P_n \odot K_2) = \lceil \frac{3n}{2} \rceil$.*

**Proof.** Let $\{x_1, x_2, ..., x_{n-1}\}$ be the edges of $P_n$ and $\{z_1, z_2, ..., z_n\}$ denote the edges of the $n$ copies of $K_2$ and $\{y_1, y_2, ..., y_{2n}\}$ be the edges connecting the $n$ vertices of $P_n$ and the $n$ pairs of edges of the $n$ copies of $K_2$ such that $y_{2i-1}, y_{2i}$ are the edges joining the $i^{th}$ vertex of $P_n$ and the $i^{th}$ copy of $K_2$ for $1 \leq i \leq n$. Define a function

$f : E(P_n \odot K_2) \rightarrow \{0, 1, 2\}$ as follows:

$$f(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd or } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

$f(y_i) = 0, \forall\ i$ and $f(z_i) = 1, \forall\ i$.
Then, $\sum f(e) \leq n + \lceil \frac{n}{2} \rceil \leq \lceil \frac{3n}{2} \rceil$ so that $\gamma_I'(P_n \odot K_2) \leq \lceil \frac{3n}{2} \rceil$.

Let $f$ be any minimum EIDF on $(P_n \odot K_2)$. By Theorem 1.1, $\gamma_I'(P_n) = \lceil \frac{n}{2} \rceil$. So, $f$ can assign the weights 1 and 0 alternatively to the edges of the path $P_n$. Then $f$ cannot assign the weight 0 to any of the $n$ copies of $K_2$, as in that case each of these edges must be adjacent to an edge of weight 2 or adjacent to two edges of weight 1 each, which violates the minimality of $f$. So, $f$ must assign the minimum positive weight 1 to all the $n$ copies of $K_2$. Then, each of the remaining edges of $P_n \odot K_2$ is adjacent to exactly two edges of weight 1 each and hence can get the weight 0. So, $\sum f(e) \geq n + \lceil \frac{n}{2} \rceil \geq \lceil \frac{3n}{2} \rceil$. Thus, $\gamma_I'(P_n \odot K_2) = \lceil \frac{3n}{2} \rceil$.

**Theorem 2.4.** *For any cycle $C_n$, $\gamma_I'(C_n \odot K_2) = \lceil \frac{3n}{2} \rceil$.*
**Proof.** Let $x_1, x_2, ..., x_n$ be the edges of $C_n$ and $y_1, y_2, ..., y_{2n}$ be the edges connecting $C_n$ and $K_2$. Also let $z_1, z_2, ..., z_n$ denote the $n$ copies of $K_2$.
Define a function $f : E(C_n \odot K_2) \rightarrow \{0, 1, 2\}$ by

$$f(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$f(y_i) = 0, \forall\ i$ and $f(z_i) = 1, \forall\ i$.
Then, $\gamma_I'(C_n \odot K_2) \leq \sum f(e) \leq n + \lceil \frac{n}{2} \rceil \leq \lceil \frac{3n}{2} \rceil$.

Let $f$ be any minimum EIDF on $C_n \odot K_2$. Then by a similar argument used in the second part of the above theorem we can say that $f$ must assign the weight 1 to all the $n$ copies of $K_2$. Also, since by theorem 1.2, $\gamma_I'(C_n) = \lceil \frac{n}{2} \rceil$, $f$ can assign the weights 1 and 0 alternatively to the edges of $C_n$. Then, each of the remaining edges of $C_n \odot K_2$ is adjacent to at least two edges of weight 1 each and hence can get the weight 0. Hence, $\sum f(e) \geq n + \lceil \frac{n}{2} \rceil \geq \lceil \frac{3n}{2} \rceil$. Thus, $\gamma_I'(C_n \odot K_2) = \lceil \frac{3n}{2} \rceil$.

**Theorem 2.5.** *Let $C_m$ be a cycle on $m$ vertices and $P_n$ be a path on $n$ vertices, then,*

$$\gamma_I'(C_m \odot P_n) = \gamma_I'(C_m) + m\gamma_I'(P_n)$$

**Proof.** Let $X = \{x_1, x_2, ..., x_m\}$ be the edges of $C_m, Y = \{y_{i1}, y_{i2}, ..., y_{in-1}\}$ be the edges of $i^{th}$ copy of $P_n$ and $Z = \{z_1, z_2, ..., z_n, z_{n+1}, z_{n+2}, ..., z_{2n}, z_{2n+1}, ..., z_{3n}, ..., z_{mn}\}$ be the edges connecting the vertices of $C_m$ and the $m$ copies of $P_n$.

Define a function $f : E(C_m \odot P_n) \to \{0, 1, 2\}$ by

$$f(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

$$f(y_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd or } j = n-1 \\ 0 & \text{otherwise} \end{cases}$$

$f(z_i) = 0, \forall\ i.$
Then, $\sum f(e) \leq \lceil \frac{n}{2} \rceil + m \lceil \frac{n}{2} \rceil = \gamma_I'(C_m) + m\gamma_I'(P_n)$.
The corona, $C_m \odot P_n$ contains $m$ copies of $P_n$ joined to $C_m$. Since $C_m$ and the $m$ copies of $P_n$ are induced sub-graphs of the corona $C_m \odot P_n$, the weight $\sum f(e)$ of any EIDF is at least $\gamma_I'(C_m) + m\gamma_I'(P_n)$. Hence, $\sum f(e) \geq \gamma_I'(C_m) + m\gamma_I'(P_n)$.
Therefore, $\gamma_I'(C_m \odot P_n) = \gamma_I'(C_m) + m\gamma_I'(P_n)$.

**Proposition 2.6.** *For $n \geq 2$, $\gamma_I'(P_2 \times P_n) = 2\lceil \frac{n}{2} \rceil$.*
**Proof.** The graph, $P_2 \times P_n$, is a disconnected graph with exactly two components isomorphic to $P_n$. So, using theorem 1.1, $\gamma_I'(P_2 \times P_n) = 2\lceil \frac{n}{2} \rceil$.

**Proposition 2.7.** *For $n \geq 2$, $\gamma_I'(P_3 \times P_3) = 4$.*
**Proof.** $P_3 \times P_3$ is a disconnected graph with two components, one is a cycle of length 4 and the other one is the star graph $K_{1,4}$. By theorems 1.2 and 1.3, we have $\gamma_I'(C_n) = \lceil \frac{n}{2} \rceil$ and $\gamma_I'(K_{1,n}) = 2$. So, $\gamma_I'(P_3 \times P_3) = \gamma_I'(C_4) + \gamma_I'(K_{1,4}) = 2 + 2 = 4$.

**Theorem 2.8.** *For $n \geq 2$, $\gamma_I'(P_3 \times P_n) = 2n - 2$.*
**Proof.** Let $n = 2k$ or $n = 2k + 1$ according as $n$ is even or odd. We prove the result by induction on $k$. By propositions 2.6 and 2.7, the result is true for $k = 1$. Assume that the result is true for the positive integer, $k$.
That is, $\gamma_I'(P_3 \times P_n) = \gamma_I'(P_3 \times P_{2k}) = 2(2k) - 2$ and
$\gamma_I'(P_3 \times P_n) = \gamma_I'(P_3 \times P_{2k+1}) = 2(2k + 1) - 2$.
We will prove that the result is true for the integer, $k + 1$. The graph product, $P_3 \times P_n$ is a disconnected graph with exactly two components. Further when $n$ is even, the two components will be isomorphic graphs.
**Case 1.** $n = 2k$ is even.
The result is true for $k$ by induction assumption. That is, $\gamma_I'(P_3 \times P_{2k}) = 2(2k) - 2$.
If $n = 2k$ is even, then, the graph products $P_3 \times P_{2(k+1)}$ and $P_3 \times P_{2k}$ differ only in an induced copy of $C_4$ in each of its two components. Take any minimum EIDF of $P_3 \times P_{2k}$. Then at most two of the four edges of the induced copy of $C_4$ can be given weight 0 and the other two edges can be given the weight 1 to get a minimum EIDF of $P_3 \times P_{2(k+1)}$, which has only 4 weights more than that of $P_3 \times P_{2k}$.
Hence, $\gamma_I'(P_3 \times P_{2k+1}) = \gamma_I'(P_3 \times P_{2k}) + 4 = [2(2k) - 2] + 4 = [2(2k + 1)] - 2$.

So, the result is true for $k+1$.
**Case 2.** $n = 2k+1$ is odd.
The result is true for $k$ by induction assumption. That is, $\gamma_I'(P_3 \times P_{2k+1}) = 2(2k+1)-2$. If $n = 2k+1$, is odd, the graph products $P_3 \times P_{2(k+1)+1}$ and $P_3 \times P_{2k+1}$ differ only in an induced copy of $C_4$ in each of its two components. Take any minimum EIDF of $P_3 \times P_{2k}$. Then at most two of the four edges of the induced copy of $C_4$ can be given weight 0 and the other two edges can be given the weight 1 to get a minimum EIDF of $P_3 \times P_{2(k+1)+1}$, which has only 4 weights more than that of $P_3 \times P_{2k+1}$.
Hence,

$$\begin{aligned}\gamma_I'(P_3 \times P_{2(k+1)+1}) &= \gamma_I'(P_3 \times P_{2k+1}) + 4\\ &= [2(2k+1)-2]+4\\ &= [2(2(k+1)+1)]-2.\end{aligned}$$

So, the result is true for $k+1$. Hence, by induction, the result is true for all integers, $n$.

**Theorem 2.9.** *For* $\geq 2$, $\gamma_I'(P_n \square P_2) = n$.
**Proof.** The Cartesian product of $P_n$ and $P_2$ is the ladder graph $L_n$ and it has $n$ middle step edges. Every other edge of the ladder is adjacent to at least two of these middle edges. Let $g$ be an EIDF on $P_n \square P_2$ in which all the middle edges are given the weight 1 and all other edges the weight 0. Then we get a minimum EIDF and $\sum g(e) \leq n$.

For the lower bound, consider the two copies $P_n$ and $P_n'$ of the paths forming $P_n \square P_2$ where $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(P_n') = \{v_1', v_2', v_3', ..., v_n'\}$. The vertices $\{v_1, v_2, v_3, ..., v_n, v_n', ..., v_2', v_1'\}$ form a cycle of length $2n$. Let $g$ be a minimum EIDF on $P_n \square P_2$ in which the weights 1 and 0 are given alternatively to the edges forming this cycle. Then each of the middle step edge is adjacent to at least two edges of weights 1 each and hence get the weight 0. By Theorem 1.1, $\gamma_I'(C_{2n}) = \lceil \frac{2n}{2} \rceil = n$. Hence, $\sum g(e) \geq n$. Therefore $\gamma_I'(P_n \square P_2) = \gamma_I'(L_m) = n$.

**Theorem 2.10.** *For* $n \geq 2$, $\gamma_I'(P_n \square P_m) = \lceil \frac{mn}{2} \rceil = \begin{cases} \frac{mn+1}{2}, & \textit{if } m \textit{ and } n \textit{ are odd} \\ \frac{mn}{2}, & \textit{otherwise} \end{cases}$
**Proof.** Let $u_1, u_2, u_3, ..., u_n$ be the vertices of $P_n$ and $v_1, v_2, v_3, ..., v_m$ be the vertices of $P_m$.
**Case 1. Both $m$ and $n$ are odd**
Define a function $f : E(P_n \square P_m) \to \{0, 1, 2\}$ by
$f(e) = 1$, when $e = \{(u_i, v_j)(u_i, v_{j+1})\}$ where $i = 1, 2, 3, ..., n$ and $j = 1, 3, 5, ..., m-2$.

$f(e) = 1$ when $e = \{(u_i, v_m)(u_{i+1}, v_m)\}$, where $i = 1, 3, 5, ..., n-2$ and
$f(e) = 1$ when $e = \{(u_{n-1}, v_m)(u_n, v_m)\}$
$f(e) = 0$, otherwise
Then, $\sum f(e) \leq m.\left(\frac{n-1}{2}\right) + \frac{m-1}{2} + 1 = \frac{mn+1}{2}$.

Let $f$ be a minimum EIDF on $P_n \Box P_m$. Consider the $n$ paths induced by the vertices $(u_i v_1), (u_i v_2), (u_i v_3), ..., (u_i v_{m-1})$, where $i = 1, 2, 3, ..., n$. Since $\gamma_I^{'}(P_{m-1}) = \lceil \frac{m-1}{2} \rceil$, $f$ can assign the weights 1 and 0 alternatively to the edges of these paths. Using a similar argument, the weights 1 and 0 can be assigned alternatively to the edges of the path induced by the $n$ vertices $(u_1 v_m), (u_2 v_m), (u_3 v_m), ..., (u_m v_m)$. Then all the remaining edges can get the weight 0 as each of them is incident with at least two edges of weight 1. $\sum f(e) \geq n.\frac{m-1}{2} + \frac{n+1}{2} = \frac{mn+1}{2}$. Thus, we get, $\gamma_I^{'}(P_n \Box P_m) = \frac{mn+1}{2}$.

**Case 2. $m$ is even and $n$ is even or odd**
Define a function $f : E(P_n \Box P_m) \rightarrow \{0, 1, 2\}$ by
$f(e) = 1$, if $e = \{(u_i, v_j)(u_i, v_{j+1})\}$ where $i = 1, 2, 3, ..., n$ and $j = 1, 3, 5, ..., m-1$.
$f(e) = 0$, otherwise
Then $f$ is a EIDF of the graph, $P_n \Box P_m$ so that $\gamma_I^{'}(P_n \Box P_m) \leq \sum f(e) \leq m.\frac{n}{2} = \frac{mn}{2}$.

For the lower bound, consider any minimum EIDF, $f$, on $P_n \Box P_m$. For $i = 1, 2, 3, ..., n$ the set of vertices $(u_i v_1), (u_i v_2), (u_i v_3), ..., (u_i v_m)$ induces $n$ paths $P_m$. Using a similar argument used in the previous case, we get $\sum f(e) \geq n\frac{m}{2} = \frac{mn}{2}$. Therefore, $\gamma_I^{'}(P_n \Box P_m) = \frac{mn}{2}$.

**Corollary 2.11.** *For $n \geq 2$, the edge Italian domination number of the strong product of the path graphs $P_n$ and $P_m$ is given by,*

$$\gamma_I^{'}(P_n \boxtimes P_m) = \left\lceil \frac{mn}{2} \right\rceil = \begin{cases} \frac{mn+1}{2}, & \text{if both m and n are odd} \\ \frac{mn}{2}, & \text{otherwise} \end{cases}$$

**Proof.** Consider the strong product $P_n \boxtimes P_m$. We know that $P_n \Box P_m$ is an induced subgraph of $P_n \boxtimes P_m$. So, consider the minimum EIDF for $P_n \Box P_m$ as defined in the proof of the theorem 2.10 and extend it to $P_n \boxtimes P_m$. In this EIDF, the edges of $P_n \boxtimes P_m$, which are not the edges of $P_n \Box P_m$ can be given the weight 0, as each of them is incident with at least two edges of $P_n \Box P_m$, having weight 1 and hence the weight of these edges of $P_n \boxtimes P_m$ do not affect the minimum weight for $P_n \boxtimes P_m$. Hence, $\gamma_I^{'}(P_n \boxtimes P_m) = \lceil \frac{mn}{2} \rceil$.

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