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UNICITY OF MEROMORPHIC FUNCTION WITH THEIR SHIFT OPERATOR SHARING SMALL FUNCTION

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Abstract: In this paper, we introduce a new notation of reduced linear shift operator $L_c^r(\phi)$, and with the aid of this new operator, we study the uniqueness of meromorphic functions $\phi(z)$ and $L_c^r(\phi)$ share ∞ CM in the extended complex plane. The results obtained in the paper significantly improve a existing result. Further, using the notion of sets, we deal the same problem. We exhibit a handful result to justify certain statements relevant to the content of the paper.

Keywords and Phrases: Uniqueness, Sharing value, Meromorphic Functions, Small Function and shift operator.

2020 Mathematics Subject Classification: 30D35.

1. Introduction and Preliminaries

We assume in this paper that the readers are familiar with the fundamental concepts of Nevanlinna value distribution theory, see ([15, 25]). A meromorphic function is one that is meromorphic across the entire complex plane. By $S^*(\sigma, \phi)$, we denote any quantity satisfying $S(r,\phi) = o(T(r,\phi))$ as $r \longrightarrow \infty$ outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function α is said to be a small function of ϕ if it satisfies $T(r,\alpha) = o(T(r,\phi))$. We say that two non-constant meromorphic functions ϕ and ψ share small function α IM(CM) if $\phi - \alpha$ and $\psi - \alpha$ have the same zeros ignoring multiplicities (counting multiplicities). Let ϕ be a non-constant meromorphic function. We denote by $N_1(r, 1/\phi)$ the counting function of simple zeros of ϕ.

Let ϕ be non-constant meromophic function.
The order of ϕ is defined by

$$\lambda = \overline{\lim_{r\longrightarrow\infty}} \frac{\log^+ T(r,\phi)}{\log r}.$$

Definition 1.1. *Let α be a small function of ϕ and ψ and let $S(\phi = \alpha = \psi)$ be the set of all commom zeros of $\phi - \alpha$ and $\psi - \alpha$ counting multiplicities. We say that two non-constant meromorphic functions ϕ and ψ share small function α CM almost if*

$$N\left(r, \frac{1}{\phi - \alpha}\right) + N\left(r, \frac{1}{\psi - \alpha}\right) - 2N(r, \phi = \alpha = \psi) = S(r,\phi) + S(r,\psi).$$

Definition 1.2. [18] *We denote by $N_2(r,\alpha;\phi)$ the sum $\overline{N}(r,\alpha;\phi) + \overline{N}(r,\alpha;\phi| \geq 2)$. Let c be a nonzero complex constant, and let $\phi(z)$ be a meromorphic function. The shift operator is denoted by $\phi(z+c)$. Also, we use the notations $\Delta_c\phi$ and $\Delta_c^k\phi$ to denote the difference and kth-order difference operators of $\phi(z)$, which are defined respectively by*

$$\Delta_c\phi(z) = \phi(z+c) - \phi(z), \quad \Delta_c^k\phi(z) = \Delta_c(\Delta_c^{k-1}\phi(z)), \quad k \in \mathcal{N}, k \geq 2.$$

Carefully observing the definitions, we see that all the variants of difference operators are nothing but linear combinations of different shift operators. So generalizing $\Delta_c^k\phi$, it will be reasonable to introduce the linear c-shift operator $L_c(\phi) = L_c(\phi)(z)$ as follows:

$$L_c\phi = L_c(\phi)(z) = \sum_{j=0}^{k} \alpha_j \phi(z + c_j),$$

where $\alpha_j \in \mathcal{C}$ for $j = 1, 2, ..., k$ with $\alpha_k \neq 0$. For convenience, putting $\alpha_k = \beta_k, \alpha_{k-1} = \beta_{k-1}, ..., \alpha_0 = (-1)^k\beta_0$, where β_i are nonzero complex constants with $\Sigma_{j=0}^k(-1)^{k-j}\beta_j = 0$, we get a special operator denoted by $L_c^r\phi = L_c^r(\phi)(z)$ and call it the reduced linear c-shift operator.

Putting $\beta_k = \binom{k}{k}, \beta_{k-1} = \binom{k}{k-1}, \beta_{k-2} = \binom{k}{k-2}, ..., \beta_0 = \binom{k}{0}$ *in* $L^r_c(\phi)(z)$*, we easily verify that* $L^r_c(\phi)(z) = \Delta^k_c\phi$*.*

For c-shift operator of meromorphic functions and its certain properties, we refer to the articles [1], [2], [3], [6], [16], [17]. For recent development in operator sharing small function aspect of it, we referred to the articles [4], [5], [7], [8].

Nevanlinna [24] proved the following famous five-value theorem.

Theorem A. *Let* $\phi(z)$ *and* $\psi(z)$ *be two non-constant meromorphic functions, and let* $\alpha_j(j = 1, 2, 3, 4, 5)$ *be five distinct values in the extended complex plane. If* $\phi(z)$ *and* $\psi(z)$ *share* $\alpha_j(j = 1, 2, 3, 4, 5)$ *IM, then* $\phi(z) \equiv \psi(z)$*.*

In 2000, Li and Qiao [20] proved that Theorem A is still valid for five small functions, they proved.

Theorem B. *Let* $\phi(z)$ *and* $\psi(z)$ *be two non-constant meromorphic functions, and let* $\alpha_j(z)(j = 1, 2, 3, 4, 5)$ *(one of them can be* ∞*) be five distinct small functions of* $\phi(z)$ *and* $\psi(z)$*. If* $\phi(z)$ *and* $\psi(z)$ *share* $\alpha_j(z)(j = 1, 2, 3, 4, 5)$ *IM, then* $\phi(z) \equiv \psi(z)$*.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see [9].

In 2012, Chen and Chen [9] proved.

Theorem C. *Let* $\phi(z)$ *be a non-constant meromorphic function of finite order, let* α, c *be two nonzero finite values, and let* $n \geq 7$ *be positive integer. If* $[\phi(z)]^n$ *and* $[\Delta\phi(z)]^n$ *share* α *CM,* $\phi(z)$ *and* $\Delta\phi(z)$ *share* ∞ *CM, then* $\phi(z) \equiv \tau\Delta\phi(z)$*, where* $\tau^n = 1, \tau \neq 1$*.*

In 2018, Qi, Li and Yang [22] proved.

Theorem D. *Let* $\phi(z)$ *be a non-constant meromorphic function of finite order, let* α, c *be two nonzero finite values, and let* $n \geq 9$ *be positive integer. If* $[\phi^{'}(z)]^n$ *and* $[\phi(z + c)]^n$ *share* α *CM,* $\phi^{'}(z)$ *and* $\phi(z + c)$ *share* ∞ *CM, then* $\phi^{'}(z) \equiv \tau\phi(z + c)$*, where* $\tau^n = 1$*.*

Theorem E. *Let* $\phi(z)$ *be a non-constant entire function of finite order, let* α, c *be two nonzero finite values, and let* $n \geq 5$ *be positive integer. If* $[\phi^{'}(z)]^n$ *and* $[\phi(z + c)]^n$ *share* α *CM,* $\phi^{'}(z)$ *and* $\phi(z + c)$ *share* ∞ *CM, then* $\phi^{'}(z) \equiv \tau\phi(z + c)$*, where* $\tau^n = 1$*.*

In 2020, Wang and Fang [23] removed the condition that the function $\phi(z)$ is of finite order in Theorems D and E, and proved.

Theorem F. *Let* $\phi(z)$ *be a non-constant meromorphic function, let* α, c *be two nonzero finite values, and let* $n \geq 5, k$ *be positive integers. If* $[\phi^{(k)}(z)]^n$ *and* $[\phi(z + c)]^n$ *share* α *CM,* $\phi^{(k)}(z)$ *and* $\phi(z + c)$ *share* ∞ *CM, then* $\phi^k(z) \equiv \tau\phi(z + c)$*, where* $\tau^n = 1$*.*

By above theorems, we naturally pose following problem:

Problem 1. Are Theorem C, Theorem D and Theorem F still valid if the constant α is replaced by a small function $\alpha(z)$ of $\phi(z)$?

In this paper, we study the problem and obtain the following results.

Theorem 1.1. *Let $\phi(z)$ be a non-constant meromorphic function, let c be two nonzero finite value, and let $n \geq 10$ be positive integer, and let $\alpha(z)(\not\equiv 0)$ be a small function of $\phi(z)$. If $[\phi(z)]^n$ and $[L_c^r(\phi)(z)]^n$ share a CM, $\phi(z)$ and $L_c^r(\phi)(z)$ share ∞ CM, then $\phi(z) \equiv \tau L_c^r(\phi)(z)$, where $\tau^n = 1, \tau \neq 1$.*

Hence, Theorem C is still valid if the constant α is replaced by a small function $\alpha(z)$ of $\phi(z)$.

Theorem 1.2. *Let $\phi(z)$ be a non-constant meromorphic function, let c be two nonzero finite value, and let $n \geq 3 + 2m$ be positive integer, and let $\alpha(z)(\not\equiv 0)$ be a small function of $\phi(z)$. If $[\phi(z)]^n P(\phi)$ and $[\phi(z+c)]^n P(\phi)$ share $a(z)$ CM, $\phi(z)P(\phi)$ and $\phi(z+c)P(\phi)$ share ∞ CM, then either $\phi(z)P(\phi) \equiv \tau\phi(z+c)P(\phi)$, where $\tau^{n+m} = 1$ or $[\phi(z)]^n P(\phi)[\phi(z+c)]^n P(\phi) \equiv \alpha^2(z)$.*

2. Lemmas

Lemma 2.1. [24, 25] *Let $\phi(z)$ be a non constant meromorphic function, and let k be positive integer. Then*

$$m\left(r, \frac{\phi^{(k)}}{\phi}\right) = S(r, \phi).$$

Lemma 2.2. [19] *Let $\phi(z)$ be a non constant meromorphic function, and let $n \geq 2$ be a positive integer. If ϕ and $\phi^{(n)}$ have finite many zeros, then ϕ is of finite order.*

Lemma 2.3. [24] *Let*

$$M = \left(\frac{X''}{X'} - \frac{2X'}{X-1}\right) - \left(\frac{Y''}{Y'} - \frac{2Y'}{Y-1}\right),$$

where X and Y are two non-constant meromorphic functions. If X and Y share 1 CM and $M \not\equiv 0$, then

$$N_1\left(r, \frac{1}{X-1}\right) \leq N(r, M) + S(r, X) + S(r, Y).$$

Remark 2.1. *We know from the proof in [24] that Lemma 2.3 is valid when X and Y share 1 CM almost.*

Lemma 2.4. [12, 13] *Let $\phi(z)$ be a non constant meromorphic function of finite order, let c be a nonzero complex number. Then*

$$m\left(r, \frac{\phi(z+c)}{\phi(z)}\right) = S(r,\phi),$$

for all r outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.5. [10, 14] *Let $\phi(z)$ be a non constant meromorphic function of finite order, and let c be a nonzero complex number. Then*

$$T(r,\phi(z+c)) = T(r,\phi) + S(r,\phi),$$
$$N(r,\phi(z+c)) = N(r,\phi) + S(r,\phi),$$
$$N(r,\frac{1}{\phi(z+c)}) = N(r,\frac{1}{\phi}) + S(r,\phi).$$

Lemma 2.6. [11, 12] *Let $\phi(z)$ be a non constant meromorphic function of finite order, and let c be a nonzero complex number. If $\phi(z+c) \equiv \phi(z)$, then ϕ is of order at least 1.*

Lemma 2.7. [5] *Let $\phi(z)$ be a non constant meromorphic function of finite order, and let $c \in \mathcal{C} \setminus \{0\}$ be fixed. Then*

$$T(r, L_c^r\phi) = (k+1)T(r,\phi) + S(r,\phi).$$

3. Proof of Main Results

Proof of Theorem 1.1.
Let

$$X = \frac{\phi^n}{\alpha} \;\; and \;\; Y = \frac{[L_c^r\phi]^n}{\alpha}. \tag{3.1}$$

Since ϕ^n and $[L_c^r\phi]^n$ share α CM, we know that X and Y share 1 CM almost. Set

$$\Phi = \frac{X'}{X(X-1)} - \frac{Y'}{Y(Y-1)}. \tag{3.2}$$

We discuss from following two cases.
Case 1: $\Phi \equiv 0$. By (3.1) we have

$$\frac{X-1}{X} \equiv A\frac{Y-1}{Y}, \tag{3.3}$$

where A is a nonzero value.

If $A = 1$, then from (3.3) we get $\phi^n \equiv [L_c^r\phi]^n$, that $f \equiv \tau L_c^r\phi$, where τ is a complex number such that $\tau^n = 1$.

If $A \neq 1$, then from (3.3) we have

$$\frac{1}{X} - \frac{A}{Y} \equiv 1 - A. \tag{3.4}$$

By (3.4) we can obtain

$$T(r,\phi) = T(r, L_c^r\phi) + S(r,\phi),$$

$$S(r,\phi) = S(r, L_c^r\phi). \tag{3.5}$$

According to (3.1), (3.4), (3.5), Nevanlinna's Second Fundamental Theorem ([2], page 19, Theorem 1.6) and Lemma 2.7 we get

$$\begin{aligned} nT(r,\phi) =& T(r,\phi) + S(r,\phi) \\ \leq& \overline{N}(r,X) + \overline{N}(r,\frac{1}{X}) + \overline{N}(r,\frac{1}{X - \frac{1}{1-A}}) + S(r,\phi) \\ \leq& \overline{N}(r,\phi) + \overline{N}(r,\frac{1}{\phi}) + \overline{N}(r, L_c^r\phi) + S(r,\phi) \end{aligned}$$

$$nT(r,\phi) = (k+3)T(r,\phi) + S(r,\phi), \tag{3.6}$$

it follows from (3.6) and $n \geq 6$ taht $T(r,\phi) = S(r,\phi)$, a contradiction.

Case 2: $\Phi \not\equiv 0$. Let z_0 be a common pole of ϕ and $L_c^r\phi$ with multiplicity l, then by (3.2) we know that z_0 is the zero of φ, and the multiplicity is at least $nl - 1$. Since ϕ and $L_c^r\phi$ share ∞ CM, then

$$\begin{aligned} \overline{N}(r,X) = \overline{N}(r,Y) &\leq \frac{1}{nl-1} N(r,\frac{1}{\varphi}) + S(r,\phi) \\ &\leq \frac{1}{nl-1} T(r,\varphi) + S(r,\phi) \end{aligned}$$

By using Lemma 2.3 we get,

$$\overline{N}(r,X) \leq \frac{1}{n-1}[N(r,\frac{1}{X}) + N(r,\frac{1}{Y})] + S(r,\phi). \tag{3.7}$$

Let M be defined as in Lemma 2.3. Suppose that $M \not\equiv 0$, by Lemma 2.3 and Remark 2.1 we have

$$N(r,M) \leq \overline{N}(r,\frac{1}{X}) + \overline{N}(r,\frac{1}{Y}) + N_0(r,\frac{1}{X'}) + N_0(r,\frac{1}{Y'}) + S(r,\phi). \tag{3.8}$$

where $N_0(r, \frac{1}{X'})$ denotes the counting function corresponding to the zeros of X' which are not the zeros of X and $X-1$; $N_0(r, \frac{1}{Y'})$ denotes the counting function corresponding to the zeros of Y' which are not the zeros of Y and $Y-1$. By Nevanlinna's Second Fundamental Theorem, we get

$$T(r,X)+T(r,Y) \le \overline{N}(r,X)+\overline{N}(r,\tfrac{1}{Y})$$

$$+\overline{N}(r,\frac{1}{X-1})+\overline{N}(r,Y)+\overline{N}(r,\frac{1}{Y})+\overline{N}(r,\frac{1}{Y-1})+N_0(r,\frac{1}{X'})+N_0(r,\frac{1}{Y'})+S(r,\phi). \tag{3.9}$$

Since X and Y share 1 almost CM, we have

$$\overline{N}(r,\frac{1}{X-1})+\overline{N}(r,\frac{1}{Y-1}) \le N_1(r,\frac{1}{X-1})+\frac{1}{2}\left(\overline{N}(r,\frac{1}{X-1})+\overline{N}(r,\frac{1}{Y-1})\right). \tag{3.10}$$

By (3.8)-(3.10) we have

$$T(r,X)+T(r,Y) \le \overline{N}(r,X)$$

$$+2\overline{N}(r,\frac{1}{X})+\overline{N}(r,Y)+2\overline{N}(r,\frac{1}{Y})+\frac{1}{2}\left(\overline{N}(r,\frac{1}{X-1})+\overline{N}(r,\frac{1}{Y-1})\right)+S(r,\phi). \tag{3.11}$$

By Nevanlinna's First Fundamental Theorem ([2], Page 12, Theorem 1.2), we have

$$\overline{N}(r,\frac{1}{X-1})+\overline{N}(r,\frac{1}{Y-1}) \le T(r,X)+T(r,Y)+S(r,\phi). \tag{3.12}$$

By (3.7), (3.11), (3.12) and Lemma 2.7, we can obtain

$$T(r,X)+T(r,Y) \le 4\overline{N}(r,\frac{1}{\phi})+4\overline{N}(r,\frac{1}{L_c^r\phi})+2\overline{N}(r,\phi)+2\overline{N}(r,L_c^r\phi)+S(r,\phi)$$

$$T(r,X)+T(r,Y) \le \left(6+\frac{6(k+1)}{n-1}\right)(T(r,X)+T(r,Y)+S(r,\phi)). \tag{3.13}$$

Obviously, by (3.1) we have

$$\begin{aligned}
\overline{N}(r,\frac{1}{X}) &= \overline{N}(r,\frac{1}{\phi})+S(r,\phi),\\
\overline{N}(r,X) &= \overline{N}(r,\phi)+S(r,\phi),\\
\overline{N}(r,\frac{1}{X}) &= \overline{N}(r,\frac{1}{L_c^r\phi})+S(r,\phi),\\
\overline{N}(r,Y) &= \overline{N}(r,L_c^r\phi)+S(r,\phi),\\
T(r,X) &= nT(r,\phi)+S(r,\phi),\\
T(r,Y) &= nT(r,L_c^r\phi)+S(r,\phi).
\end{aligned}$$

Hence, by above formulas, (3.13) and Nevanlinna's First Fundamental Theorem, we get

$$\begin{aligned} n(T(r,X)+T(r,L_c^r\phi)) &\le \left(6+\frac{6(k+1)}{n-1}\right)(\overline{N}(r,\frac{1}{\phi})+\overline{N}(r,\frac{1}{L_c^r\phi}))+S(r,\phi)) \\ &\le \left(6+\frac{6(k+1)}{n-1}\right)(T(r,\phi)+T(r,L_c^r\phi))+S(r,\phi), \end{aligned}$$

and it follows that

$$\frac{n^2-7n-6k}{n-1}(T(r,\phi)+T(r,L_c^r\phi)) \le S(r,\phi). \tag{3.14}$$

Thus by (3.14) and $n \ge 10$, we get $T(r,\phi)=S(r,\phi)$. a contradiction.
Hence, $M \equiv 0$. Thus we have

$$\frac{X''}{X'}-2\frac{X'}{X-1}=\frac{Y''}{Y'}-2\frac{Y'}{Y-1}.$$

Solving above equation, we get

$$\frac{1}{X-1}=\frac{A}{Y-1}+B, \quad \frac{A}{Y-1}=\frac{1+B-BX}{X-1}, \tag{3.15}$$

where $A(\neq 0)$ and B are constants.
Case 2.1: $B \neq 0,-1$. It follows from (3.15) that

$$\begin{aligned} T(r,L_c^r\phi) &= T(r,\phi)+S(r,\phi), \\ \overline{N}\left(r,\frac{1}{X-\frac{B+1}{B}}\right) &= \overline{N}(r,Y). \end{aligned} \tag{3.16}$$

So by (3.15), (3.16), Nevanlinna's Second Fundamental Theorem, Lemma 2.7 and the fact that ϕ and $L_c^r\phi$ share ∞ CM, we get

$$\begin{aligned} nT(r,\phi) &\le T(r,X)+S(r,\phi) \\ &\le \overline{N}(r,X)+\overline{N}(r,\frac{1}{X})+\overline{N}(r,\frac{1}{X-\frac{B+1}{B}})+S(r,\phi) \\ &\le \overline{N}(r,\frac{1}{X})+\overline{N}(r,X)+\overline{N}(r,Y)+S(r,\phi) \\ &\le \overline{N}(r,\frac{1}{\phi})+\overline{N}(r,\phi)+\overline{N}(r,L_c^r\phi)+S(r,\phi) \end{aligned}$$

$$nT(r,\phi) = (k+3)T(r,\phi) + S(r,\phi). \tag{3.17}$$

Therefore, by (3.17) and $n \geq 10$, we can get $T(r,\phi) = S(r,\phi)$, a contradiction.
Case 2.2: $B = 0$, By (3.15) we obtain

$$X = \frac{Y + (A-1)}{A}, \quad Y = AX - (A-1). \tag{3.18}$$

If $A \neq 1$, by (3.18) we get

$$\overline{N}\left(r, \frac{1}{X - \frac{A-1}{A}}\right) = \overline{N}(r, \frac{1}{Y}) = \overline{N}(r, \frac{1}{L_c^r \phi}) + S(r,\phi). \tag{3.19}$$

By (3.16), (3.19), Nevanlinna's Second Fundamental Theorem, Lemma 2.7 and the fact that ϕ and $L_c^r\phi$ share ∞ CM, we get

$$\begin{aligned} nT(r,\phi) &\leq T(r,X) + S(r,\phi) \\ &\leq \overline{N}(r,X) + \overline{N}(r,\frac{1}{X}) + \overline{N}\left(r, \frac{1}{X - \frac{A-1}{A}}\right) + S(r,\phi) \\ &\leq \overline{N}(r,\frac{1}{X}) + \overline{N}(r,X) + \overline{N}(r,\frac{1}{Y}) + S(r,\phi) \\ &\leq \overline{N}(r,\frac{1}{\phi}) + \overline{N}(r,\phi) + \overline{N}(r,\frac{1}{L_c^r\phi}) + S(r,\phi) \end{aligned}$$

$$nT(r,\phi) = (k+3)T(r,\phi) + S(r,\phi). \tag{3.20}$$

Therefore, by (3.20) and $n \geq 10$, we can get $T(r,\phi) = S(r,\phi)$, a contradiction.

Hence $A = 1$. It follows from (3.18) that $X \equiv Y$. Thus by (3.1) we deduce that $\phi \equiv \tau L_c^r\phi$, where $\tau^n = 1, \tau \neq -1$.
Case 2.3: $B = -1$, by (3.15) we have

$$X = \frac{A}{-Y + A + 1}, \quad Y = \frac{(A+1)X - A}{X}. \tag{3.21}$$

If $A \neq 1$, we get from (3.19) that $\overline{N}\left(r, \frac{1}{X - \frac{A}{A+1}}\right) = \overline{N}(r, \frac{1}{Y})$. Using the same arguement as in the Case 2.1, we get a contradiction. Thus, $A = -1$.
By (3.21), we get $XY \equiv 1$. It follows from $XY \equiv 1$ and (3.1) that

$$\phi^n [L_c^r \phi]^n \equiv \alpha^2. \tag{3.22}$$

Set $\phi L_c^r\phi = \beta$, then we get $\beta^n = \alpha^2$. It follows that $T(r,\beta) = \frac{2}{n}T(r,\alpha)$. Thus $\beta \not\equiv 0$ is a small function of ϕ. Since ϕ and $L_c^r\phi$ share ∞ CM, we deduce from $\phi L_c^r\phi = \beta$ that

$$N(r,\frac{1}{\phi}) \leq N(r,\frac{1}{\beta}) \leq T(r,\beta) + O(1) = S(r,\phi), \tag{3.23}$$

$$N(r,\phi) \leq N(r,\beta) \leq T(r,\beta) + O(1) = S(r,\phi). \tag{3.24}$$

Thus by Nevanlinna's Second Fundamental Theorem, (3.23), (3.24) and Lemma 2.5, we get

$$\begin{aligned} 2T(r,\phi) = T(r,\phi^2) &\leq T(r,\frac{\phi^2}{\beta}) + T(r,\beta) + O(1) \\ &\leq T(r,\frac{\phi^2}{\beta}) + T(r,\frac{\beta}{\phi^2}) + N\left(r,\frac{1}{\frac{\phi^2}{\beta}-1}\right) + S(r,\phi) \end{aligned}$$

$$2T(r,\phi) = N(r,\frac{\beta}{\phi L_c^r}) + S(r,\phi) \leq S(r,\phi), \tag{3.25}$$

that is $T(r,\phi) = S(r,\phi)$, a contradiction.

Hence, we prove that $\phi \equiv \tau L_c^r\phi$, where $\tau^n = 1$.

Proof of Theorem 1.2.
Let

$$X = \frac{(\phi(z))^n P(\phi)}{\alpha}, \quad Y = \frac{[\phi(z+c)]^n P(\phi)}{\alpha}. \tag{3.26}$$

Since $\phi^n P(\phi)$ and $[\phi(z+c)]^n P(\phi)$ share α CM, we know that X and Y share 1 CM almost. Set

$$\Psi = \left(\frac{X''}{X'} - \frac{2X'}{X-1}\right) - \left(\frac{Y''}{Y'} - \frac{2Y'}{Y-1}\right) \tag{3.27}$$

we discuss from following two cases.
Case 1: $\Psi \equiv 0$. By (3.27) we have

$$\frac{1}{X-1} \equiv \frac{A}{Y-1} + B, \tag{3.28}$$

where A is a nonzero value.

If $A = 1$, then from (3.28) we get $\phi^n P(\phi) \equiv [\phi(z+c)]^n P(\phi)$, that is $\phi P(\phi) \equiv \tau\phi_c P(\phi)$, where τ is a complex number such that $\tau^{n+m} = 1$.

If $A \neq 1$, then from (3.28) we have

$$X = \frac{Y-1+A}{A}, \quad Y = AX-(A-1). \tag{3.29}$$

By (3.29) we can obtain

$$T(r,\phi P(\phi)) = T(r,\phi_c P(\phi)) + S(r,\phi),$$
$$S(r,\phi) = S(r,\phi_c P(\phi)). \tag{3.30}$$

According to (3.26), (3.29), (3.30) and Nevalinna's Second Fundamental Theorem ([2], Page 19, Theorem 1.6) we get

$$\begin{aligned}(n+m)T(r,\phi) &= T(r,X) + S(r,\phi)\\ &\leq \overline{N}(r,X) + \overline{N}(r,\frac{1}{X}) + \overline{N}\left(r,\frac{1}{X-\frac{A-1}{A}}\right) + S(r,\phi)\\ &\leq \overline{N}(r,\phi P(\phi)) + \overline{N}\left(r,\frac{1}{\phi P(\phi)}\right) + \overline{N}(r,\phi_c P(\phi)) + S(r,\phi)\end{aligned}$$

$$(n+m)T(r,\phi) = 3(1+m)T(r,\phi) + S(r,\phi), \tag{3.31}$$

it follows from (3.31) and $n \geq 3+2m$ that $T(r,\phi) = S(r,\phi)$, a contradiction.
Case 2: $\Psi \not\equiv 0$. Let z_0 be a common pole of $\phi P(\phi)$ and $\phi_c P(\phi)$ with multiplicity 1. then by (3.2) we know that z_0 is the zero of ξ and the multiplicity is at least $2n-1$. Then we use the same argument as in the proof of Theorem 1.1 and note that (3.26) is replaced by the following formula.
Since $\phi P(\phi)$ and $\phi_c P(\phi)$ share ∞ CM, then

$$\begin{aligned}\overline{N}(r,\frac{1}{X}) &= \overline{N}(r,\frac{1}{Y})\\ &\leq \frac{1}{2n-1}N(r,\frac{1}{\xi}) + S(r,\phi)\\ &\leq \frac{1}{2n-1}T(r,\xi) + S(r,\phi)\end{aligned}$$

$$\overline{N}(r,\frac{1}{X}) \leq \frac{1}{2n-1}[\overline{N}(r,\frac{1}{X}) + \overline{N}(r,\frac{1}{Y})] + S(r,\phi), \tag{3.32}$$

and we prove either $\phi P(\phi) \equiv \tau\phi_c P(\phi)$, with $\tau^{n+m} = 1$, or $(\phi P(\phi).\phi_c P(\phi)) \equiv \alpha^2$.

References

[1] Ahamed M. B., An Investigation on the Conjecture of Chen and Yi, Results Math, 74 (2019), 122.

[2] Ahamed M. B., The class of meromorphic functions sharing values with their dierence polynomials, Indian J Pure Appl. Math., (2022). doi.org/10.1007/s13226-022-00329-3

[3] Ahamed M. B., Class of Meromorphic Functions Partially Shared Values with Their Dierences or Shifts, Kyungpook Math. J., 61 (4), (2021), 745-763.

[4] Banerjee A. and Ahamed M. B., Results on meromorphic function sharing two sets with its Linear c-difference operator, J. Contemp. Mathemat. Anal., 55 (2020), 143-155.

[5] Banerjee Abhijit and Bhattacharyya Saikat, Uniqueness of meromorphic functions with their reduced linear c-shift operators sharing two or more values or sets, Springer, 509, (2019).

[6] Chaithra C. N., Naveenkumar S. H. and Jayarama H. R., Uniqueness of meromorphic function sharing two values concerning differential-difference polynomial and it's kth derivative, Mathematics in Engineering, Science and Aerospace, 14 (1), (2023), 129-144.

[7] Chaithra C. N., Naveenkumar S. H. and Jayarama H. R., On the Transcendental solution of the Fermat type q-shift Equation, Electronic Journal of Mathematical Analysis and Applications, 11 (2), (2023), 1-7.

[8] Chaithra C. N., Naveenkumar S. H. and Rajeshwari S., Results on the uniqueness of difference polynomials with diffrence operator of entire functions sharing small function, Communications in Mathematics and Applications, 14 (2), (2023), 775-790.

[9] Chen B. Q. and Chen Z. X., Meromorphic functions sharing two sets with its difference operator, Bull Malays Math Sci Soc, 35 (2012), 765-774.

[10] Chen Z. X., Complex differences and difference equations, Mathematics Momograph Series, Science Press, Beijing, 29, 2014.

[11] Chen Z. X., On growth, zeros and poles of meromorphic solutions of linear and nonlinear difference equations, Sci China Math., 54 (2011), 2123-2133.

[12] Chiang Y. M. and Feng S. J., On the Nevanlinna characteristic of $f(z+n)$ and difference equations in the complex plane, Ramanujan J., 16 (2008), 105-129.

[13] Halburd R. G. and Korhonen R. J., Difference analogue of the lemma on the logarithemic derivative with applications to difference equations, J Math Anal Appl., 314 (2006), 477-487.

[14] Halburd R. G., Korhonen R. J. and Tohge K., Holomorphic curves with shift-invariant hyperplane pre-images, Trans Am Math Soc., 366 (2014), 4267-4298.

[15] Hayman W. K., Meromorphic functions, Oxford Mathematical Momographs, Clarendon Press, Oxford, 1964.

[16] Jayarama H. R., Naveenkumar S. H. and Chaithra C. N., Uniqueness of L-functions relating to possible Differential polynomial share with some finite weight, Ganita, 72 (2), (2022), 93-101.

[17] Jayarama H. R., Naveenkumar S. H., Rajeshwari S. and Chaithra C. N., Uniqueness of Transcendental meromorphic functions and certain Differential polynomials, J. Appl. Math. and Informatics, 41 (2023), 765-780.

[18] Lahiri I., Weighted sharing and uniqueness of meromorphic functions, Complex Var., 46 (2001), 241-253.

[19] Langley J., Zeros of derivatives meromorphic functions, Comput Methods Funct Theory, 10 (2010), 421-439.

[20] Li Y. H. and Qiao J. Y., Uniqueness of meromorphic functions concerning small functions, Sci China Ser A, 43 (2000), 581-590.

[21] Mallick S. and Ahamed M. B., On Uniqueness of a meromorphic function and its higher difference operators sharing two sets, Anal. Math. Phys., 12 (2022), 78.

[22] Qi X. G., Li N., and Yang L. Z., Uniqueness of meromorphic functions concerning their differences and solutions of difference Painleve equations, Comput Methods Funct Theory, 18 (2018), 567-582.

[23] Wang P. L., and Fang M. L., Unicity of meromorphic functions concerning derivatives and differences, Acta Math Sin Chin Ser., 63 (2020), 171-180.

[24] Yang C. C. and Yi H. X., Uniqueness theory of meromorphic functions, Mathematics and Its Applications, Springer, Netherlands, 2003.

[25] Yang L., Value distribution theory, Mathematics and Its Applications, Springer-Verlag, Berlin, 1993.