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DIFFERENTIAL SUBORDINATIONS AND FUZZY DIFFERENTIAL SUBORDINATIONS USING HILBERT SPACE OPERATOR

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Abstract: Differential subordination has recently been extended from the geometric function theory to the fuzzy set theory by several authors. In this paper, we use the notion of fuzzy differential subordination to introduce certain fuzzy classes using Hilbert Space Operator. Certain interesting results are established for these classes.

Keywords and Phrases: Differential subordination, fuzzy subordination, fuzzy set, fuzzy best dominant, subordinations.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

S. S. Miller and P. T. Mocanu introduced differential subordination and derived some properties associated with it [11]. Then developed by many authors see also [12, 13]. Fuzzy subordination and fuzzy differential subordination was first studied by G. I. Oros and Gh. Oros [14]. Authors have extended the notion

of subordination from the geometric theory of analytic functions of one complex variable to the fuzzy set theory. In [15] the authors have defined the notion of fuzzy differential subordination.
Fuzzy differential subordination theory represents a generalization of the classical concept of differential subordination which emerged in recent years as a result of embedding the concept of fuzzy sets into geometric function theory. In this paper we have studied differential subordination and fuzzy differential subordination for certain classes of holomorphic functions.
Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H(U)$ denote the class of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by $H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\}$ and $\mathcal{A}_n$=$\{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + ..., z \in U\}$ with $\mathcal{A}_1$=$\mathcal{A}$.

Definition 1.1. [11] *For the functions f and g analytic in U, we say that the function f is subordinate g in U and written as $f \prec g$, if there exist a Schwartz function w analytic in U with $w(0) = 0, |w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0),\ f(U) \subset g(U)$.*

Definition 1.2. [22] *Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise would be the following:*
A pair (S, F_S), where F_S: $X \to [0, 1]$ and supp$(S, F_S) = \{x \in X : 0 < F_S(x) \le 1\}$ is called fuzzy subset. The function F_S is called membership function of the fuzzy subset (S, F_S).

Definition 1.3. [14] *Let two fuzzy subsets of X be (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x),\ x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x),\ x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*
Assume that D is a set in $\mathbb{C}$ and f, g are holomorphic functions. We indicate by

$$f(D) = supp(f(D), F_{f(D)}) = \{f(z) : 0 < F_{f(D)}(f(z)) \le 1, z \in D\}$$

and

$$g(D) = supp(g(D), F_{g(D)}) = \{g(z) : 0 < F_{g(D)}(g(z)) \le 1, z \in D\}.$$

Definition 1.4. [14] *Suppose that D is a set in $\mathbb{C}$, $z_0 \in D$ is a fixed point and let the functions $f, g \in H(D)$. The function f is named a fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if*
(1) $f(z_0) = g(z_0)$,

(2) $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)),\ z \in D.$

Definition 1.5. [15] *Let h be univalent in U and $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$. If $\mathcal{P}$ holomorphic in U satisfies the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3.U)}(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_{h(U)}(h(z)), \tag{1.1}$$

i.e.,

$$\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z),\ z \in U$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P}(z) \prec_F q(z),\ z \in U$ for all $\mathcal{P}$ satisfying (1.1). *A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z),\ z \in U$ for all fuzzy dominant q of* (1.1) *is said to be the fuzzy best dominant of* (1.1).

Definition 1.6. *Suppose that H is the complex Hilbert space and the algebra of all bounded linear operators is denoted by L(H). Let T be the bounded linear operator and σ(T) the spectrum on the complex plane. The operator f(T) on H is known as Riesz-Dunford integral* [3] *given by*

$$f(T) = \frac{1}{2\pi i}\int_c (zI - T)^{-1} f(z)dz \tag{1.2}$$

where I is the identity operator on H and C is the simple smooth closed contour, positive oriented and consists the spectrum σ(T) [5]. *The operator f(T) converges in the norm topology and can be written as:*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} T^n \tag{1.3}$$

For $f \in \mathcal{A}$, $0 \leq \theta \leq 1$ *and* $0 \leq \mu \leq 1$, *Dunford* [3] *defined the operator $\mathcal{R}_\mu^\theta : \mathcal{A} \to \mathcal{A}$*

$$\begin{aligned} \mathcal{R}_\mu^\theta f(z) &= \frac{1}{(1-\mu)^{\theta+1}\Gamma(\theta+1)}\int_0^\infty t^{\theta+1} e^{-(t/(1-\mu))} f(zt)dt \\ &= z + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} a_n z^n \end{aligned} \tag{1.4}$$

In [2] they mentioned different types of intuitionistic fuzzy continuities and intuitionistic fuzzy boundedness in intuitionistic fuzzy pseudo normed linear spaces which we used at the time of extending the hilbert space operator for fuzzy notions.

We need the following lemmas in investigating our main results.

Lemma 1.1. [11] (Hallenbeck and Ruscheweyh) *Let h be a convex function with h(0) = a, and let* $\gamma \in \mathbb{C}^*$ *be a complex number with* $\Re(\gamma) \geq 0$. *If* $\mathcal{P} \in H[a,n]$ *and*

$$\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z) \prec h(z),\ \ z \in U$$

then

$$\mathcal{P}(z) \prec g(z) \prec h(z),\ \ z \in U$$

where

$$g(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt,\ \ z \in U.$$

Lemma 1.2. [11] *Let g be a convex function in U and let* $h(z) = g(z) + n\alpha zg'(z)$, *for* $z \in U$, *where* $\alpha > 0$ *and n is positive integer. If* $\mathcal{P}(z) = g(0) + \mathcal{P}_n z^n + \mathcal{P}_{n+1} z^{n+1} + \ldots,\ z \in U$ *is holomorphic in U and*

$$\mathcal{P}(z) + \alpha z\mathcal{P}'(z) \prec h(z),\ \ z \in U$$

then

$$\mathcal{P}(z) \prec g(z),\ \ z \in U,$$

and the result is sharp.

Lemma 1.3. [16] *Let h be a convex function with h(0) = a, and let* $\gamma \in \mathbb{C}^*$ *be a complex number with* $\Re(\gamma) \geq 0$. *If* $\mathcal{P} \in H[a,n]$ *with* $\mathcal{P}(0) = a$ *and* $\Psi : \mathbb{C}^2.U \to \mathbb{C}, \psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z)$ *is holomorphic in U, then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \frac{1}{\gamma} z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)) \leq F_{h(U)}(h(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt$$

the function q is convex and is the fuzzy best (a,n)-dominant.

Lemma 1.4. [16] *Suppose that q is convex function in U, let* $h(z) = g(z) + n\gamma zg'(z)$,

$\gamma > 0$ *and* $n \in \mathbb{N}$. *If* $\mathcal{P} \in H[q(0), n]$ *and* $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \gamma z\mathcal{P}'(z)$ *is holomorphic in* U, *then*

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + \gamma z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)),$$

implies

$$F_{\mathcal{P}(U)}(\mathcal{P}(z)) \leq F_{q(U)}(q(z)), z \in U,$$

i.e.,

$$\mathcal{P}(z) \prec_F q(z)$$

and q *is the fuzzy best dominant.*

Recently Oros and Oros [15, 16], Lupas [6, 7, 8], Hyder [7], Wanas [20], Wanas and Bulut [19], Altınkaya and Wanas [1] and Wanas and Majeed [18, 21] have obtained fuzzy differential subordination result for certain classes of holomorphic functions. In [6] determined certain coefficient inequalities for the classes of q-starlike and q-convex function. We used to extend this is in fuzzy form. In [4] defined subclasses of analytic functions which are based upon operators on Hilbert space involving Wright's generalized hypergeometric function. Here we used the Hilbert space operator defined in (1.4), we study differential subordinations and fuzzy differential subordinations properties associated with it.

2. Subordination Results

Theorem 2.1. *Let* g *be a convex function,* $g(0) = 1$ *and let* h *be the function* $h(z) = g(z) + \frac{z}{\zeta}g'(z)$, $z \in U$, *if* $a, \zeta > 0, n \in \mathbb{N}, f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}_\mu^\theta f(z)\right)' \prec h(z), z \in U, \tag{2.1}$$

then

$$\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta} \prec g(z), z \in U,$$

and this result is sharp.

Proof. Let

$$\mathcal{R}_\mu^\theta f(z) = z + \sum_{n=2}^{\infty}\frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)}a_n z^n, (0 \leq \theta \leq 1, 0 \leq \mu \leq 1)$$

Consider

$$\mathcal{P}(z) = \left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta} = 1 + \mathcal{P}_\zeta z^\zeta + \mathcal{P}_{\zeta+1}z^{\zeta+1} + ..., \ z \in U.$$

We deduce that $\mathcal{P} \in H[1,\zeta]$. Differentiating above equation, we get

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta-1} \left(\mathcal{R}_{\mu}^{\theta} f(z)\right)' = \mathcal{P}(z) + \frac{1}{\zeta} z\mathcal{P}'(z),\ \ z \in U,$$

then (2.1) becomes

$$\mathcal{P}(z) + \frac{1}{\zeta} z\mathcal{P}'(z) \prec h(z) = g(z) + \frac{z}{\zeta} g'(z), z \in U.$$

By using Lemma 1.2, we have

$$\mathcal{P}(z) \prec g(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \prec g(z),\ \ z \in U.$$

It can be observed that this result is sharp.

Theorem 2.2. *Let h be a holomorphic function which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, z \in U,$ and $h(0) = 1$. if $a, \zeta > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and satisfies the differential subordination*

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta-1} \left(\mathcal{R}_{\mu}^{\theta} f(z)\right)' \prec h(z), z \in U \tag{2.2}$$

then

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \prec q(z), z \in U,$$

where

$$q(z) = \frac{\zeta}{z^{\zeta}} \int_0^z h(t) t^{\zeta-1} dt.$$

The function q is convex and it is the best dominant.

Proof. Let

$$\mathcal{P}(z) = \left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta}$$

$$= \left(\frac{z + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} a_n z^n}{z} \right)^{\zeta}$$

$$= \left(1 + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} a_n z^{(n-1)} \right)^{\zeta}$$

$$= 1 + \sum_{j=\zeta+1}^{\infty} \mathcal{P}_j z^{j-1}$$

$$= 1 + \zeta \left(\sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} a_n z^{(n-1)} \right) +$$

$$+ \frac{\zeta(\zeta-1)}{2!} \left(\sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} a_n z^{(n-1)} \right)^2 + ...,$$

for $z \in U,\ \mathcal{P} \in H[1, \zeta]$. Differentiating $\mathcal{P}$, we obtain

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z} \right)^{\zeta-1} \left(\mathcal{R}_{\mu}^{\theta} f(z) \right)' = \mathcal{P}(z) + \frac{1}{\zeta} z \mathcal{P}'(z), z \in U,$$

and (2.2) becomes

$$\mathcal{P}(z) + \frac{1}{\zeta} z \mathcal{P}'(z) \prec h(z), z \in U.$$

Using Lemma 1.1, we have

$$\mathcal{P}(z) \prec q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z} \right)^{\zeta} \prec q(z) = \frac{\zeta}{z^{\zeta}} \int_0^z h(t) t^{\zeta-1} dt,\ \ z \in U,$$

and q is the best fuzzy dominant.

Corollary 2.1. *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U*, where* $0 \leq \beta < 1$*. If* $a, \zeta \geq 0, n \in \mathbb{N}, f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z} \right)^{\zeta-1} \left(\mathcal{R}_{\mu}^{\theta} f(z) \right)' \prec h(z), z \in U, \tag{2.3}$$

then

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z} \right)^{\zeta} \prec q(z), z \in U,$$

where q is given by

$$q(z) = (2\beta - 1) + \frac{2(1-\beta)\zeta}{z^{\zeta}} \int_0^z \frac{t^{\zeta-1}}{1+t} dt, z \in U.$$

The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of the Theorem 2.2 and considering $\mathcal{P}(z) = \left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta}$, the differential subordination (2.3) becomes

$$\mathcal{P}(z) + \frac{1}{\zeta} z\mathcal{P}'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1+z}, z \in U.$$

By using Lemma 1.2, we have $\mathcal{P}(z) \prec q(z)$ i.e.,

$$\begin{aligned} \left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \prec q(z) =& \frac{\zeta}{z^{\zeta}} \int_0^z h(t) t^{\zeta-1} dt \\ =& \frac{\zeta}{z^{\zeta}} \int_0^z t^{\zeta-1} \frac{(1 + (2\beta - 1)t)}{1+t} dt \\ =& \frac{\zeta}{z^{\zeta}} \int_0^z \left[(2\beta - 1) t^{\zeta-1} + 2(1-\beta) \frac{t^{\zeta-1}}{1+t} \right] dt. \end{aligned}$$

Therefore

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \prec q(z) = (2\beta - 1) + \frac{2(1-\beta)\zeta}{z^{\zeta}} \int_0^z \frac{t^{\zeta-1}}{1+t} dt, z \in U.$$

3. Fuzzy Subordination Results

Theorem 3.1. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and*

$$\frac{1}{z}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right) + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1} \Gamma(n+\theta+2)}{\Gamma(\theta+1)} (n-1) a_n z^{(n-1)} + z\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)''$$

is holomorphic in U. If

$$F_{\psi(\mathbb{C}^2.U)}\left[\frac{1}{z}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right) + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} (n-1) a_n z^{(n-1)} + z\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)''\right] \le F_{h(U)}(h(z)), \tag{3.1}$$

then

$$F_{(\mathcal{R}_{\mu}^{\theta} f)'(U)}(\mathcal{R}_{\mu}^{\theta} f(z))' \le F_{q(U)} q(z) \le F_{h(U)} h(z),$$

i.e.,

$$(\mathcal{R}_{\mu}^{\theta} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ *is convex and is the fuzzy best dominant.*

Proof. Assume that

$$\mathcal{P}(z) = (\mathcal{R}_{\mu}^{\theta} f(z))'. \tag{3.2}$$

Then $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$. therefore, we have

$$\begin{aligned}
\mathcal{P}(z) + z\mathcal{P}'(z) &= (\mathcal{R}_{\mu}^{\theta} f(z))' + z(\mathcal{R}_{\mu}^{\theta} f(z))'' \\
&= 1 + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} n a_n z^{(n-1)} \\
&+ z\left(\sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} n(n-1) a_n z^{(n-2)}\right) \\
&= 1 + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} n^2 a_n z^{(n-1)} \\
&= \frac{1}{z}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right) + \sum_{n=2}^{\infty} \frac{(1-\mu)^{n+1}\Gamma(n+\theta+2)}{\Gamma(\theta+1)} (n-1) a_n z^{(n-1)} + z\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)''.
\end{aligned} \tag{3.3}$$

According to (3.1) and (3.3), we deduce that

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \le F_{h(U)}(h(z)).$$

Thus by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \le F_{q(U)}q(z) \le F_{h(U)}h(z), z \in U.$$

From (3.2), we find that

$$F_{(\mathcal{R}_{\mu}^{\theta} f)'(U)}(\mathcal{R}_{\mu}^{\theta} f(z))' \le F_{q(U)}q(z) \le F_{h(U)}h(z),$$

i.e.,

$$(\mathcal{R}_{\mu}^{\theta} f(z))' \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

For $a = c$ and $h(z) = \frac{1+(2p-1)z}{1+z}(0 \le p < 1)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1. *Let $f \in \mathcal{A}$ and $zf''(z) + f'(z)$ is holomorphic in U. If*

$$zf''(z) + f'(z) \prec_F \frac{1 + (2p-1)z}{1+z},$$

then

$$f'(z) \prec_F q(z) \prec_F \frac{1 + (2p-1)z}{1+z},$$

where $q(z) = 2p - 1 + \frac{2(1-p)}{z} ln(1+z)$ is convex and the fuzzy best dominant.

Theorem 3.2. *Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and $(\mathcal{R}_\mu^\theta f(z))'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[(\mathcal{R}_\mu^\theta f(z))'\right] \leq F_{h(U)} h(z), \tag{3.4}$$

then

$$F_{(\mathcal{R}_\mu^\theta f)(U)}\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right). \tag{3.5}$$

It is clear that $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + z\mathcal{P}'(z) = (\mathcal{R}_\mu^\theta f(z))'. \tag{3.6}$$

In view of (3.6), the fuzzy differential subordination (3.4) becomes

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)).$$

Thus by applying Lemma 1.3 with $\gamma = 1$, we obtain

$$F_{\mathcal{P}(U)}\mathcal{P}(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U.$$

From (3.5), we get

$$F_{(\mathcal{R}_\mu^\theta f)(U)}\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

i.e.,

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right) \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

For $a = c$ and $h(z) = e^{bz}, |b| \le 1$ in Theorem 3.2, we obtain the following

Corollary 3.2. *Let $f \in \mathcal{A}$, $f'(z)$ is holomorphic in U. If $f'(z) \prec_F \ e^{bz}$, then*

$$\frac{f(z)}{z} \prec_F q(z) \prec_F e^{bz},$$

where $q(z) = \frac{e^{bz}-1}{bz}$ is convex and the fuzzy best dominant.

Theorem 3.3. *Let g be a convex function, $g(0)=1$ and let h be the function $h(z)= g(z)+\frac{z}{\zeta}g'(z), z \in U$. If $a, \zeta > 0, n \in \mathbb{N}, f \in \mathcal{A}$ and $\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)'\right] \le F_{h(U)}h(z), \tag{3.7}$$

then

$$F_{(\mathcal{R}_{\mu}^{\theta} f)^{\zeta}(U)}\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \le F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta} \prec_F g(z),$$

and this result is sharp.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta}. \tag{3.8}$$

Then $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, therefore in view of (1.4) and (3.8). We have

$$\mathcal{P}(z) + \frac{1}{\zeta}z\mathcal{P}'(z) = \left(\frac{\mathcal{R}_{\mu}^{\theta} f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}_{\mu}^{\theta} f(z)\right)'. \tag{3.9}$$

According to (3.7) and (3.9), We obtained

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z)+\frac{1}{\zeta}z\mathcal{P}'(z)\right]\leq F_{h(U)}h(z),$$

then by applying Lemma 1.4 with $\gamma=\zeta$ we have

$$F_{\mathcal{P}(U)}(\mathcal{P}(z))\leq F_{q(U)}(q(z))\leq F_{h(U)}(h(z)), z\in U.$$

From (3.8) we obtain

$$F_{(\mathcal{R}^{\theta}_{\mu}f)^{\zeta}(U)}\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta}\leq F_{g(U)}g(z),$$

i.e.,

$$\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta}\prec_F g(z), z\in U.$$

and this result is sharp.

Theorem 3.4. *Let h be a holomorphic function which satisfies the inequality $Re\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2},\ z\in U$ and $h(0)=1$ if $a,\zeta>0,\ n\in\mathbb{N}, f\in\mathcal{A}$ and $\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}^{\theta}_{\mu}f(z)\right)'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)}\left[\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta-1}\left(\mathcal{R}^{\theta}_{\mu}f(z)\right)'\right]\leq F_{h(U)}h(z), \tag{3.10}$$

then

$$F_{(\mathcal{R}^{\theta}_{\mu}f)^{\zeta}(U)}\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta}\prec F_{q(U)}q(z), z\in U,$$

i.e.,

$$\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta}\prec_F q(z), z\in U,$$

where $q(z)=\frac{\zeta}{z^{\zeta}}\int_0^z h(t)t^{\zeta-1}dt$ the function q is convex and it is the best dominant.

Proof. Assume that

$$\mathcal{P}(z)=\left(\frac{\mathcal{R}^{\theta}_{\mu}f(z)}{z}\right)^{\zeta}. \tag{3.11}$$

It is clear that $\mathcal{P} \in H[1,1]$ and $\mathcal{P}(0) = 1$, we find that

$$\mathcal{P}(z) + \frac{1}{\zeta} z\mathcal{P}'(z) = \left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta-1} \left(\mathcal{R}_\mu^\theta f(z)\right)'. \tag{3.12}$$

According to (3.10) and (3.12), we obtain

$$F_{\psi(\mathbb{C}^2.U)} \left[\mathcal{P}(z) + \frac{1}{\zeta} z\mathcal{P}'(z)\right] \leq F_{h(U)} h(z).$$

Then by applying Lemma 1.3 with $\gamma = \zeta$, we have

$$F_{\mathcal{P}(U)} \mathcal{P}(z) \leq F_{q(U)} q(z) \leq F_{h(U)} h(z), z \in U.$$

From (3.11), we get

$$F_{\psi(\mathbb{C}^2.U)} \left[\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta-1} \left(\mathcal{R}_\mu^\theta f(z)\right)'\right] \leq F_{q(U)} q(z) \leq F_{h(U)} h(z),$$

and

$$\left(\frac{\mathcal{R}_\mu^\theta f(z)}{z}\right)^{\zeta} \prec_F q(z), \ z \in U,$$

where $q(z) = \frac{1}{z} \int_0^z h(t)dt$ is convex and is the fuzzy best dominant.

Theorem 3.5. *Suppose q is a convex function in U such that $q(0) = 1$, $h(z) = q(z) + zq'(z)$. Let $f \in \mathcal{A}$ and $\left(\frac{z\mathcal{R}_\mu^{\theta+1} f(z)}{\mathcal{R}_\mu^\theta f(z)}\right)'$ is holomorphic in U. If*

$$F_{\psi(\mathbb{C}^2.U)} \left[\left(\frac{z\mathcal{R}_\mu^{\theta+1} f(z)}{\mathcal{R}_\mu^\theta f(z)}\right)'\right] \leq F_{h(U)} h(z), \tag{3.13}$$

then

$$F_{\left(\frac{\mathcal{R}_\mu^{\theta+1} f}{\mathcal{R}_\mu^\theta f}\right)(U)} \left(\frac{\mathcal{R}_\mu^{\theta+1} f(z)}{\mathcal{R}_\mu^\theta f(z)}\right) \prec F_{q(U)} q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{R}_\mu^{\theta+1} f(z)}{\mathcal{R}_\mu^\theta f(z)}\right) \prec_F q(z), z \in U,$$

and q is fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z) = \left(\frac{\mathcal{R}_{\mu}^{\theta+1} f(z)}{\mathcal{R}_{\mu}^{\theta} f(z)}\right). \tag{3.14}$$

It is clear that $\mathcal{P} \in H[1,1]$. Differentiating both sides of (3.14) with respect to z, it yields

$$\mathcal{P}'(z) = \left(\frac{(\mathcal{R}_{\mu}^{\theta+1} f(z))'}{\mathcal{R}_{\mu}^{\theta} f(z)}\right) - \mathcal{P}(z)\left(\frac{(\mathcal{R}_{\mu}^{\theta} f(z))'}{\mathcal{R}_{\mu}^{\theta} f(z)}\right)$$

Then

$$\mathcal{P}(z) + z\mathcal{P}'(z) = \left(\frac{z\mathcal{R}_{\mu}^{\theta+1} f(z)}{\mathcal{R}_{\mu}^{\theta} f(z)}\right)' \tag{3.15}$$

Using (3.15) in (3.14), we get

$$F_{\psi(\mathbb{C}^2.U)}\left[\mathcal{P}(z) + z\mathcal{P}'(z)\right] \leq F_{h(U)}(h(z)).$$

Thus by applying Lemma 1.4 with $\gamma = 1$, we obtain

$$F_{\left(\frac{\mathcal{R}_{\mu}^{\theta+1} f}{\mathcal{R}_{\mu}^{\theta} f}\right)(U)}\left(\frac{\mathcal{R}_{\mu}^{\theta+1} f(z)}{\mathcal{R}_{\mu}^{\theta} f(z)}\right) \prec F_{q(U)} q(z), z \in U,$$

i.e.,

$$\left(\frac{\mathcal{R}_{\mu}^{\theta+1} f(z)}{\mathcal{R}_{\mu}^{\theta} f(z)}\right) \prec_F q(z), z \in U,$$

and q is fuzzy best dominant.

4. Conclusions

In the present work, we have introduced some properties of fuzzy differential subordination and subordinations of analytic functions by using Hilbert Space Operator.

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