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ON CERTAIN RESULTS INVOLVING SQUARE OF RAMANUJAN'S MOCK THETA FUNCTIONS

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Abstract: In this paper, making use of an identity deduced from Bailey's transform, certain results have been established involving the square of Ramanujan's mock theta functions.

Keywords and Phrases: Bailey's transform, identity, mock theta function, partial mock theta function.

2020 Mathematics Subject Classification: 33C20, 33D45.

1. Introduction, Notations and Definitions

For $|q| < 1$, the $q-$shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)...(1-aq^{n-1}), & n \in N. \end{cases}$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{r=0}^{\infty}(1-aq^r).$$

Also, if $A = \sum_{n=0}^{\infty} B_n$ is a mock theta function then $B_m = \sum_{n=0}^{m} B_n$ is called partial mock theta function. For the definitions of mock theta functions of order three, five and seven one is refereed chapters 2 and 3 of the 'Resonance of Ramanujan's Mathematics, Vol. II', due to Agarwal R. P. [1] and also one can refers the some results established on mock theta functions in [2, 4, 5, 6, 7, 8] .

In 1949 Bailey [3] established following simple but very useful transform, viz., If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{1.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.3}$$

where u_r, v_r, α_r and δ_r are arbitrary functions of r alone.
Choosing $u_r = v_r = 1$ in above Bailey's transform, it takes the form,

If $\beta_n = \sum_{r=0}^{n} \alpha_r$ and $\gamma_n = \sum_{r=n}^{\infty} \delta_r$ then under suitable convergence conditions we have

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.4}$$

By simple manipulation (1.4) can be expressed as

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r$$

$$\text{or} \quad \sum_{n=0}^{\infty} \alpha_n \left\{ \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n \right\} = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r$$

$$\text{or} \quad \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r + \sum_{n=0}^{\infty} \alpha_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r \tag{1.5}$$

Taking $\delta_r = \alpha_r$ in (1.5) we get the identity,

$$\left(\sum_{n=0}^{\infty} \alpha_n \right)^2 + \sum_{n=0}^{\infty} (\alpha_n)^2 = 2 \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \alpha_r. \tag{1.6}$$

In next section we shall establish main results by making use of the identity (1.6) and Ramanujan's mock theta functions.

2. Main Results

Here we shall establish results for mock theta functions of order three.

(a) Putting $\frac{q^{n^2}}{(-q;q)_n^2}$ for α_n in (1.6) we get

$$f^2(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q)_n^4}=2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}f_n(q). \tag{2.1}$$

(b) Taking $\alpha_n=\frac{q^{n^2}}{(-q^2;q^2)_n}$ in (1.6) we have

$$\Phi^2(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q^2;q^2)_n^2}=2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}\Phi_n(q). \tag{2.2}$$

(c) For $\alpha_n=\frac{q^{n^2}}{(q;q^2)_n}$, (1.6) yields,

$$\Psi^2(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n^2}=2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}\Psi_n(q). \tag{2.3}$$

(d) Taking $\alpha_n=\frac{q^{n^2}}{(-\omega q,-\omega^2 q;q)_n}$, (where ω is the cube root of the unity) in (1.6) we get,

$$\chi^2(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-\omega q,-\omega^2 q;q)_n^2}=2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-\omega q,-\omega^2 q;q)_n}\chi_n(q). \tag{2.4}$$

(e) Choosing $\alpha_n=\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}$ in (1.6) we obtain,

$$\omega^2(q)+\sum_{n=0}^{\infty}\frac{q^{4n(n+1)}}{(q;q^2)_{n+1}^4}=2\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}\omega_n(q). \tag{2.5}$$

(f) Taking $\alpha_n=\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$ in (1.6) we get,

$$\nu^2(q)+\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(-q;q^2)_{n+1}^2}=2\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}\nu_n(q). \tag{2.6}$$

(g) For $\alpha_n = \frac{q^{2n(n+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}}$ (1.6) gives,

$$\rho^2(q) + \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}^2} = 2\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(\omega q, \omega^2 q; q^2)_{n+1}} \rho_n(q). \tag{2.7}$$

3. Mock Theta Functions of order Five

In this section we shall use the identity (1.6) in order to establish results involving mock theta functions of order five. See chapter 3, page 92 of the book "Resonance of Ramanujan mathematics, Volume II," due to Agarwal R. P. [1]

(a) Putting $\alpha_n = \frac{q^{n^2}}{(-q;q)_n}$ in (1.6) we find

$$f_0^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_n^2} = 2\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} f_{0,n}(q). \tag{3.1}$$

(b) Putting $\alpha_n = q^{(n+1)(n+2)/2}(-q;q)_n$ in (1.6) we get,

$$\Psi_0^2(q) + \sum_{n=0}^{\infty} q^{(n+1)(n+2)}(-q;q)_n^2 = 2\sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}(-q;q)_n \Psi_{0,n}(q). \tag{3.2}$$

(c) Taking $\alpha_n = \frac{q^n}{(q^{n+1};q)_n}$ in (1.6) we obtain,

$$\chi_0^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^{n+1};q)_n^2} = 2\sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} \chi_{0,n}(q). \tag{3.3}$$

(d) For $\alpha_n = q^{n^2}(-q;q^2)_n$, (1.6) yields,

$$\Phi_0^2(q) + \sum_{n=0}^{\infty} q^{2n^2}(-q;q^2)_n^2 = 2\sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n \Phi_{0,n}(q). \tag{3.4}$$

(e) Choosing $\alpha_n = q^{2n^2}(q;q^2)_n$ in (1.6) we find,

$$F_0^2(q) + \sum_{n=0}^{\infty} q^{4n^2}(q;q^2)_n^2 = 2\sum_{n=0}^{\infty} q^{2n^2}(q;q^2)_n F_{0,n}(q). \tag{3.5}$$

(f) Taking $\alpha_n = \dfrac{q^{n(n+1)}}{(-q;q)_n}$ in (1.6) we get,

$$f_1^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(-q;q)_n^2} = 2\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n} f_{1,n}(q). \tag{3.6}$$

(g) For $\alpha_n = q^{(n+1)^2}(-q;q^2)_n$, (1.6) yields,

$$\Phi_1^2(q) + \sum_{n=0}^{\infty} q^{2(n+1)^2}(-q;q^2)_n^2 = 2\sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n \Phi_{1,n}(q). \tag{3.7}$$

(h) Putting $\alpha_n = q^{n(n+1)/2}(-q;q)_n$ in (1.6) we get,

$$\Psi_1^2(q) + \sum_{n=0}^{\infty} q^{n(n+1)}(-q;q)_n^2 = 2\sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n \Psi_{1,n}(q). \tag{3.8}$$

(i) Taking $\alpha_n = \dfrac{q^{2n(n+1)}}{(q;q^2)_{n+1}}$ in (1.6) we find,

$$F_1^2(q) + \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}}{(q;q^2)_{n+1}^2} = 2\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} F_{1,n}(q). \tag{3.9}$$

(j) Lastly, taking $\alpha_n = \dfrac{q^n}{(q^{n+1};q)_{n+1}}$ in (1.6) we have,

$$\chi_1^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^{n+1};q)_{n+1}^2} = 2\sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}} \chi_{1,n}(q). \tag{3.10}$$

4. Mock Theta Functions of order Seven

In this section we shall establish results involving mock theta functions of order seven. For mock theta functions of order seven see Agarwal R. P. [1 ; chapter 3, page 125].

(a) Taking $\alpha_n = \dfrac{q^{n^2}(q;q)_n}{(q;q)_{2n}}$ in (1.6) we have,

$$\mathcal{F}_0^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q)_n^2}{(q;q)_{2n}^2} = 2\sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q;q)_{2n}} \mathcal{F}_{0,n}(q). \tag{4.1}$$

(b) For $\alpha_n = \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}}$, (1.6) yields

$$\mathcal{F}_1^2(q) + \sum_{n=0}^{\infty} \frac{q^{2(n+1)^2}}{(q^{n+1};q)_{n+1}^2} = 2\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}} \mathcal{F}_{1,n}(q). \tag{4.2}$$

(c) Putting $\alpha_n = \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}}$ in (1.6) we obtain,

$$\mathcal{F}_2^2(q) + \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^{n+1};q)_{n+1}^2} = 2\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}} \mathcal{F}_{2,n}(q). \tag{4.3}$$

References

[1] Agarwal, R. P., Resonance of Ramanujan's Mathematics, Volume II, New Age International (P) Limited, New Delhi, 1996.

[2] Ahmad Ali, S., A bilateral extension of second order mock theta functions, South East Asian Journal of Mathematics and Mathematical Sciences, Vol. 6, No. 2 (2008), 121-122.

[3] Bailey, W. N., Identities of Rogers-Ramanujan type, Proc. London Mathematical Society, Volume s2-50, Issue 1 (1948), 1–10.

[4] Chand, K. B., Pant, G. S., Pande, V. P., Product Formulas for mock theta functions, South East Asian Journal of Mathematics and Mathematical Sciences, Vol. 12, No. 1 (2016), 87-94.

[5] Denis, Remy Y., Singh, S. N. and Singh, S. P., On single series representation of mock theta functions of fifth and seventh order, Ital. J. Pure Appl. Math., No. 23 (2008), 67-74.

[6] Pant, G. S., Pande, V. P. and Mohammad, Shahjade, Transformation Formula involving partial mock-theta functions, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 5, No. 2 (2016), 99-112.

[7] Singh, Satya Prakash, Mishra, Bindu Prakash, On Certain Results Involving Mock- Theta Functions, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 1, No. 1 (2012), 7-16.

[8] Singh, Satya Prakash, A note on mock theta functions of order three and continued fractions, Proc. Nat. Acad. Sci. India Sect. A Phys. Sci., 76, No. 3 (2006), 205-207.