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FRACTIONAL CALCULUS OPERATORS OF THE GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND RELATED JACOBI TRANSFORMS

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Abstract: Our aim is to obtain certain image formulas of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\alpha,\beta,p}(z)$ by using Saigo's hypergeometric fractional integral and differential operators. Corresponding assertions for the classical Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are established. All the results are represented in terms of the Hadamard product of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\alpha,\beta,p}(z)$.

Keywords and Phrases: Fractional Calculus operators, Fox-Wright function, Generalized hypergeometric function, Extended Mittag-Leffler function, Gegenbauer and Legendre transforms.

2020 Mathematics Subject Classification: Primary 26A33, 33B20, 33C20; Secondary 26A09, 33B15, 33C05.

1. Introduction and Preliminaries

We recall Saigo's fractional integral and differential operators involving the Gauss's hypergeometric function ${}_2F_1$ as kernel. Let $\alpha,\ \beta,\ \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $x > 0$, then the Saigo's fractional integral and differential operators $\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x)$, $\left(I_{-}^{\alpha,\beta,\eta} f\right)(x)$ and $\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x)$, $\left(D_{-}^{\alpha,\beta,\eta} f\right)(x)$ are defined as (see, e.g., [12, 13, 24, 25, 30]):

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\, \mathrm{d}t, \tag{1.1}$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t \tag{1.2}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) &= \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \quad (n = [\Re(\alpha)]+1),\end{aligned} \tag{1.3}$$

$$\begin{aligned}\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) &= \left(I_{-}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) \\ &= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x) \quad (n = [\Re(\alpha)]+1),\end{aligned} \tag{1.4}$$

respectively. When $\beta = -\alpha$, (1.1), (1.2), (1.3) and (1.4) coincide with the classical Riemann-Liouville fractional integrals and derivatives of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) and $x > 0$ (see, e.g., [12, 13, 25]):

$$\left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0+}^{\alpha} f\right)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t, \tag{1.5}$$

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, \mathrm{d}t \tag{1.6}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{0+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)\, \mathrm{d}t \\ &= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha} f\right)(x) \qquad (n = [\Re(\alpha)]+1),\end{aligned} \tag{1.7}$$

$$\left(D_{-}^{\alpha,-\alpha,\eta}f\right)(x)=\left(D_{-}^{\alpha}f\right)(x)=(-1)^{n}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}(t-y)^{n-\alpha-1}f(t)\,\mathrm{d}t$$
$$=(-1)^{n}\left(\frac{d}{dx}\right)^{n}\left(I_{-}^{n-\alpha}f\right)(x)\qquad(n=[\Re(\alpha)]+1),\tag{1.8}$$

respectively, where $[\Re(\alpha)]$ is the integral part of $\Re(\alpha)$.

If $\beta=0$, (1.1), (1.2), (1.3) and (1.4) are the so-called Erdélyi-Kober fractional integrals and derivatives defined for $\alpha\in\mathbb{C}$ ($\Re(\alpha)>0$) and $x>0$ (see, e.g., [12, 13, 25]):

$$\left(I_{0+}^{\alpha,0,\eta}f\right)(x)=\left(I_{\eta,\alpha}^{+}f\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}t^{\eta}f(t)\,\mathrm{d}t,\tag{1.9}$$

$$\left(I_{-}^{\alpha,0,\eta}f\right)(x)=\left(K_{\eta,\alpha}^{-}f\right)(x)\equiv\frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)\,\mathrm{d}t\tag{1.10}$$

and

$$\left(D_{0+}^{\alpha,0,\eta}f\right)(x)=\left(D_{\eta,\alpha}^{+}f\right)(x)$$
$$=\left(\frac{d}{dx}\right)^{n}\left(I_{0+}^{-\alpha+n,-\alpha,\alpha+\eta-n}f\right)(x)\quad(n=[\Re(\alpha)]+1),\tag{1.11}$$

$$\left(D_{-}^{\alpha,0,\eta}f\right)(x)=\left(D_{\eta,\alpha}^{-}f\right)(x)$$
$$=(-1)^{n}\left(\frac{d}{dx}\right)^{n}\left(I_{-}^{-\alpha+n,-\alpha,\alpha+\eta}f\right)(x)\quad(n=[\Re(\alpha)]+1),\tag{1.12}$$

$$\left(D_{\eta,\alpha}^{+}f\right)(x)=x^{-\eta}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}t^{\alpha+\eta}(x-t)^{n-\alpha-1}f(t)\,\mathrm{d}t\quad(n=[\Re(\alpha)]+1),\tag{1.13}$$

$$\left(D_{\eta,\alpha}^{-}f\right)(x)=x^{\eta+\alpha}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}t^{-\eta}(t-x)^{n-\alpha-1}f(t)\,\mathrm{d}t\quad(n=[\Re(\alpha)]+1),\tag{1.14}$$

respectively. In recent years, extensions of a number of well-known special functions have been investigated and studied the (p,q)–variant, and in turn, when $p=q$ the p–variant together with the set of related higher transcendental hypergeometric type special functions (see, for details, [1, 2, 3, 4, 7, 8, 11, 14, 15, 17, 19, 21, 27]). In particular, Choi *et al.* [9, p. 201, Eq. (2.1)] introduced and studied the p-extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ in the form:

$$\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma;p)_n}{\Gamma(\lambda n+\mu)} \frac{z^n}{n!} \tag{1.15}$$

$$(z,\, \mu,\, \gamma \in \mathbb{C};\ \Re(\lambda) > 0,\ \Re(\mu) > 0;\ \Re(p) \geq 0),$$

provided that the series on the right-hand side converges.

Clearly, when $p = 0$ in (1.15) yields the generalized Mittag-Leffler function introduced by Prabhakar [22]

$$\mathcal{E}^{\gamma}_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n+\mu)} \frac{z^n}{n!} \tag{1.16}$$

when $(\Re(p) > 0$. They developed and studied its certain basic properties, Mellin transform, Euler-Beta transform, Laplace transform and Whittaker transform, and so on. The concept of the Hadamard product (or convolution) of two analytic functions is required in our current investigation. It can aid in the decomposition of a newly emerged function into two known functions. If one of the power series, in particular, describes an entire function, then the Hadamard product series also defines an entire function. If we assume

$$g(z) := \sum_{n=0}^{\infty} c_n z^n \ (|z| < R_f) \ \text{ and } \ h(z) := \sum_{n=0}^{\infty} d_n z^n \ (|z| < R_g)$$

two given power series and whose radii of convergence are given by R_f and R_g, respectively. Then their Hadamard product(or convolution) is the power series defined by(see also, [26])

$$(g \,*\, h)(z) := \sum_{n=0}^{\infty} c_n\, d_n z^n = (h \,*\, g)(z) \ (|z| < R) \tag{1.17}$$

where

$$R = \lim_{n\to\infty} \left| \frac{c_n\, d_n}{c_{n+1}\, d_{n+1}} \right| = \left(\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \right) \cdot \left(\lim_{n\to\infty} \left| \frac{d_n}{d_{n+1}} \right| \right) = R_f \,.R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

The Fox-Wright function${}_r\Psi_s(z)$ $(r,\ s\in\mathbb{N}_0)$, which is a generalization of hypergeometric function, is defined as follows (see, for details, [12, 16]; see also [25, 29]):

$$
{}_r\Psi_s\left[\begin{array}{c}(a_1,A_1),\cdots,(a_s,A_s);\\(b_1,B_1),\cdots,(b_s,B_s);\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+A_1n)\cdots\Gamma(a_r+A_rn)}{\Gamma(b_1+B_1n)\cdots\Gamma(b_s+B_sn)}\,\frac{z^n}{n!} \tag{1.18}
$$

$$
\left(A_\ell\in\mathbb{R}^+\ (\ell=1,\dots,r);\ B_\ell\in\mathbb{R}^+\ (\ell=1,\dots,s);\ 1+\sum_{\ell=1}^{s}B_\ell-\sum_{\ell=1}^{r}A_\ell\geqq 0\right),
$$

where the equality in the convergence condition holds true for

$$
|z|<\nabla:=\left(\prod_{\ell=1}^{r}A_\ell^{-A_\ell}\right)\ .\ \left(\prod_{\ell=1}^{s}B_j^{B_\ell}\right).
$$

The Fox–Wright function extends the generalized hypergeometric function ${}_pF_q[z]$ which power series form reads

$$
{}_rF_s\left[\begin{array}{c}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{array}\Big|z\right]=\sum_{k\geq 0}\frac{\prod\limits_{l=1}^{r}(a_l)_k}{\prod\limits_{l=1}^{s}(b_l)_k}\frac{z^k}{k!}, \tag{1.19}
$$

where, as usual, we make use of the Pochhammer symbol (or raising factorial)

$$
(\tau)_0=1;\quad (\tau)_k=\tau(\tau+1)\cdots(\tau+k-1)=\frac{\Gamma(\tau+k)}{\Gamma(\tau)},\qquad k\in\mathbb{N}.
$$

In the special case $A_i=B_j=1; i=1,\cdots,r; j=1,\cdots,s$, the Fox–Wright function ${}_r\Psi_s[z]$ reduces (up to the multiplicative constant) to the generalized hypergeometric function

$$
{}_r\Psi_s\left[\begin{array}{c}(a_1,1),\cdots,(a_r,1)\\(b_1,1),\cdots,(b_s,1)\end{array}\Big|z\right]=\frac{\Gamma(a_1)\cdots\Gamma(a_r)}{\Gamma(b_1)\cdots\Gamma(b_s)}\,{}_rF_s\left[\begin{array}{c}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{array}\Big|z\right].
$$

The H-function is defined as the Mellin-Barnes type path integral (see, for details, [16]):

$$
\begin{aligned}
H_{u,\,v}^{m,\,n}(z)=H_{u,\,v}^{m,\,n}\left[z\,\middle|\,\begin{array}{c}(a_u,A_u)\\(b_v,B_v)\end{array}\right]&=H_{u,\,v}^{m,\,n}\left[z\,\middle|\,\begin{array}{c}(a_1,A_1),\cdots,(a_u,A_u)\\(b_1,B_1),\cdots,(b_v,B_v)\end{array}\right]\\
&=\frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(s)\,z^{-s}\,ds
\end{aligned} \tag{1.20}
$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{v} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{u} \Gamma(a_j + A_j s)} \tag{1.21}$$

and $\mathcal{L}$ is a suitable contour of the Mellin-Barnes type separating the poles of $\Gamma(b_j + B_j s)$ $(j = 1, \cdots, m)$ from those of $\Gamma(1-a_j-A_j s)$ $(j = 1, \cdots, n)$. An empty product is interpreted as 1, the integers $m,\ n,\ u,\ v$ satisfy the inequalities $0 \le m \le\ v$ and $0 \le n \le\ u$, the coefficient A_j $(j = 1, \cdots, u)$ and B_j $(j = 1, \cdots, v)$ are positive real numbers, and the complex parameters a_j $(j = 1, \cdots, u)$ and b_j $(j = 1, \cdots, v)$ are so constrained that no poles of the integrand coincide. Also, the Mellin-Barnes contour integral representing in the H-function converges absolutely and defines an analytic function for $|\arg(z)| < \frac{\pi}{2}\Omega$, where

$$\Omega = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0.$$

In this and other instances, the sets of positive integers, integers, real numbers, and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively.

In this paper, we obtain certain image formulas of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ by using Saigo's hypergeometric fractional calculus (integral and differential) operators (1.1), (1.2), (1.3) and (1.4). Corresponding assertions for the classical Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$.

2. Fractional integration of the $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$

We begin the main results exposition with presenting a composition formulas of generalized fractional integrals (1.1) and (1.2) involving p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.17) of p–extended Mittag-Leffler function (1.15) and Fox-Wright function ${}_r\Psi_s(z)$ (1.18).

Lemma 1. *Let $\alpha,\ \beta,\ \eta \in \mathbb{C}$. Then there exists the relation*

(a) *If* $\Re(\alpha) > 0$ *and* $\Re(\sigma) > \max[0, \Re(\beta - \eta)]$, *then*

$$(I_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1} \tag{2.1}$$

In particular, for $x > 0$ *we have*

$$(I_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)} x^{\sigma+\alpha-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > 0), \tag{2.2}$$

$$(I_{\eta,\alpha}^{+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)} x^{\sigma-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\eta)). \tag{2.3}$$

(b) *If* $\Re(\alpha) > 0$ *and* $\Re(\sigma) < 1 + \min[\Re(\beta), \Re(\eta)]$, *then*

$$(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)} x^{\sigma-\beta-1}. \tag{2.4}$$

In particular, for $x > 0$ *we have*

$$(I_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)} x^{\sigma+\alpha-1} \qquad (0 < \Re(\alpha) < 1 - \Re(\sigma)), \tag{2.5}$$

$$(K_{\eta,\alpha}^{-} t^{\sigma-1})(x) = \frac{\Gamma(\eta-\sigma+1)}{\Gamma(\alpha+\eta-\sigma+1)} x^{\sigma-1} \qquad (\Re(\sigma) < 1 + \Re(\sigma)). \tag{2.6}$$

Theorem 1. *Let* $\rho, \alpha, \beta, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$ *be such that* $\Re(p) > 0$, $\Re(\alpha) > 0$ *and* $\Re(\sigma) > \max[0, \Re(\beta - \eta)]$. *Then the following Saigo hypergeometric fractional integral* $I_{0+}^{\alpha,\beta,\eta}$ *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})$ *holds true:*

$$\left(I_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x) = x^{\sigma-\beta-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega x^{\rho}) * {}_3\Psi_2\left[\begin{matrix}(1,1),(\sigma,\rho),(\sigma+\eta-\beta,\rho);\\(\sigma-\beta,\rho),(\sigma+\alpha+\eta,\rho);\end{matrix}\ \omega x^{\rho}\right], \tag{2.7}$$

where it is assumed that the left-sided hypergeometric fractional integral in (2.7) *exists.*

Proof. Applying definition (1.15), using (1.1) and (2.1) and changing the orders of integration and summation, we find for $x > 0$

$$\left(I_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x)$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)} \frac{\omega^k}{k!} \left(I_{0+}^{\alpha,\beta,\eta} t^{\sigma+\rho k-1}\right)(x)$$

$$= x^{\sigma-\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)\,k!} \frac{\Gamma(1+k)\Gamma(\sigma+\rho k)\Gamma(\sigma+\eta-\beta+\rho k)}{\Gamma(\sigma+\alpha+\eta+\rho k)\Gamma(\sigma-\beta+\rho k)\,k!} (\omega x^{\rho})^k. \tag{2.8}$$

By applying the Hadamard product (1.17) in (2.8), which in view of (1.15) and (1.18), yields the desired formula (2.7).

Theorem 2. *Let* $\rho, \alpha, \beta, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$ *be such that* $\Re(p) > 0$, $\Re(\alpha) > 0$ *and* $\Re(\sigma) < 1+\min[\Re(\beta), \Re(\eta)]$. *Then the following Saigo hypergeometric fractional integral* $I_{-}^{\alpha,\beta,\eta}$ *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)$ *holds true:*

$$\left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x) = x^{\sigma-\beta-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{x^{\rho}}\right) * {}_{3}\Psi_{2}\left[\begin{array}{l}(1,1),(1+\beta-\sigma,\rho),(1+\eta-\sigma,\rho);\\(1-\sigma,\rho),(1+\alpha+\beta+\eta-\sigma,\rho);\end{array}\ \frac{\omega}{x^{\rho}}\right], \tag{2.9}$$

where it is assumed that the right-sided hypergeometric fractional integral in (2.9) *exists.*

Proof. Applying definition (1.15), using (1.2) and (2.4) and changing the orders of integration and summation, we find for $x > 0$

$$\begin{aligned}&\left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)\\&=\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)\,k!}(\omega)^k\ \left(I_{-}^{\alpha,\beta,\eta}t^{\sigma-k-1}\right)(x)\\&=x^{\sigma-\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)\,k!}\,\frac{\Gamma(1+k)\Gamma(1+\beta-\sigma+\rho k)\Gamma(1+\eta-\sigma+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1+\alpha+\beta+\eta-\sigma+\rho k)\,k!}\left(\frac{\omega}{x^{\rho}}\right)^k.\end{aligned} \tag{2.10}$$

By applying the Hadamard product (1.17) in (2.10), which in view of (1.15) and (1.18), yields the desired formula (2.9).

Corollary 2.1. *Let* $\rho, \alpha, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$, *be such that* $\Re(p) > 0$, $,\Re(\alpha) > 0, \Re(\sigma) > 0$. *Then the following Riemann-Liouville fractional integral* I_{0+}^{α} *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})$ *holds true:*

$$\left(I_{0+}^{\alpha}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x) = x^{\sigma+\alpha-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega x^{\rho}) * {}_{2}\Psi_{1}\left[\begin{array}{l}(1,1),(\sigma,\rho);\\(\sigma+\alpha,\rho);\end{array}\ \omega x^{\rho}\right], \tag{2.11}$$

where it is assumed that the left-sided Riemann-Liouville fractional integral in (2.11) *exists.*

Corollary 2.2. *Let* $\rho, \alpha, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$, *be such that* $\Re(p) > 0$, $\Re(\alpha) > 0$

and $\Re(\sigma) > -\Re(\eta)$. *Then the following Erdélyi-Kober fractional integral* $I^{+}_{\eta,\alpha}$ *of* $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t^{\rho})$ *holds true:*

$$\left(I^{+}_{\eta,\alpha}\left\{t^{\sigma-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t^{\rho})\right\}\right)(x) = x^{\sigma-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega x^{\rho}) * {}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(\sigma+\eta,\rho);\\(\sigma+\alpha+\eta,\rho);\end{array}\omega x^{\rho}\right], \tag{2.12}$$

where it is assumed that the left-sided Erdélyi-Kober fractional integral in (2.12) *exists.*

Corollary 2.3. *Let* $\rho,\alpha,\sigma,\omega,\mu,\gamma\in\mathbb{C}$, $\rho>0$ *be such that* $\Re(p)>0$, $0<\Re(\alpha)<1-\Re(\sigma)$. *Then the following Riemann-Liouville fractional integral* I^{α}_{-} *of* $\mathcal{E}^{\gamma}_{\lambda,\mu,p}\left(\frac{\omega}{t^{\rho}}\right)$ *holds true:*

$$\left(I^{\alpha}_{-}\left\{t^{\sigma-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x) = x^{\sigma+\alpha-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}\left(\frac{\omega}{x^{\rho}}\right) * {}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(1-\alpha-\sigma,\rho);\\(1-\sigma,\rho);\end{array}\frac{\omega}{x^{\rho}}\right], \tag{2.13}$$

where it is assumed that the right-sided Riemann-Liouville fractional integral in (2.13) *exists.*

Corollary 2.4. *Let* $\rho,\alpha,\eta,\sigma,\omega,\mu,\gamma\in\mathbb{C}$, $\rho>0$ *be such that* $\Re(p)>0$, $\Re(\alpha)>0$ *and* $\Re(\sigma)<1+\Re(\eta)$. *Then the following Erdélyi-Kober fractional integral* $K^{-}_{\eta,\alpha}$ *of* $\mathcal{E}^{\gamma}_{\lambda,\mu,p}\left(\frac{\omega}{t^{\rho}}\right)$ *holds true:*

$$\left(K^{-}_{\eta,\alpha}\left\{t^{\sigma-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\frac{\omega}{t^{\rho}})\right\}\right)(x) = x^{\sigma-1}\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\frac{\omega}{x^{\rho}}) * {}_{2}\Psi_{1}\left[\begin{array}{c}(1,1),(1+\eta-\sigma,\rho);\\(1+\alpha+\eta-\sigma,\rho);\end{array}\frac{\omega}{x^{\rho}}\right], \tag{2.14}$$

where it is assumed that the right-sided Erdélyi-Kober fractional integral in (2.14) *exists.*

3. Fractional differentiation of the $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$

In this section, we obtain a composition formulas of generalized fractional differentiation (1.3) and (1.4) involving p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.17) of p–extended Mittag-Leffler function and Fox-Wright function ${}_{p}\Psi_{q}(z)$.

Lemma 2. *Let* α, β, $\eta,\mu,\gamma\in\mathbb{C}$, $\rho>0$. *Then there exists the relations*

(a) *If* $\Re(\alpha) > 0$ *and* $\Re(\sigma) > -\min[0, \Re(\alpha+\beta+\eta)]$, *then*

$$(D_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)} x^{\sigma+\beta-1} \tag{3.1}$$

In particular, for $x > 0$ *we have*

$$(D_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} x^{\sigma-\alpha-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > 0), \tag{3.2}$$

$$(D_{\eta,\alpha}^{+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)} x^{\sigma-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\alpha+\eta)). \tag{3.3}$$

(b) *If* $\Re(\alpha) > 0, \Re(\sigma) < 1 + \min[\Re(-\beta-n), \Re(\alpha+\eta)]$ *and* $n = [\Re(\alpha)] + 1$, *then*

$$(D_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\beta)} x^{\sigma+\beta-1}. \tag{3.4}$$

In particular, for $x > 0$ *we have*

$$(D_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)} x^{\sigma-\alpha-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) < 1 + \Re(\alpha) - n), \tag{3.5}$$

$$(D_{\eta,\alpha}^{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma-\eta)} x^{\sigma-1} \ (\Re(\alpha) > 0,\ \Re(\sigma) < 1 + \Re(\alpha+\eta) - n). \tag{3.6}$$

Theorem 3. *Let* $\rho, \alpha, \beta, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C},\ \rho > 0$ *be such that* $\Re(p) > 0,\ \Re(\alpha) \geq 0$ *and* $\Re(\sigma) > -\min[0, \Re(\alpha+\beta+\eta)]$. *Then the following Saigo hypergeometric fractional differentiation* $D_{0+}^{\alpha,\beta,\eta}$ *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})$ *holds true:*

$$\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x) = x^{\sigma+\beta-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega x^{\rho}) * {}_3\Psi_2\left[\begin{array}{c}(1,1),(\sigma,\rho),(\alpha+\sigma+\eta+\beta,\rho);\\(\sigma+\beta,\rho),(\sigma+\eta,\rho);\end{array}\ \omega x^{\rho}\right], \tag{3.7}$$

where it is assumed that the left-sided hypergeometric fractional derivative in (3.7) *exists.*

Proof. By virtue of the formulas (1.3) and (1.15), the term-by-term fractional differentiation and the application of the relation (3.1), yields for $x > 0$

$$\begin{aligned}&\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x)\\&=\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)}\frac{\omega^k}{k!}\left(D_{0+}^{\alpha,\beta,\eta}t^{\sigma+\rho k-1}\right)(x)\\&=x^{\sigma+\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)\,k!}\frac{\Gamma(1+k)\Gamma(\sigma+\rho k)\Gamma(\sigma+\alpha+\eta+\beta+\rho k)}{\Gamma(\sigma+\beta+\rho k)\Gamma(\sigma+\eta+\rho k)\,k!}(\omega x^{\rho})^k.\end{aligned} \tag{3.8}$$

By applying the Hadamard product (1.17) in (3.8), which in view of (1.15) and (1.18), yields the desired formula (3.7).

Theorem 4. *Let $\rho,\alpha,\beta,\eta,\sigma,\omega,\mu,\gamma\in\mathbb{C}$, $\rho>0$ be such that $\Re(p)>0$, $\Re(\alpha)\geq 0$ and $\Re(\sigma)<1+\min[\Re(-\beta-n),\Re(\alpha+\eta)]$, $n=[\Re(\alpha)]+1$. Then the following Saigo hypergeometric fractional differentiation $D_{-}^{\alpha,\beta,\eta}$ of $\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)$ holds true:*

$$\begin{aligned}\left(D_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)&=x^{\sigma+\beta-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{x^{\rho}}\right)\\&*\ {}_3\Psi_2\left[\begin{matrix}(1,1),(1-\sigma-\beta,\rho),(1+\alpha+\eta-\sigma,\rho);\\(1-\sigma,\rho),(1-\beta+\eta-\sigma,\rho);\end{matrix}\ \frac{\omega}{x^{\rho}}\right],\end{aligned} \tag{3.9}$$

where it is assumed that the right-sided hypergeometric fractional derivative in (3.9) *exists.*

Proof. By virtue of the formulas (1.3) and (1.15), the term-by-term fractional differentiation and the application of the relation (3.4), yields for $x > 0$

$$\begin{aligned}&\left(D_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x)\\&=\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)\,k!}(\omega)^k\left(D_{-}^{\alpha,\beta,\eta}t^{\sigma-\rho k-1}\right)(x)\\&=x^{\sigma+\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)k!}\frac{\Gamma(1+k)\Gamma(1-\beta-\sigma+\rho k)\Gamma(1+\alpha+\eta-\sigma+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1-\sigma+\eta-\beta+\rho k)k!}\left(\frac{\omega}{x^{\rho}}\right)^k.\end{aligned} \tag{3.10}$$

By applying the Hadamard product (1.17) in (3.10), which in view of (1.15) and (1.18), yields the desired formula (3.9).

Corollary 3.1. *Let $\rho,\alpha,\sigma,\omega,\mu,\gamma\in\mathbb{C}$, $\rho>0$, be such that $\Re(p)>0$, and $\Re(\alpha)\geq$*

$0, \Re(\sigma) > 0$. *Then the following Riemann-Liouville fractional differentiation* D_{0+}^{α} *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})$ *holds true:*

$$\left(D_{0+}^{\alpha}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})\right\}\right)(x) = x^{\sigma-\alpha-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega x^{\rho}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma,\rho);\\(\sigma-\alpha,\rho);\end{array}\omega x^{\rho}\right], \tag{3.11}$$

where it is assumed that the left-sided Riemann-Liouville fractional derivative in (3.11) *exists.*

Corollary 3.2. *Let* $\rho, \alpha, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$, *be such that* $\Re(p) > 0$, $\Re(\alpha) \geq 0$ *and* $\Re(\sigma) > -\Re(\eta+\alpha)$. *Then the following Erdélyi-Kober fractional differentiation* $D_{\eta,\alpha}^{+}$ *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t^{\rho})$ *holds true:*

$$\left(D_{\eta,\alpha}^{+}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega t)\right\}\right)(x) = x^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\omega x^{\rho}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma+\alpha+\eta,\rho);\\(\sigma+\eta,\rho);\end{array}\omega x^{\rho}\right], \tag{3.12}$$

where it is assumed that the left-sided Erdélyi-Kober fractional derivative in (3.12) *exists.*

Corollary 3.3. *Let* $\rho, \alpha, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$ *be such that* $\Re(p) > 0$, $\Re(\alpha) \geq 0, \Re(\sigma) < \Re(\alpha) - [\Re(\alpha)]$. *Then the following Riemann-Liouville fractional differentiation* D_{-}^{α} *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)$ *holds true:*

$$\left(D_{-}^{\alpha}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\frac{\omega}{t^{\rho}})\right\}\right)(x) = x^{\sigma-\alpha-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\frac{\omega}{x^{\rho}}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\alpha-\sigma,\rho);\\(1-\sigma,\rho);\end{array}\frac{\omega}{x^{\rho}}\right], \tag{3.13}$$

where it is assumed that the right-sided Riemann-Liouville fractional derivative in (3.13) *exists.*

Corollary 3.4. *Let* $\rho, \alpha, \eta, \sigma, \omega, \mu, \gamma \in \mathbb{C}$, $\rho > 0$ *be such that* $\Re(p) > 0$, $\Re(\alpha) \geq 0$ *and* $\Re(\sigma) < \Re(\alpha+\eta) - [\Re(\alpha)]$. *Then the following Erdélyi-Kober fractional differentiation* $D_{\eta,\alpha}^{-}$ *of* $\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)$ *holds true:*

$$\left(D_{\eta,\alpha}^{-}\left\{t^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}\left(\frac{\omega}{t^{\rho}}\right)\right\}\right)(x) = x^{\sigma-1}\mathcal{E}_{\lambda,\mu,p}^{\gamma}(\frac{\omega}{x^{\rho}}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\alpha-\sigma+\eta,\rho);\\(1-\sigma-\eta,\rho);\end{array}\frac{\omega}{x^{\rho}}\right], \tag{3.14}$$

where it is assumed that the right-sided Erdélyi-Kober fractional derivative in (3.14) *exists.*

4. Jacobi and Related Integral Transforms

In this section, we obtain Jacobi and related integral transforms of the p-extended Mittag-Leffler functions (1.15). The classical orthogonal Jacobi polynomials $P_n^{(\varpi,\theta)}(t)$ is defined by (see, for details, [23, 28, 29]):

$$P_n^{(\varpi,\theta)}(t) = (-1)^n(-t) = \binom{\varpi+n}{n} {}_2F_1\left[\begin{matrix} -n, \varpi+\theta+n+1 \\ \varpi+1 \end{matrix} \,\Big|\, \frac{1-t}{2}\right], \tag{4.1}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function [23].

Definition 1. (see, for example, [10, p. 501]) *The Jacobi transform of a function $f(t)$ is defined as follows:*

$$\mathbb{J}^{(\varpi,\theta)}[f(t);n] = \int_{-1}^{1} (1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\,f(t)\mathrm{d}t \tag{4.2}$$

$$(\min\{\Re(\varpi),\,\Re(\theta),\} > -1;\; n \in \mathbb{N}_0),$$

provided that the function $f(t)$ is so constrained that the integral in (4.2) *exists.*

The Jacobi transform of the power function $t^{\rho-1}$ (see, for example, [10]) is given by

$$\begin{aligned}\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1};\,n] &= \int_{-1}^{1} (1-t)^{\alpha-1}\,(1+t)^{\eta-1}\,P_n^{(\varpi,\theta)}(t)\,t^{\rho-1}\,\mathrm{d}t \\ &= 2^{\alpha+\eta-1}\binom{\varpi+n}{n}\mathrm{B}(\alpha,\eta)F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha : -n, \varpi+\theta+n+1; 1-\rho; \\ \alpha+\eta : \quad \varpi+1; \quad -; \end{matrix}\; 1,2\right]\end{aligned} \tag{4.3}$$

$$(\min\{\Re(\alpha),\,\Re(\eta),\} > 0;\; \rho,\mu,\gamma \in \mathbb{C};\; n \in \mathbb{N}_0),$$

where $F^{q:m;\nu}_{p:l;\beta}$ denotes the Kampé de Fériet's function in two variables (see, e.g., [29, p. 22, Eq. 1.3(2)] and [29, p. 37, Eq. 1.4(21)]). In particular, upon setting $\alpha = \varpi+1$ and $\eta = \theta+1$, this last integral formula (4.3) would reduce immediately to the following form:

$$\begin{aligned}\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1};\,n] &= \int_{-1}^{1} (1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\,t^{\rho-1}\,\mathrm{d}t \\ &= 2^{\varpi+\theta+1}\binom{\varpi+n}{n}\mathrm{B}(\varpi+1,\theta+1)F^{1:2;1}_{1:1;0}\left[\begin{matrix}\varpi+1 : -n, \varpi+\theta+n+1; 1-\rho; \\ \varpi+\theta+2 : \quad \varpi+1; \quad -; \end{matrix}\; 1,2\right]\end{aligned} \tag{4.4}$$

$$(\min\{\Re(\varpi),\ \Re(\theta),\} > -1;\ \rho, \mu, \gamma \in \mathbb{C};\ n \in \mathbb{N}_0),$$

Indeed, in its further special case when $\rho = m + 1\,(m \in \mathbb{N}_0)$, (4.4) yields the following well-known result for the Jacobi transform of t^m $(m \in \mathbb{N}_0)$, which is given by (see, for example, [23, p. 261, Eq. (14) and (15)])

$$\begin{aligned} &\mathbb{J}^{(\varpi,\theta)}[t^m;\, n] \\ &= \int_{-1}^{1} (1-t)^{\varpi}\,(1+t)^{\theta}\, P_n^{(\varpi,\theta)}(t)\, t^m\, \mathrm{d}t \\ &= \begin{cases} 0 & (m = 0, 1, 2, \cdots, n-1) \\ 2^{\varpi+\theta+n+1}\,\mathrm{B}(\varpi+n+1, \theta+n+1) & (m = n) \\ 2^{\varpi+\theta+n+1}\,\binom{m}{n}\mathrm{B}(\varpi+n+1, \theta+n+1) & \\ \cdot\ {}_2F_1\left[\begin{matrix} n-m, \varpi+n+1 \\ \varpi+\theta+2n+2 \end{matrix}\,\middle|\,2\right], & (m = n+1, n+2, n+3, \cdots) \end{cases} \end{aligned} \tag{4.5}$$

$(\min\{\Re(\varpi),\ \Re(\theta),\} > -1;\ m, n \in \mathbb{N}_0)$

For various choices of the parameters ϖ and θ, the Jacobi polynomials $P_n^{(\varpi,\theta)}(t)$ contain, as their special cases, such other classical orthogonal polynomials as (for example) the Gegenbauer (or Ultraspherical) polynomials $C_n^{\nu}(t)$, the Legendre (or spherical) polynomials $P_n(t)$, and the Tchebycheff polynomials $T_n(t)$ and $U_n(t)$ of the first and second kind (see, for details, [29]). In fact, we have the following relationships with the Gegenbauer polynomials $C_n^{\nu}(z)$ and the Legendre polynomials $P_n(z)$:

$$C_n^{\nu}(t) = \binom{\nu+n-\frac{1}{2}}{n}^{-1} \binom{2\nu+n-1}{n} P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(t) \tag{4.6}$$

and

$$P_n(t) = C_n^{\frac{1}{2}}(t) = P_n^{(0,0)}(t), \tag{4.7}$$

respectively, which, in conjunction with (4.2), yields the corresponding Gegenbauer transform $\mathbb{G}^{(\nu)}[f(t);n]$ given by

$$\begin{aligned}&\mathbb{G}^{(\nu)}[f(t);n]\\&=\binom{\nu+n-\frac{1}{2}}{n}^{-1}\binom{2\nu+n-1}{n}\mathbb{J}^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}[f(t);n]\\&=\int_{-1}^{1}(1-t^2)^{\nu-\frac{1}{2}}\,C_n^{\nu}(z)\,f(t)\mathrm{d}t\quad(\Re(\nu)>-\frac{1}{2};\,n\in\mathbb{N}_0),\end{aligned}\tag{4.8}$$

and the corresponding Legendre transform $\mathbb{L}[f(t);n]$ defined by

$$\mathbb{L}[f(t);n]=\mathbb{G}^{(\frac{1}{2})}[f(t);n]=\int_{-1}^{1}P_n(t)\,f(z)\mathrm{d}t\quad(n\in\mathbb{N}_0).\tag{4.9}$$

Now, we prove three results which exhibit the connections between the Jacobi, Gegenbauer and Legendre transforms with the following p-extended Gauss' hypergeometric function (1.15).

Theorem 5. *Under the condition stated in* (1.15)*, the following Jacobi transform formula holds true:*

$$\begin{aligned}&\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1}\,\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t);\,n]\\&=2^{\varpi+\theta+1}\binom{\varpi+n}{n}\mathrm{B}(\varpi+1,\theta+1)\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)}\\&\quad\cdot F^{1:2;1}_{1:1;0}\left[\begin{matrix}\varpi+1:-n,\varpi+\theta+n+1;1-\rho-k;\\ \varpi+\theta+2:\qquad\varpi+1;\quad —;\end{matrix}\;1,2\right]\frac{\omega^k}{k!},\end{aligned}\tag{4.10}$$

$$(\Re(p)>0;\;m,\,n\in\mathbb{N}_0;\min\{\Re(\varpi),\,\Re(\theta),\}>-1;\;\rho,\mu,\gamma\in\mathbb{C}),$$

where it is assumed that the Jacobi transforms in (4.4) *exists.*

Proof. By applying the definition (4.2) in conjunction with (1.15), we have

$$\begin{aligned}&\mathbb{J}^{(\varpi,\theta)}[t^{\rho-1}\,{}_r\,\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t);\,n]\\&=\int_{-1}^{1}t^{\rho-1}(1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\;\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t)\,\mathrm{d}t\\&=\int_{-1}^{1}t^{\rho-1}(1-t)^{\varpi}\,(1+t)^{\theta}\,P_n^{(\varpi,\theta)}(t)\;\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)}\,\frac{(\omega t)^k}{k!}\,\mathrm{d}t,\end{aligned}\tag{4.11}$$

Now, upon changing the order of integration and summation (which can be justified easily by absolute convergence), we make use of the Jacobi transform formula (4.4) with the parameter ρ replaced by $\rho + k\,(\rho, \mu, \gamma \in \mathbb{C}; k \in \mathbb{N}_0)$.

By applying the Jacobi transform formula (4.5), we can simplify the assertion (4.10) of Theorem 5 in their special case when $\rho = m + 1\,(m \in \mathbb{N}_0)$. Moreover, in view of the relationship (4.6), Theorem 5 yields the following corollary by setting $\varpi = \theta = \nu - \frac{1}{2}$.

Corollary 4.1. *Under the condition stated in* (1.15)*, the following Gegenbauer transform formula holds true:*

$$\begin{aligned}&\mathbb{G}^{(\nu)}[t^{\rho-1}\ \mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t);\, n]\\ &= 2^{2\nu}\binom{2\nu+n-1}{n}\mathrm{B}(\nu+\frac{1}{2},\nu+\frac{1}{2})\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)}\\ &\qquad\cdot F^{1:2;1}_{1:1;0}\left[\begin{matrix}\nu+\frac{1}{2}: -n, 2\nu+n; 1-\rho-k; \\ \\ 2\nu+1: \qquad \nu+\frac{1}{2}; \quad —;\end{matrix}\ 1,2\right]\frac{\omega^k}{k!},\end{aligned} \tag{4.12}$$

$$(\Re(p) > 0;\ m,\, n \in \mathbb{N}_0;\ \rho, \mu, \gamma \in \mathbb{C}),$$

where it is assumed that the Gegenbauer transforms in (4.12) *exists.*

For the Legendre transform defined by (4.9), a special case of Theorem 5 when $\varpi = \theta = 0$ (or, alternatively, Corollary 4.1 with $\nu = \frac{1}{2}$) yields the following result.

Corollary 4.2. *Under the condition stated in* (1.15)*, the following Legendre transform formula holds true:*

$$\begin{aligned}&\mathbb{L}[t^{\rho-1}\,{}_r\,\mathcal{E}^{\gamma}_{\lambda,\mu,p}(\omega t);\, n]\\ &= 2\sum_{k=0}^{\infty}\frac{(\gamma;p)_k}{\Gamma(\lambda k+\mu)}\ F^{1:2;1}_{1:1;0}\left[\begin{matrix}1: -n, n+1; 1-\rho-k; \\ \\ 2: \qquad 1; \quad —;\end{matrix}\ 1,2\right]\frac{\omega^k}{k!},\end{aligned} \tag{4.13}$$

$$(\Re(p) > 0;\ m,\, n \in \mathbb{N}_0;\ \rho, \mu, \gamma \in \mathbb{C}p),$$

where it is assumed that the Legendre transform in (4.13) *exists.*

5. Concluding Remarks

In this paper, we obtain certain image formulas of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ by using Saigo's hypergeometric fractional calculus (integral and differential) operators (1.1), (1.2), (1.3) and (1.4). Corresponding assertions for the classical Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$ and Fox-Wright function ${}_r\Psi_s(z)$. We also established Jacobi and its particular assertions for the Gegenbauer and Legendre transforms of the p–extended Mittag-Leffler function $\mathcal{E}^{\gamma}_{\lambda,\mu,p}(z)$.

References

[1] Agarwal R., Chandola A., Pandey R. M., Nisar K. S., m-Parameter Mittag-Leffler function, its various properties, and relation with fractional calculus operators, Math Meth Appl Sci., 44 (2021), 5365-5384.

[2] Ayub U., Mubeen S., Abdeljawad T., Rahman G., Nisar K. S., The New Mittag-Leffler Function and Its Applications, Journal of Mathematics, 2020, (2020), Article ID 2463782, 8 pages.

[3] Chaudhry, M. A., Qadir, A., Rafique M. and Zubair, S. M., Extension of Euler's Beta function, J. Comput. Appl. Math., 78 (1997), 19-32.

[4] Chaudhry, M. A., Qadir, A., Srivastava H. M. and Paris, R. B., Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159 (2004), 589-602.

[5] Choi J. and Parmar, R. K., Fractional Integration And Differentiation of the (p, q)–extended Bessel function, Bulletin of the Korean Mathematical Society, 55 (2) (2018), 599-610.

[6] Choi J. and Parmar, R. K., Fractional calculus of the (p, q)–extended Struve function, Far East Journal of Mathematical Sciences, 103 (2) (2018), 541-559.

[7] Choi J., Parmar, R. K. and Pogány, T. K., Mathieu-type series built by (p, q)–extended Gaussian hypergeometric function, Bull. Korean Math. Soc., 54 (3) (2017), 789-797.

[8] Choi J., Rathie A. K. and Parmar, R. K., Extension of extended beta hypergeometric and confluent hypergeometric functions, Honam Mathematical J., 36 (2) (2014) 339-367.

[9] Choi J., Parmar, R. K. and Chopra P., Extended Mittag-Leffler Function and Associated Fractional Calculus Operators, Georgian Math. J., 27 (2) (2020), 199-209.

[10] Debnath L. and Bhatta D., Integral Transforms and Their Applications, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, 2014.

[11] Jangid K., Purohit S. D., Nisar K. S. and Araci S., Chebyshev type inequality containing a fractional integral operator with a multi-index Mittag-Leffler function as a kernel, Analysis, 41 (1) (2021), 61-67.

[12] Kilbas, A. A.. Srivastava, H. M. and Trujillo,J. J., Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[13] Kiryakova, V., Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series, 301, Longman Scientific and Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York, 1994.

[14] Luo, M. J., Parmar, R. K. and Raina, R. K., On extended Hurwitz–Lerch zeta function, J. Math. Anal. Appl., 448 (2017), 1281-1304.

[15] Maširević, D. Jankov, Parmar, R. K. and Pogány, T. K., (p, q)–extended Bessel and modified Bessel functions of the first kind, Results in Mathematics, (2017). doi:10.1007/s00025-016-0649-1.

[16] Mathai, A. M., Saxena, R. K. and Haubold, H. J., The H-Functions: Theory and Applications, Springer, New York, 2010.

[17] R. Nadeem, T. Usman, K. S. Nisar, T. Abdeljawad, A new generalization of Mittag-Leffler function via q-calculus, Adv. Differ. Equ., 2020, 695 (2020).

[18] Parmar, R. K., Pogány, T. K. and Saxena, R. K., On properties and applications of (p, q)-extended τ-hypergeometric functions, Comptes Rendus. Mathématique, 356 (3) (2018), 278-282.

[19] Parmar, R. K. and Pogány, T. K., Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities, J. Contemp. Math. Anal., 52(6) (2017), 261-272.

[20] Parmar, R. K. and Chopra P. and Paris, R. B., On an extension of extended beta and hypergeometric functions, Journal of Classical Analysis, 11(2) (2017), 91-106.

[21] Pogány, T. K. and Parmar, R. K., On p–extended Mathieu series, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. 22=534 (2018), 107-117.

[22] Prabhakar, T. R., A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7-15.

[23] Rainville, E. D., Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[24] Saigo, M., A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ., 11 (1977/78), 135-143.

[25] Samko, S. G., Kilbas, A. A. and Marichev, O. I., Fractional Integrals and Derivatives: Theory and Applications, Translated from the Russian: Integrals and Derivatives of Fractional Order and Some of Their Applications ("Nauka i Tekhnika", Minsk, 1987); Gordon and Breach Science Publishers: Reading, UK, 1993.

[26] Saxena R. K. and Parmar, R. K., Fractional Integration and Differentiation of the Generalized Mathieu Series, Axioms, 6(3), (2017), 1-11.

[27] Sharma U. P., Agarwal R. and Nisar K. S., Bicomplex two-parameter Mittag-Leffler function and properties with application to the fractional time wave equation, Palestine Journal of Mathematics, Vol. 12(1) (2023), 462-481.

[28] Sneddon, I. N., The Use of the Integral Transforms. Tata McGraw-Hill, New Delhi, 1979.

[29] Srivastava H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, Halsted Press, (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[30] Srivastava H. M. and Saxena, R. K., Operators of fractional integration and their applications, Applied Mathematics and Computation, 118 (2001), 1-52.

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