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ON TRANSFORMED GRAPHS

Ishita Sarkar, Manjunath Nanjappa* and Ivan Gutman**

Department of Mathematics,
CHRIST (Deemed to be University),
Bengaluru - 560029, INDIA

E-mail : ishita.sarkar@res.christuniversity.in

*School of Engineering and Technology,
CHRIST (Deemed to be University),
Bengaluru - 560074, INDIA

E-mail : manjunath.nanjappa@christuniversity.in

**Faculty of Science,
University of Kragujevac,
Kragujevac, 34000, SERBIA

E-mail : gutman@kg.ac.rs

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Abstract: The network systems and graphical analysis through the study of structural characteristics is a vast field of growing importance in research. Topological indices have a significant and crucial role in the study of structure–property relationships. In this paper, we examine constructional transformed networks constructed by unique vertex-edge incidence and mutual adjacency associations. Expressions for the first and second hyper Zagreb indices and co-indices of these transformed networks and their complements are obtained.

Keywords and Phrases: Hyper Zagreb Indices, Transformation Graphs, Co-indices.

2020 Mathematics Subject Classification: 05C09, 05C90, 05C92.

1. Introduction

A graph H can be transformed into another network $T(H)$, such that the transformed graph T and the original graph H are in a one-to-one correspondence. More detailed information on this matter is found in [4, 15, 19, 21, 27]. Topological indices are numerical values calculated from the structure of the underlying graph, and reflect certain structural features of the graph. Topological indices have found numerous applications in chemistry, physics, pharmacology, toxicology and elsewhere.

In the present work, we are considering simple, connected and undirected graphs. The vertex and edge set of a graph H are $V(H)$ and $E(H)$ respectively. $N(v) = \{w \in V(H) | vw \in E(H)\}$ and $N[v] = N(v) \cup \{v\}$ indicate the open and closed neighbourhoods of the vertex v respectively. In H, $d_H(w)$ denotes the degree of vertex w and $e = vw$ is the edge joining the vertices v and w. The degree of an edge e is defined as the number of neighbours of e, i.e., $d_H(e) = d_H(v) + d_H(w) - 2$. The minimum and maximum vertex degree of the graph H are denoted by δ_H and Δ_H, respectively.

The First (M_1) and the Second (M_2) Zagreb Indices belong among the oldest and best studied vertex degree-based topological indices [13, 14]. In terms of edge degrees, reformulations of the Zagreb indices have been proposed in [20] as

$$EM_1(H) = \sum_{e \in E(H)} d_H(e)^2 \quad \text{and} \quad EM_2(H) = \sum_{e \sim f \in E(H)} d_H(e) d_H(f)$$

The first reformulated Zagreb co-index $(\overline{EM_1})$ has been studied in [11], where its main properties are determined. The second reformulated Zagreb co-index $(\overline{EM_2})$ is then defined analogously:

$$\overline{EM_1}(H) = \sum_{e \notin E(H)} d_H(e)^2 \quad \text{and} \quad \overline{EM_2}(H) = \sum_{e \nsim f \in E(H)} d_H(e) d_H(f)$$

Here and later, $e \nsim f$ indicates pairs of non-incident edges e and f, i.e., edges that have no common end vertex.

Another vertex-degree-based graph invariant, called forgotten topological index, has also been put forward [9], defined as:

$$F(H) = \sum_{w \in V(H)} d_H(w)^3 \quad \text{or} \quad F(H) = \sum_{vw \in E(H)} \left(d_H(v)^2 + d_H(w)^2 \right)$$

Numerous graph operational expressions have been determined with respect to this index [7].

In [25], the study on the first hyper Zagreb index (HM_1) was initiated and certain results relevant to the mathematical techniques of graph operations were obtained. The study on another variant, the second hyper Zagreb index (HM_2), was initiated in [8] and computations pertaining to various classes graphs have been reported in [1, 10].

$$HM_1(H) = \sum_{vw \in E(H)} (d_H(v) + d_H(w))^2 \text{ and } HM_2(H) = \sum_{vw \in E(H)} \big(d_H(v).d_H(w)\big)^2$$

The so-called reformulated forgotten index has also been proposed as a structure-descriptor [2], quite similar to the notion of the reformulations for first and second Zagreb indices.

$$RF(H) = \sum_{e \sim f \in E(H)} \big(d_H(e)^2 + d_H(f)^2\big) \quad \text{or} \quad RF(H) = \sum_{e \in E(H)} d_H(e)^3$$

The reformulated F-coindex of the graph H is defined as:

$$\overline{RF}(H) = \sum_{e \nsim f \in E(H)} \big(d_H(e)^2 + d_H(f)^2\big)$$

The first and second K indices and coindices (Kulli indices and coindices) have been defined as the reformulations of the first and second hyper Zagreb indices in terms of the edge degrees [16, 17].

$$K^1(H) = \sum_{e \sim f \in E(H)} (d_H(e) + d_H(f))^2 \text{ and } K^2(H) = \sum_{e \sim f \in E(H)} (d_H(e).d_H(f))^2$$

$$\overline{K}^1(H) = \sum_{e \nsim f \in E(H)} (d_H(e) + d_H(f))^2 \text{ and } \overline{K}^2(H) = \sum_{e \nsim f \in E(H)} (d_H(e).d_H(f))^2$$

Consequently, on generalizing the Zagreb-index concept [3] another descriptor called the generalized Zagreb index was conceived:

$$Z_{a,b}(H) = \sum_{vw \in E(H)} \Big(d_H(v)^a d_H(w)^b + d_H(v)^b d_H(w)^a\Big)$$

Various properties and computations related to this index have been reported in [23, 24].

Now, the generalized Zagreb index and co-index is formulated where summation is based on edge degrees:

$$EZ_{a,b}(H) = \sum_{e\sim f\in E(H)} \Big(d_H(e)^a d_H(f)^b + d_H(e)^b d_H(f)^a\Big)$$

$$\overline{EZ_{a,b}}(H) = \sum_{e\nsim f\in E(H)} \Big(d_H(e)^a d_H(f)^b + d_H(e)^b d_H(f)^a\Big)$$

Works pertaining to structure-descriptors of the transformation graphs have been carried out in [16, 18]. In the present paper, exact expressions for the first and second Hyper-Zagreb indices of the total transformation graphs are determined.

At this point, we recall that the total graph includes edges of subdivision and line graph along with the edges of the base graph [5, 22].

2. Methodology

We begin with ordinary graphs and we end up by obtaining a complex network structure from the base graphs. The base graph can be transformed by implementing any of the following combinations [26]:
We have, $x, y, z \in \{+, -\}$ and $v, w \in V(T_{xyz}(H))$;

(i) If $x = +$, v and w are adjacent in H & if $x = -$, v and w are not adjacent in H; for $v, w \in V(H)$.

(ii) If $y = +$, v and w are adjacent in H & if $y = -$, v and w are not adjacent in H; for $v, w \in E(H)$.

(iii) If $z = +$, v is incident on w in H & if $z = -$, v is not incident on w in H; for $v \in V(H), w \in E(H)$.

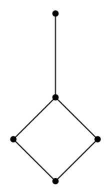

Figure 1: Base graph, H

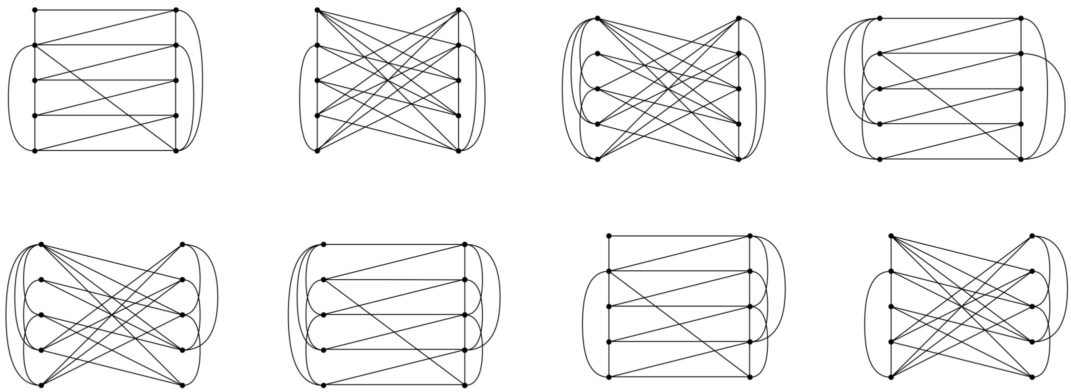

Figure 2: Transformation Graphs, $T_{+++}(H), T_{++-}(H), T_{-+-}(H), T_{-++}(H)$(first row, left to right); $T_{---}(H), T_{--+}(H), T_{+-+}(H), T_{+--}(H)$(second row, left to right)

For a graph in figure (1), there are eight distinct permutations of $\{+, -\}$. The respective eight graphs in figure (2) are constructed from a single base graph, H. Also, $\overline{T_{-++}(H)} \cong T_{+--}(H)$, $\overline{T_{+-+}(H)} \cong T_{-+-}(H)$, $\overline{T_{++-}(H)} \cong T_{--+}(H)$, and $\overline{T_{+++}(H)} \cong T_{---}(H)$ are constructed accordingly.

Lemma 1. *(from [27]) Let H be a graph with size q and order p. Then $|V(T_{xyz}(H))| = q + p$ and the edge cardinality is given as:*

$$|E(T_{xyz}(H))| = \begin{cases} \frac{1}{2}\Big[M_1(H) + 4q\Big] & ; x = +, y = +, z = +, \\ \frac{1}{2}\Big[M_1(H) + 2q(p-2)\Big] & ; x = +, y = +, z = -, \\ \frac{1}{2}\Big[q^2 + 7q - M_1(H)\Big] & ; x = +, y = -, z = +, \\ \frac{1}{2}\Big[M_1(H) + p(p-1)\Big] & ; x = -, y = +, z = +, \\ \frac{1}{2}\Big[(q+p)^2 - 5q - p - M_1(H)\Big] & ; x = -, y = -, z = -, \\ \frac{1}{2}\Big[q^2 + p(p-1) + 3q - M_1(H)\Big] & ; x = -, y = -, z = +, \\ \frac{1}{2}\Big[M_1(H) + 2q(p-4) + p(p-1)\Big] & ; x = -, y = +, z = -, \\ \frac{1}{2}\Big[q^2 + 2qp - q - M_1(H)\Big] & ; x = +, y = -, z = -, \end{cases}$$

2.1. First Hyper Zagreb Index of Transformation Graphs

Theorem 1. *Let H be a graph with size q and order p. Then;*

$$\begin{aligned} HM_1(T_{+++}(H)) = {} & 8M_1(H) + 12M_2(H) + 10F(H) + 4HM_1(H) + 8EM_1(H) \\ & + 2EM_2(H) + RF(H) - 16 \end{aligned}$$

Proof. The vertex-degree behaviour in the graph $T_{+++}(H)$ is :

$$d_{T_{+++}(H)}(v) = \begin{cases} 2d_H(v) & , \text{ if } v \in V(H) \\ d_H(v) + 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$\begin{aligned} HM_1(T_{+++}(H)) = & \sum_{vw \in E(T_{+++}(H)) \cap E(H)} \Big(d_{T_{+++}(H)}(v) + d_{T_{+++}(H)}(w) \Big)^2 \\ & + \sum_{vw \in E(T_{+++}(H)) \cap E(L(H))} \Big(d_{T_{+++}(H)}(v) + d_{T_{+++}(H)}(w) \Big)^2 \\ & + \sum_{vw \in E(T_{+++}(H)) \setminus (E(H) \cap E(L(H))} \Big(d_{T_{+++}(H)}(v) + d_{T_{+++}(H)}(w) \Big)^2 \\ & = f_1 + f_2 + f_3 \end{aligned}$$

Now for the computation of f_1,

$$\begin{aligned} f_1 &= \sum_{vw \in E(T_{+++}(H)) \cap E(H)} \Big(d_{T_{+++}(H)}(v) + d_{T_{+++}(H)}(w) \Big)^2 \\ &= \sum_{vw \in E(H)} \Big(2d_H(v) + 2d_H(w) \Big)^2 = \sum_{vw \in E(H)} 4\Big(d_H(v) + d_H(w) \Big)^2 \\ &= 4HM_1(H) \end{aligned}$$

For the formulation of f_2,

$$\begin{aligned} f_2 &= \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_H(v) + 2 + d_H(w) + 2 \Big)^2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_H(v) + d_H(w) + 4 \Big)^2 \\ &= RF(H) + 2EM_2(H) + 16|E(L(H))| + 8EM_1(H) \end{aligned}$$

For the computation of f_3,

$$\begin{aligned} f_3 &= \sum_{\substack{v \in V(H) \\ w \in E(H) \\ v \sim w}} \Big(2d_H(v) + d_H(w) + 2 \Big)^2 = \sum_{v \in V(H)} \sum_{x \in N_H(v)} \Big(3d_H(v) + d_H(x) \Big)^2 \\ &= 10F(H) + 12M_2(H) \end{aligned}$$

Hence from all the computations,

$$\begin{aligned} HM_1(T_{+++}(H)) &= 8M_1(H) + 12M_2(H) + 10F(H) + 4HM_1(H) + 8EM_1(H) \\ &+ 2EM_2(H) + RF(H) - 16 \end{aligned}$$

This concludes the proof.

Theorem 2. *Let H be a graph with size q and order p. Then;*

$$\begin{aligned} HM_1(T_{++-}(H)) &= (p-2)\Big(5EM_1(H) + 2(2p+q-4)M_1(H)\Big) + 2EM_2(H) \\ &+ RF(H) + q\Big[(q+p-2)(q+p-6) - 4(p-2)\Big](p-2) + 4q^3 \end{aligned}$$

Proof. The vertex-degree behaviour in the graph $T_{++-}(H)$ is :

$$d_{T_{++-}(H)}(v) = \begin{cases} q & , \text{ if } v \in V(H) \\ d_H(v) + p - 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$\begin{aligned} HM_1(T_{++-}(H)) &= \sum_{vw \in E(H)} \Big(d_{T_{++-}(H)}(v) + d_{T_{++-}(H)}(w)\Big)^2 \\ &= \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} \Big(d_H(v) + d_H(w)\Big)^2 + \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_H(v) + d_H(w)\Big)^2 \\ &+ \sum_{\substack{v \in V(H) \\ w \in E(H) \\ v \sim w}} \Big(d_H(v) + d_H(w)\Big)^2 \\ &= f_1 + f_2 + f_3 \end{aligned}$$

For the computation of f_1,

$$f_1 = \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} \Big(d_H(v) + d_H(w)\Big)^2 = \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} 4q^2$$

$$\Longrightarrow f_1 = 4q^3$$

For the computation of f_2,

$$f_2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_H(v) + d_H(w)\Big)^2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_H(v) + d_H(w) + 2p - 4\Big)^2$$

$$\Longrightarrow f_2 = RF(H) + 2EM_2(H) + 4(p-2)EM_1(H) + 2(p-2)^2 M_1(H) - 4q(p-2)^2$$

Also for the computation of f_3,

$$f_3 = \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_H(v)+d_H(w)\Big)^2 \;=\sum_{w\in E(H)}\sum_{\substack{v\in V(H)\\ v\nsim w}} \Big(d_H(w)+q+p-2\Big)^2$$

$$= \sum_{w\in E(H)} (p-2)\Big[d_H(w)^2 + 2(q+p-2)d_H(w) + (q+p-2)^2\Big]$$

$$= (p-2)\Big[EM_1(H) + 2(q+p-2)\Big(M_1(H)-2q\Big) + q(q+p-2)^2\Big]$$

Hence from all the computations,

$$HM_1(T_{++-}(H)) = (p-2)\Big(5EM_1(H) + 2(2p+q-4)M_1(H)\Big) + 2EM_2(H) + RF(H) + q\Big[(q+p-2)(q+p-6) - 4(p-2)\Big](p-2) + 4q^3$$

This concludes the proof.

Theorem 3. *Let H be a graph with size q and order p. Then;*

$$HM_1(T_{+-+}(H)) = 4\Big(HM_1(H) - M_2(H)\Big) + 2\Big(F(H) + \overline{EM_2}(H)\Big) - 2(q+1)\Big[(q+1)M_1(H) - 2\overline{EM_1}(H)\Big] + \overline{RF}(H) + 2q\Big[(q+1)^3 + (q+3)^2\Big]$$

Proof. The vertex-degree behaviour in the graph $T_{+-+}(H)$ is :

$$d_{T_{+-+}(H)}(v) = \begin{cases} 2d_H(v) & ,\ \text{if } v\in V(H) \\ q+1-d_H(v) & ,\ \text{if } v\in E(H)\end{cases}$$

Therefore;

$$HM_1(T_{+-+}(H)) = \sum_{vw\in E(T_{+-+}(H))} \Big(d_{T_{+-+}(H)}(v) + d_{T_{+-+}(H)}(w)\Big)^2$$

$$= \sum_{vw\in E(H)} \Big(2d_H(v)+2d_H(w)\Big)^2 + \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big(q+1-d_H(v)+q+1-d_H(w)\Big)$$

$$+ \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(2d_H(v)+q+1-d_H(w)\Big)^2$$

$$= f_1 + f_2 + f_3$$

For the computation of f_1,

$$f_1 = \sum_{vw\in E(H)} \Big(2d_H(v) + 2d_H(w)\Big)^2 \;\; = 4 \sum_{vw\in E(H)} \Big(d_H(v) + d_H(w)\Big)^2$$
$$= 4HM_1(H)$$

For the computation of f_2,

$$f_2 = \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big(q + 1 - d_H(v) + q + 1 - d_H(w)\Big)$$
$$= \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big[\big(d_H(v) + d_H(w)\big)^2 - 2(2q + 2)(d_H(v) + d_H(w)) + (2q + 2)^2\Big]$$
$$= \overline{RF}(H) + 2\overline{EM_2}(H) - 2(q + 1)\Big((q + 1)M_1(H) + 2\overline{EM_1}(H)\Big) + 2q(q + 1)^2(q + 1)$$

For the computation of f_3,

$$f_3 = \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(2d_H(v) + q + 1 - d_H(w)\Big)^2 \;\; = \sum_{v\in V(H)} \sum_{x\in N_H(v)} \Big(d_H(v) - d_H(x) + q + 3\Big)^2$$
$$= 2\Big[F(H) - 2M_2(H) + q(q + 3)^2\Big]$$

Hence from all the computations,

$$HM_1(T_{+-+}(H)) = 4\Big(HM_1(H) - M_2(H)\Big) + 2\Big(F(H) + \overline{EM_2}(H)\Big) - 2(q + 1)\Big[(q + 1)M_1(H) - 2\overline{EM_1}(H)\Big] + \overline{RF}(H) + 2q\Big[(q + 1)^3 + (q + 3)^2\Big]$$

This concludes the proof.

Theorem 4. *Let H be a graph with size q and order p. Then;*

$$HM_1(T_{-++}(H)) = 4(p + 1)M_1(H) + 4M_2(H) + 2F(H) + 8EM_1(H) + 2EM_2(H) + RF(H) + 2(p - 1)^2\Big(q + p(p - 1) - 2p\Big) - 16q$$

Proof. The vertex-degree behaviour in the graph $T_{-++}(H)$ is :

$$d_{T_{-++}(H)}(v) = \begin{cases} p - 1 & , \text{ if } v \in V(H) \\ d_H(v) + 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$HM_1(T_{-++}(H)) = \sum_{vw\in E(T_{-++}(H))} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2$$

$$= \sum_{vw\notin E(H)} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2 + \sum_{\substack{v,w\in E(H)\\ v\sim w}} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2$$

$$+ \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2$$

$$= f_1 + f_2 + f_3$$

For the computation of f_1,

$$f_1 = \sum_{vw\notin E(H)} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2 = \sum_{vw\notin E(H)} \Big(d_H(v) + d_H(w)\Big)^2$$

$$= \sum_{vw\notin E(H)} (2p-2)^2$$

$$\implies f_1 = (2p-2)^2\Big[\frac{p(p-1)}{2} - p\Big]$$

In order to compute f_2,

$$f_2 = \sum_{\substack{v,w\in E(H)\\ v\sim w}} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2 = \sum_{\substack{v,w\in E(H)\\ v\sim w}} \Big(d_H(v) + d_H(w) + 4\Big)^2$$

$$\implies f_2 = 8EM_1(H) + 2EM_2(H) + RF(H) + 16|E(L(H))|$$

For the computation of f_3,

$$f_3 = \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_{T_{-++}(H)}(v) + d_{T_{-++}(H)}(w)\Big)^2 = \sum_{v\in V(H)} \sum_{\substack{w\in E(H)\\ v\sim w}} \big(d_H(w) + p + 1\big)^2$$

$$= \sum_{v\in V(H)} \sum_{x\in N_H(v)} \Big(d_H(v) + d_H(x) + p - 1\Big)^2$$

$$\implies f_3 = 4(p-1)M_1(H) + 4M_2(H) + 2F(H) + 2q(p-1)^2$$

Hence from all the computations,

$$HM_1(T_{-++}(H)) = 4(p+1)M_1(H) + 4M_2(H) + 2F(H) + 8EM_1(H)$$
$$+ 2EM_2(H) + RF(H) + 2(p-1)^2\Big(q + p(p-1) - 2p\Big) - 16q$$

This concludes the proof.

Theorem 5. *Let H be a graph with size q and order p. Then;*

(1)

$$HM_1(T_{---}(H)) = M_1(H)\Big[M_1(H) + 6(q+p)^2 + 8(2q+p) - 26\Big] + 5(q+p)$$
$$\Big(2M_2(H) + F(H)\Big) - 8\Big[3M_2(H) + 2F(H)\Big] - 2\Big[4EM_1(H) + EM_2(H)$$
$$+ 2HM_1(H)\Big] - RF(H) + 2(q+p-1)^2\Big[(q+p)(q+p-1) + 12q\Big] + 16q(q+1)$$

(2)

$$HM_1(T_{--+}(H)) = \Big(M_1(H) + 2q(p-2)\Big)\Big[6(q+p-1)^2 + M_1(H) + 2q(p-2)\Big]$$
$$-(p-2)\Big[(p-2)\Big\{2M_1(H) - 4q\Big\} + (q+p-2)\Big\{q(q+p-2) + 2M_1(H)$$
$$-4q\Big\} + 5EM_1(H)\Big] - RF(H) - 2EM_2(H) + (5q+5p-6)\Big[2(p-4)M_1(H)$$
$$+ 2M_2(H) + F(H) + q\Big\{p(q+p-8) + 16\Big\}\Big] + 2(q+p)(q+p-1)^3 - 4q^3$$

(3)

$$HM_1(T_{-+-}(H)) = (q^2 + 7q - M_1(H))\Big[q^2 + 7q - M_1(H) - 6(q+p-1)^2\Big]$$
$$+(q+1)\Big[2(q+1)\Big\{M_1(H) - q(q+1)\Big\} + 4\overline{EM_1}(H)\Big] + (5q+5p-6)\Big[q(q+3)^2$$
$$-2(q+1)M_1(H) + 2M_2(H) + F(H)\Big] - 4\Big(HM_1(H) - M_2(H)\Big) - 2\Big(F(H)$$
$$+\overline{EM_2}(H)\Big) - \overline{RF}(H) + 2\Big[(q+p)(q+p-1)^3 - q(q+3)^2\Big]$$

(4)

$$HM_1(T_{+--}(H)) = \Big(M_1(H) + p(p-1)\Big)\Big[M_1(H) + p(p-1) - 6(q+p-1)^2\Big]$$
$$+(5q+5p-6)\Big[p(p-1)^2 + 2M_2(H) + F(H)\Big] + 2(p-1)^2\Big[p(3-p) - q\Big]$$
$$-RF(H) - 2\Big(EM_2(H) + F(H)\Big) - 4\Big[M_2(H) + 2EM_1(H) + (p+1)M_1(H)\Big]$$
$$+2\Big((q+p)(q+p-1)^3 + 8q\Big)$$

Proof. The proof follows from the notion $\overline{T_{-++}(H)} \cong T_{+--}(H)$, $\overline{T_{+-+}(H)} \cong T_{-+-}(H)$, $\overline{T_{++-}(H)} \cong T_{--+}(H)$, and $\overline{T_{+++}(H)} \cong T_{---}(H)$ and edge cardinality cases as mentioned in lemma (1). The first Zagreb index of the transformation graphs have been determined in [15, 27]. We have

$$HM_1(\overline{H}) = 2p(p-1)^3 - 12q(p-1)^2 + 4q^2 + (5p-6)M_1(H) - HM_1(H)$$

Also, we utilise the theorem (1), (2), (3), (4) results to get the desired output.

Theorem 6. *Let H be a graph with size q and order p. Then;*

(1)

$$\overline{HM_1}(T_{+++}(H)) = \Big(M_1(H) + 4q\Big)^2 + (q+p-2)\Big[4M_1(H) + 2M_2(H) + F(H)\Big] - \Big[4HM_1(H) + RF(H) + 2EM_2(H) + 8M_1(H) + 8EM_1(H) + 10F(H) + 12M_2(H) - 16q\Big]$$

(2)

$$\overline{HM_1}(T_{++-}(H)) = \Big[M_1(H) + 2q(p-2)\Big]^2 + (q+p-2)\Big[qp(q+p-8) + 16q + 2(p-4)M_1(H) + 2M_2(H) + F(H)\Big] - \Big[5(p-2)EM_1(H) + 2EM_2(H) + RF(H) + 2(p-2)(2p+q-4)M_1(H) + q(p-2)\{(q+p-2)(q+p-6) - 4(p-2)\} + 4q^3\Big]$$

(3)

$$\overline{HM_1}(T_{+-+}(H)) = ((q^2 + 7q - M_1(H))^2 + (q+p-2)\Big(q(q+3)^2 - 2(q+1)M_1(H) + 2M_2(H) + F(H)\Big) - \Big[4(HM_1(H) - M_2(H)) + 2(F(H) + \overline{EM_2}(H))) - 2(q+1)\Big\{(q+1)M_1(H) - 2\overline{EM_1}(H)\Big\} + \overline{RF}(H) + 2q[(q+1)^3 + (q+3)^2]\Big]$$

(4)

$$\overline{HM_1}(T_{-++}(H)) = \Big(M_1(H) + p(p-1)\Big)^2 + (q+p-2)\Big[p(p-1)^2 + 2M_2(H) + F(H)\Big] - \Big[2p(p-1)^3 - 4p(p-1)^2 - 16q + 2q(p-1)^2 + RF(H) + 2EM_2(H) + 8EM_1(H) + 2F(H) + 4M_2(H) + 4(p+1)M_1(H)\Big]$$

(5)

$$\overline{HM_1}(T_{---}(H)) = \Big[(q+p)^2 - 5q - p - M_1(H)\Big]^2 + (q+p-2)\Big[(q+p)\{(q+p)^2 - 10q - 2p + 1\} + 8q - 2(q+p-3)M_1(H) + 2M_2(H) + F(H)\Big] - \Big[M_1(H)\Big[M_1(H) + 6(q+p)^2 + 8(2q+p) - 26\Big] + 5(q+p)\Big(2M_2(H) + F(H)\Big) - 8\Big[3M_2(H) + 2F(H)\Big] - 2\Big[4EM_1(H) + EM_2(H) + 2HM_1(H)\Big] - RF(H) + 16q(q+1) + 2\Big[(q+p)(q+p-1) + 12q\Big](q+p-1)^2\Big]$$

(6)

$$\overline{HM_1}(T_{--+}(H)) = \Big[q^2 + p(p-1) + 3q - M_1(H)\Big]^2 + (q+p-2)\Big[p(p-1)^2 + q(q+3)^2 - (2q+6)M_1(H) + 2M_2(H) + F(H)\Big] - \Big[(M_1(H) + 2q(p-2))(6(q+p-1)^2 + M_1(H) + 2q(p-2)) - (p-2)\{(p-2)(2M_1(H) - 4q) + 5EM_1(H) + (q+p-2)\{q(q+p-2) + 2M_1(H) - 4q\}\} - RF(H) - 2EM_2(H) + (5q+5p-6)\{2(p-4)M_1(H) + 2M_2(H) + F(H) + q(p(q+p-8) + 16)\} + 2(q+p)(q+p-1)^3 - 4q^3\Big]$$

(7)

$$\overline{HM_1}(T_{-+-}(H)) = \Big(M_1(H) + 2q(p-4) + p(p-1)\Big)^2 + (q+p-2)\Big[q(q+3)^2 + (q+p)(q+p-1)^2 - 2(q^2+7q)(q+p-1) + 2(p-2)M_1(H) + 2M_2(H) + F(H)\Big] - \Big[(q^2 + 7q - M_1(H))\Big[q^2 + 7q - M_1(H) - 6(q+p-1)^2\Big] + (q+1)\Big[2(q+1)\Big\{M_1(H) - q(q+1)\Big\} + 4\overline{EM_1}(H)\Big] + (5q+5p-6)\Big[q(q+3)^2 - 2(q+1)M_1(H) + 2M_2(H) + F(H)\Big] - 4\Big(HM_1(H) - M_2(H)\Big) - 2\Big(F(H) + \overline{EM_2}(H)\Big) - \overline{RF}(H) + 2\Big[(q+p)(q+p-1)^3 - q(q+3)^2\Big]\Big]$$

(8)

$$\overline{HM_1}(T_{+--}(H)) = \Big[q^2 + 2qp - q - M_1(H)\Big]^2 + (q+p-2)\Big[q\{(qp+1) + (q+p)(q+p-2)\} - 2(q+p-1)M_1(H) + 2M_2(H) + F(H)\Big] - \Big[\Big(M_1(H) + p(p-1)\Big)\Big[M_1(H) + p(p-1) - 6(q+p-1)^2\Big] + (5q+5p-6)\Big[p(p-1)^2 + 2M_2(H) + F(H)\Big] + 2(p-1)^2\Big[p(3-p) - q\Big] - RF(H) - 2\Big(EM_2(H) + F(H)\Big) - 4\Big[M_2(H) + 2EM_1(H) + (p+1)M_1(H)\Big] + 2\Big((q+p)(q+p-1)^3 + 8q\Big)\Big]$$

Proof. The proof follows from the notion of edge cardinality cases as mentioned in lemma (1). The first Zagreb index of the transformation graphs have been determined in [15]. From [12], we have

$$\overline{HM_1}(H) = 4q^2 + (p-2)M_1(H) - HM_1(H)$$

Also, we utilise the theorem (1), (2), (3), (4) results to get the desired output.

2.2. Second Hyper Zagreb Index of Transformation Graphs

Theorem 7. *Let H be a graph with size q and order p. Then;*

$$HM_2(T_{+++}(H)) = 24HM_2(H) + K^2(H) + 4\Big(4F(H) + RF(H)\Big) + 8M_1(H) + 16\Big(M_1^4(H) + EM_1(H) + EM_2(H)\Big) + 4\Big(2EZ_{3,1}(H) + 5EZ_{2,1}(H)\Big) + 4\Big(M_1^5(H) - 4q\Big)$$

Proof. The vertex-degree behaviour in the graph $T_{+++}(H)$ is :

$$d_{T_{+++}(H)}(v) = \begin{cases} 2d_H(v) & , \text{ if } v \in V(H) \\ d_H(v) + 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$\begin{aligned} HM_2(T_{+++}(H)) &= \sum_{vw \in E(T_{+++}(H))} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2 \\ &= \sum_{vw \in E(T_{+++}(H)) \cap E(H)} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2 \\ &+ \sum_{vw \in E(T_{+++}(H)) \cap (E(L(H))} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2 \end{aligned}$$

$$+ \sum_{vw \in E(T_{+++}(H)) \setminus (E(H) \cup E(L(H)))} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2$$
$$= f_1 + f_2 + f_3$$

For the computation of f_1,

$$f_1 = \sum_{vw \in E(T_{+++}(H)) \cap E(H)} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2 = \sum_{vw \in E(H)} \Big(2d_H(v).2d_H(w)\Big)^2$$
$$\Longrightarrow f_1 = 16HM_2(H)$$

In order to compute f_2,

$$f_2 = \sum_{vw \in E(T_{+++}(H)) \cap (E(L(H))} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2$$
$$= \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big[(d_H(v)+2).(d_H(w)+2)\Big]^2$$
$$\Longrightarrow f_2 = K^2(H) + 4\Big(RF(H) + EZ_{2,1}(H) + 2M_1(H)\Big) + 16\Big(EM_1(H)$$
$$+ EM_2(H) - q\Big)$$

Also, for the computation of f_3,

$$f_3 = \sum_{vw \in E(T_{+++}(H)) \setminus (E(H) \cup E(L(H)))} \Big(d_{T_{+++}(H)}(v).d_{T_{+++}(H)}(w)\Big)^2$$
$$= \sum_{\substack{v \in V(H) \\ w \in E(H) \\ v \sim w}} \Big[2d_H(v).(d_H(w)+2)\Big]^2 = \sum_{v \in V(H)} \sum_{x \in N_H(v)} \Big[2d_H(v).(d_H(v) + d_H(x) + 2)\Big]^2$$
$$\Longrightarrow f_3 = 4\Big[M_1^5(H) + 2HM_2(H) + 4F(H) + 2EZ_{3,1}(H) + 4M_1^4(H) + 4EZ_{2,1}(H)\Big]$$

Hence from all the computations,

$$HM_2(T_{+++}(H)) = 24HM_2(H) + K^2(H) + 4\Big(4F(H) + RF(H)\Big) + 8M_1(H) + 16$$
$$\Big(M_1^4(H) + EM_1(H) + EM_2(H)\Big) + 4\Big(2EZ_{3,1}(H) + 5EZ_{2,1}(H)\Big) + 4\Big(M_1^5(H) - 4q\Big)$$

This concludes the proof.

Theorem 8. *Let H be a graph with size q and order p. Then;*

$$HM_2(T_{++-}(H)) = (p-2)^2\Big[RF(H) + 4EM_2(H) + 2(p-2)EM_1(H) + 2q^2M_1(H) - 4q^3 + q^3(p-2) + (p-2)^2\Big(\frac{M_1(H)}{2} - q\Big)\Big] + K^2(H) + (p-2)\Big[2EZ_{2,1}(H) + q^2EM_1(H)\Big] + q^5$$

Proof. The vertex-degree behaviour in $T_{++-}(H)$:

$$d_{T_{++-}(H)}(v) = \begin{cases} q & , \text{ if } v \in V(H) \\ d_H(v) + p - 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$HM_2(T_{++-}(H)) = \sum_{vw \in E(T_{++-}(H))} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2$$

$$= \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2 + \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2 + \sum_{\substack{v \in V(H), w \in E(H) \\ v \nsim w}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2$$

$$= f_1 + f_2 + f_3$$

Now for the computation of f_1,

$$f_1 = \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2 = \sum_{\substack{vw \in E(H) \\ v,w \in V(H)}} (q.q)^2$$

$$\implies f_1 = q^5$$

In order to calculate f_2,

$$f_2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2$$

$$= \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big((d_H(v) + p - 2).(d_H(w) + p - 2)\Big)^2$$

$$\implies f_2 = K^2(H) + (p-2)^2\Big(RF(H) + 2EM_2(H)\Big) + (p-2)^4\Big(\frac{M_1(H)}{2} - q\Big) + 2(p-2)^2EM_2(H) + 2(p-2)^3EM_1(H) + 2(p-2)EZ_{2,1}(H)$$

Also, for the calculation of f_3,

$$f_3 = \sum_{\substack{v\in V(H), w\in E(H)\\ v\nsim w}} \Big(d_{T_{++-}(H)}(v).d_{T_{++-}(H)}(w)\Big)^2 = \sum_{w\in E(H)} \sum_{\substack{v\in V(H)\\ v\nsim w}} \Big(m.(d_H(w)+p-2)\Big)^2$$

$$\implies f_3 = q^2(p-2)\Big[EM_1(H) + q(p-2)^2 + 2(p-2)(M_1(H)-2q)\Big]$$

Hence from all the computations,

$$HM_2(T_{++-}(H)) = (p-2)^2\Big[RF(H) + 4EM_2(H) + 2(p-2)EM_1(H) + 2q^2M_1(H) - 4q^3 + q^3(p-2) + (p-2)^2\Big(\frac{M_1(H)}{2} - q\Big)\Big] + K^2(H) + (p-2)\Big[2EZ_{2,1}(H) + q^2EM_1(H)\Big] + q^5$$

This concludes the proof.

Theorem 9. *Let H be a graph with size q and order p. Then;*

$$HM_2(T_{+-+}(H)) = 24HM_2(H) + \overline{K}^2(H) + 4M_1^5(H) + (q+1)^2\Big[\overline{RF}(H) + 4\overline{EM_2}(H) + 2(q+1)\overline{EM_1}(H)\Big] + 4(q+3)\Big[(q+3)F(H) - 2(M_1^4(H) + EZ_{2,1}(H))\Big] + \frac{(q+1)^4}{2}\Big[q(q+1) - M_1(H)\Big] + 2\Big[4EZ_{3,1}(H) - (q+1)\overline{EZ_{2,1}}(H)\Big]$$

Proof. The vertex-degree behaviour in the graph $T_{+-+}(H)$ is :

$$d_{T_{+-+}(H)}(v) = \begin{cases} 2d_H(v) & , \text{ if } v \in V(H) \\ q+1-d_H(v) & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$HM_2(T_{+-+}(H)) = \sum_{vw\in E(T_{+-+}(H))} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2$$

$$= \sum_{vw\in E(H)} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2 + \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2 + \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2$$

$$= f_1 + f_2 + f_3$$

For the computation of f_1,

$$f_1 = \sum_{vw\in E(H)} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2 = \sum_{vw\in E(H)} \Big(2d_H(v).2d_H(w)\Big)^2$$
$$\Longrightarrow f_1 = 16HM_2(H)$$

In order to compute f_2,

$$\begin{aligned} f_2 &= \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(w)\Big)^2 \\ &= \sum_{\substack{v,w\in E(H)\\ v\nsim w}} \Big[(q+1-d_H(v)).(q+1-d_H(w))\Big]^2 \end{aligned}$$
$$\Longrightarrow f_2 = \frac{q(q+1)^3(q^2-1)}{2} - \frac{(q+1)^4}{2}M_1(H) + q(q+1)^4 + (q+1)^2\Big[\overline{RF}(H) + 4\overline{EM_2}(H) + 2(q+1)\overline{EM_1}(H)\Big] + \overline{K}^2(H) - 2(q+1)\overline{EZ_{2,1}}(H)$$

Also, for the computation of f_3,

$$\begin{aligned} f_3 &= \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_{T_{+-+}(H)}(v).d_{T_{+-+}(H)}(v)\Big)^2 = \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big[2d_H(v).(q+1-d_H(w))\Big]^2 \\ &= \sum_{v\in V(H)}\sum_{x\in N_H(v)} \Big[2d_H(v).(q+1-d_H(v)-d_H(x)+2)\Big]^2 \end{aligned}$$

$$\Longrightarrow f_3 = 4M_1^5(H) + 8HM_2(H) + 8EZ_{3,1}(H) + 4(q+3)\Big((q+3)F(H) - 2M_1^4(H) - 2EZ_{2,1}(H)\Big)$$

Hence from all the computations,

$$\begin{aligned} HM_2(T_{+-+}(H)) &= 24HM_2(H) + \overline{K}^2(H) + 4M_1^5(H) + (q+1)^2\Big[\overline{RF}(H) + 4\overline{EM_2}(H) \\ &+ 2(q+1)\overline{EM_1}(H)\Big] + 4(q+3)\Big[(q+3)F(H) - 2(M_1^4(H) + EZ_{2,1}(H))\Big] \\ &+ \frac{(q+1)^4}{2}\Big[q(q+1) - M_1(H)\Big] + 2\Big[4EZ_{3,1}(H) - (q+1)\overline{EZ_{2,1}}(H)\Big] \end{aligned}$$

This concludes the proof.

Theorem 10. *Let H be a graph with size q and order p. Then;*

$$HM_2(T_{-++}(H)) = 8\Big[M_1(H) + 2(EM_1(H) - q)\Big] + 4\Big[RF(H) + EZ_{2,1}(H) + 4EM_2(H)\Big] + K^2(H) + (p-1)^2\Big[2(F(H) + 2M_2(H)) + \frac{p(p-1)^2(p-3)}{2}\Big]$$

Proof. The vertex-degree behaviour in the graph $T_{-++}(H)$ is :

$$d_{T_{-++}(H)}(v) = \begin{cases} p-1 & , \text{ if } v \in V(H) \\ d_H(v) + 2 & , \text{ if } v \in E(H) \end{cases}$$

Therefore;

$$HM_2(T_{-++}(H)) = \sum_{vw \in E(T_{-++}(H))} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2$$
$$= \sum_{vw \notin E(H)} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2 + \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2$$
$$+ \sum_{\substack{v \in V(H) \\ w \in E(H) \\ v \sim w}} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2$$
$$= f_1 + f_2 + f_3$$

For the computation of f_1,

$$f_1 = \sum_{vw \notin E(H)} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2 = \sum_{vw \notin E(H)} \Big[(p-1).(p-1)\Big]^2$$
$$\Longrightarrow f_1 = \frac{1}{2}p(p-3)(p-1)^4$$

In order to compute f_2,

$$f_2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2 = \sum_{\substack{v,w \in E(H) \\ v \sim w}} \Big[(d_H(v)+2)(d_H(w)+2)\Big]^2$$
$$\Longrightarrow f_2 = K^2(H) + 4\Big[(RF(H) + EZ_{2,1}(H)) + 4EM_2(H)\Big] + 8\Big[M_1(H) + 2(EM_1(H) - q)\Big]$$

Also, to compute f_3,

$$f_3 = \sum_{\substack{v\in V(H)\\ w\in E(H)\\ v\sim w}} \Big(d_{T_{-++}(H)}(v).d_{T_{-++}(H)}(w)\Big)^2 = \sum_{v\in V(H)} \sum_{\substack{w\in E(H)\\ v\sim w}} \Big[(p-1)(d_H(w)+2)\Big]^2$$
$$= (p-1)^2 \sum_{v\in V(H)} \sum_{x\in N_H(v)} (d_H(v)+d_H(x))^2$$
$$\implies f_3 = 2(p-1)^2\Big[F(H)+2M_2(H)\Big]$$

Hence from all the computations,

$$HM_2(T_{-++}(H)) = 8\Big[M_1(H)+2(EM_1(H)-q)\Big]+4\Big[RF(H)+EZ_{2,1}(H) + 4EM_2(H)\Big]+K^2(H)+(p-1)^2\Big[2(F(H)+2M_2(H))+\frac{p(p-1)^2(p-3)}{2}\Big]$$

This concludes the proof.

Theorem 11. *Let H be a graph with size q and order p. Then;*

(1)

$$HM_2(T_{---}(H)) = M_1(H)\Big\{8(q+p-1)\Big[\frac{1}{16}(q+p-1)^3-(q+p-1)^2+(q+p-1) - (8F(H)+\chi_3(H))\Big]+8\Big\} - 4(q+p-1)M_2(H)\Big[(q+p-1)^2-8(q+p-1) + (8F(H)+\chi_3(H))\Big] - \Big\{2(q+p-1)\Big[(q+p-1)^2-8(q+p-1)+8F(H) + \chi_3(H)\Big]-16\Big\}F(H)+(q+p-1)\Big[(q+p-1)-2(4M_1(H)+2M_2(H)+F(H))\Big] \chi_3(H)+4EM_2(H)\Big[(q+p-1)^2+4\Big]+8EM_1(H)\Big[(q+p-1)^2+2\Big]+K^2(H) + 4\Big[M_1^5(H)+RF(H)+2EZ_{3,1}(H)+4M_1^4(H)+5EZ_{2,1}(H)+6HM_2(H)\Big] + 2q\Big[(q+p-1)^2((q+p-1)^2-8)-8\Big]$$

(2)

$$HM_2(T_{--+}(H)) = M_1(H)\Big\{(q+p-1)^2\Big[\frac{1}{2}(q+p-1)^2 - 4(p-4)(q+p-1)$$
$$+ 3(p-4)^2 + 2(p-2)(2q+p-2)\Big] + (p-2)^2\Big[2q^2 + \frac{(p-2)^2}{2}\Big]\Big\} + (q+p-1)^2$$
$$\Big[3(p-4) - 2(q+p-1)\Big](2M_2(H) + F(H)) + (q+p-1)^2\chi_3(H) + (p-2)$$
$$\Big[4(q+p-1)^2 + 2(p-2)^2 + q^2\Big]EM_1(H) + 4EM_2(H)\Big[(q+p-1)^2 + (p-2)^2\Big]$$
$$+ (p-2)^2 RF(H) + K^2(H) + 2(p-2)EZ_{2,1}(H) + q(q+p-1)^2\Big[(p-2)$$
$$(q+p-1)^2 - 2p(q+p-8)(q+p-1) - 32(q+p-1) + (p-4)^3 + q^2p + 4(q^2$$
$$+ (p-2)\{q(p-4) - (p-2)\})\Big] + q\Big\{(p-2)^2[q^2(p-2) - 4q^2 - (p-2)^2] + q^4\Big\}$$
$$- 2(q+p-1)\Big[qp(q+p-8) + 16q + 2(p-4)M_1(H) + 2M_2(H) + F(H)\Big]$$
$$\Big[\chi_3(H) + (3p-12)\{F(H) + 2M_2(H)\} + 3(p-4)^2M_1(H) + q(p-4)^3 + q^3p\Big]$$

(3)

$$HM_2(T_{-+-}(H)) = \frac{1}{2}(q^2 + 7q - M_1(H))(q+p-1)^4 - 2(q+p-1)^3\Big[q(q+3)^2$$
$$- 2(q+1)M_1(H) + 2M_2(H) + F(H)\Big] + (q+p-1)^2\Big[(3q+17)F(H) - \chi_3(H)$$
$$+ q(q+3)^2 - 3(q+3)^2M_1(H) + 6(q+3)M_2(H)\Big] + 4(q+p-1)^2\Big[\overline{EM_2}(H)$$
$$- (q+1)\overline{EM_1}(H) + \frac{1}{2}\{q(q+1)^3\} - \frac{1}{2}\{(q^2 - 2q - 11)M_1(H)\} - 2F(H)\Big]$$
$$- 2(q+p-1)\Big\{q(q+3)^2 - 2(q+1)M_1(H) + 2M_2(H) + F(H)\Big\}\Big\{(3q+17)F(H)$$
$$- \chi_3(H) + q(q+3)^3 - 3(q+3)^2M_1(H) + 6(q+3)M_2(H)\Big\} + 24HM_2(H) + \overline{K}^2(H)$$
$$+ 4M_1^5(H) + (q+1)^2\Big[\overline{RF}(H) + 4\overline{EM_2}(H) + 2(q+1)\overline{EM_1}(H)\Big] + 4(q+3)\Big[(q+3)$$
$$F(H) - 2(M_1^4(H) + EZ_{2,1}(H))\Big] + \frac{(q+1)^4}{2}\Big[q(q+1) - M_1(H)\Big] + 2\Big[4EZ_{3,1}(H)$$
$$- (q+1)\overline{EZ_{2,1}}(H)\Big]$$

(4)

$$HM_2(T_{+--}(H)) = \frac{1}{2}(q+p-1)^4\Big(M_1(H)+p(p-1)\Big) - 2(q+p-1)^3\Big[p(p-1)^2 + 2M_2(H)+F(H)\Big] + (q+p-1)^2\Big[p(p-1)^3+\chi_3(H)\Big] + 4(q+p-1)^2\Big[EM_2(H) + 2EM_1(H)+2pM_1(H)+\frac{1}{2}p(p-1)^3 - q(p^2-2p+5)\Big] - 2(q+p-1)\Big\{p(p-1)^2 + 2M_2(H)+F(H)\Big\}\Big\{p(p-1)^3+\chi_3(H)\Big\} + (p-1)^2\Big[\frac{1}{2}p(p-3)(p-1)^2 + 2\{F(H)+2M_2(H)\}\Big] + K^2(H) + 4\Big[RF(H)+EZ_{2,1}(H)+4EM_2(H)\Big] + 8\Big[M_1(H)+2(EM_1(H)-q)\Big] + (p-1)^2[2F(H)+4M_2(H)]$$

Proof. The proof follows from the notion $\overline{T_{-++}(H)} \cong T_{+--}(H)$, $\overline{T_{+-+}(H)} \cong T_{-+-}(H)$, $\overline{T_{++-}(H)} \cong T_{--+}(H)$, and $\overline{T_{+++}(H)} \cong T_{---}(H)$ and edge cardinality cases as mentioned in lemma (1). The first Zagreb index, the second Zagreb index and the forgotten index of the transformation graphs have been determined in [15, 27]. We have

$$\begin{aligned} HM_2(\overline{H}) &= q(p-1)^4 - 2(p-1)^3M_1(H) + (p-1)^2F(H) + 4(p-1)^2M_2(H) \\ &\quad - 2(p-1)M_1(H)F(H) + HM_2(H) \end{aligned}$$

Also, we utilise the theorems (7), (8), (9), (10), from which we get the desired result.

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