South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 2 (2023), pp. 65-76

DOI: 10.56827/SEAJMMS.2023.1902.5 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A CRYSTAL STUDY ON NEUTROSOPHIC INCLINE

P. Muralikrishna and Surya M.

PG and Research Department of Mathematics,
Muthurangam Government Arts College (Autonomous),
Vellore - 632002, Tamil Nadu, INDIA

E-mail : pmkrishna@rocketmail.com, suryamano95@gmail.com

(**Received:** *Feb. 13, 2022* **Accepted:** *Jul. 28, 2023* **Published:** *Aug. 30, 2023)*

Abstract: As fuzzy subalgebras and fuzzy ideals plays an important role in the research field of fuzzy algebraic structures. Here, an incline algebra is considered and moved forward in this paper. i.e., the concept of neutrosophic subincline and ideal of incline algebra is studied and the related properties of neutrosophic subincline and ideals on incline algebra are discussed. Furthermore, the image, preimage, level set, product are also analysed.

Keywords and Phrases: Incline algebra, neutrosophic set, neutrosophic subincline, neutrosophic ideal.

2020 Mathematics Subject Classification: 16Y60, 16Y99.

1. Introduction

Incline algebra was inititated by a Chinese Cybernetics expert Cao Zhiqiang [2]. Inclines are a generalization of Boolean algebra and fuzzy algebra a special type of a semiring and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations. An incline is a structure which has an associative, commutative addition,and a distributive multiplication such that $u + u = u, u + uv = u \ \forall \ u, v$. It has both a semiring structure and a poset structure. The ideals in a ring or semigroup form an incline, as do the topologizing filters in a ring. Incline theory is based on semiring theory and lattice theory. Also, Cao. Z. Q, Kim and Roush [6] introduced the theories of incline algebra in monograph. This theory and its related concepts are applied in many fields like nervous systen, automation theory, etc., Ahn [1] et al. also introduced the notion of quotient incline and obtained the notion of incline algebra.

Even though Lukaseicuicz and Tarski in 1920,framed the structure called fuzzy sets, its has being brought out by Zadeh [9] with a membership function, then the fuzzy set theory is also developed and other researchers has evoked with a great interest in working among different branches of mathematics. Fuzzy sets generalise the definition, allowing elements to belong to a given set with a certain degree. Instead of considering characteristic functions with value in $\{0, 1\}$ we consider now functions valued in [0,1]. Atanassov at first gave the idea of intuitionistic fuzzy sets by adding an additional component called non - membership function with a membership function and later Smarandache [6] brought a new membership function named as the degree of indeterminacy / neutrality and is defined on three components namely, $(t, i, f) = (truth, indeterminacy, falsity)$ called the neutrosophic set. And this concept is developed by Smarandache [1999] is extended to the concepts of the classic set and fuzzy sets in 2005.

The fusion of fuzzy with different algebraic structures was studied by Rosenfeld and further many correlated algebra with fuzzy sets. Jun [1, 3] applies the theory of fuzzy set to incline aglebra and came out with the concept of fuzzy subincline algebra and fuzzy ideal of incline algebra. As a motivation here the fusion of neutrosophic set with incline algebra is considered and discussed with the related results. Also, the product and level are studied in elegant results.

2. Preliminaries

This section reviews some basic facts of incline algebra and fuzzy concepts.

Definition 2.1. [9] *A function $\zeta : \mathfrak{U} \to [0, 1]$ is said to be a fuzzy set in a universal set $\mathfrak{U}$ where $\zeta(r)$ is the membership value of $r \ \forall \ r \in \mathfrak{U}$.*

Definition 2.2. [6] *A neutrosophic set $\mathscr{R}$ is of the form $\{r, \zeta_{\mathscr{R}}(r), \tau_{\mathscr{R}}(r), \upsilon_{\mathscr{R}}(r)/r \in \mathfrak{U}\}$ on the universal set $\mathfrak{U}$ with $\zeta_{\mathscr{R}}$ as a truth membership function, $\tau_{\mathscr{R}}$ as indeterminate membership function and $\upsilon_{\mathscr{R}}$ as an false membership function where $\zeta_{\mathscr{R}}, \tau_{\mathscr{R}}, \upsilon_{\mathscr{R}} : \mathfrak{U} \to [0, 1]$.*

Definition 2.3. [4,5] *A non - empty set $(\mathfrak{T}, +, *)$ is an Incline algebra if $\forall \ r, s \in \mathfrak{T}$ and the following conditions holds,*
(i) $+$ is commutative and associative
(ii) $*$ is associative and distributive (both left and right) under $+$
(iii) $r + r = r$ (idempotent)
(iv) $r + (r * s) = r$
(v) $s + (r * s) = s$.

Note.
1. Every distributive lattice is an Incline Algebra. But the converse is not true in general.
2. An Incline is a distributive lattice if and only if $r * r = r \ \ \forall\, r \in \mathfrak{T}$.
3. In an Incline the partial order is defined as $r \leq s \Leftrightarrow r{+}s = s$ and $r{\wedge}s = min(r, s)$ $r \vee s = max(r, s)\ \forall\ r, s \in \mathfrak{T}$.

Definition 2.4. [1, 5] *A subincline of an incline algebra $\mathfrak{T}$ is a non - empty subset $\mathscr{H}$ of $\mathfrak{T}$ which is closed under addition and multiplication.*

Definition 2.5. [7] *A subincline $\mathscr{H}$ of $\mathfrak{T}$ is said to be an ideal if $r \in \mathscr{H}, s \in \mathfrak{T}$ and $s \leq r$ then $s \in \mathscr{H}$.*

Definition 2.6. [3] $\mathscr{A} = \{r, \zeta_{\mathscr{A}}(r)\}$ *is said to be a fuzzy subincline of incline algebra $\mathfrak{T}$ if $\zeta_{\mathscr{A}}(r + s) \wedge \zeta_{\mathscr{A}}(r * s) \geq \zeta_{\mathscr{A}}(r) \wedge \zeta_{\mathscr{A}}(s)\ \forall\ r, s \in \mathfrak{T}$.*

Definition 2.7. [3] *A fuzzy set $\mathscr{A}$ is said to be order reversing if $\zeta_{\mathscr{A}}(r) \geq \zeta_{\mathscr{A}}(s)$ whenever $r \leq s$.*

Definition 2.8. [3] *A fuzzy subincline is called a fuzzy ideal of $\mathfrak{T}$ if its order reversing.*

3. Neutrosophic Subincline

This segregation works on the structure of neutrosophic subincline of an incline and analyses the related results in sequel.

Definition 3.1. *A neutrosophic set $\mathfrak{R} = \{r, \zeta_{\mathfrak{R}}(r), \tau_{\mathfrak{R}}(r), \upsilon_(r)/r \in \mathfrak{T}\}$ in an incline algebra $\mathfrak{T}$ is said to be a neutrosophic subincline of $\mathfrak{T}$ if the followings holds,*
(i) $\zeta_{\mathfrak{R}}(r + s) \wedge \zeta_{\mathfrak{R}}(r * s) \geq \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)$
(ii) $\tau_{\mathfrak{R}}(r + s) \wedge \tau_{\mathfrak{R}}(r * s) \geq \tau_{\mathfrak{R}}(r) \wedge \tau_{\mathfrak{R}}(s)$
(iii) $\upsilon_{\mathfrak{R}}(r + s) \vee \upsilon_{\mathfrak{R}}(r * s) \leq \upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s)\ \forall\ r, s \in \mathfrak{T}$.

Example 3.2. A incline algebra $\mathfrak{T} = \{e, \alpha, \beta, \gamma\}$ with two operations $+$ and $*$ defined on $\mathfrak{T}$ with the Cayley's table.

$+$	e	α	β	γ
e	e	α	β	γ
α	e	e	e	e
β	α	α	α	α
γ	β	e	e	β

$*$	e	α	β	γ
e	e	e	e	e
α	e	α	β	γ
β	e	α	α	e
γ	e	β	e	β

Table 1: Neutrosopohic subincline

Now $\mathfrak{R} = \{e, \alpha, \beta, \gamma\}$ is subincline on $\mathfrak{T}$ and $\mathfrak{R}$ is a fuzzy subinclines, is defined by the membership functions

$$\zeta_{\mathfrak{R}}(r) = \begin{cases} 0.4 : & r = e, \gamma \\ 0.3 : & r = \alpha, \beta \end{cases} \quad \tau_{\mathfrak{R}}(r) = \begin{cases} 0.6 : & r = e, \gamma \\ 0.2 : & r = \alpha, \beta \end{cases} \quad \upsilon_{\mathfrak{R}}(r) = \begin{cases} 0.1 : & r = e, \gamma \\ 0.5 : & r = \alpha, \beta \end{cases}$$

Thus, $\mathfrak{R}$ is a neutrosophic subincline of $\mathfrak{T}$.

Theorem 3.3. *A neutrosophici set $\mathfrak{R} = \{r, \zeta_{\mathfrak{R}}(r), \tau_{\mathfrak{R}}(r), \upsilon_{\mathfrak{R}}(r)/r \in \mathfrak{T}\}$ is a neutrosophic subincline of $\mathfrak{T}$ if and only if $\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}$ and $\overline{\upsilon}_{\mathfrak{R}} = 1 - \upsilon_{\mathfrak{R}}$ are all fuzzy subinclines of $\mathfrak{T}$.*

Proof. Let $\mathfrak{R} = (\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}, \upsilon_{\mathfrak{R}})$ be a neutrosophic subincline.
Clearly, $\zeta_{\mathfrak{R}}$ and $\tau_{\mathfrak{R}}$ are fuzzy subinclines of $\mathfrak{U}$. Now,

$$\begin{aligned} \upsilon_{\mathfrak{R}}(r+s) \vee \upsilon_{\mathfrak{R}}(r*s) &\leq \upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s) \\ \therefore\ \overline{\upsilon}_{\mathfrak{R}}(r+s) \wedge \overline{\upsilon}_{\mathfrak{R}}(r*s) &= (1 - \upsilon_{\mathfrak{R}}(r+s)) \wedge (1 - \upsilon_{\mathfrak{R}}(r*s)) \\ &= (1 - (\upsilon_{\mathfrak{R}}(r+s) \vee \upsilon_{\mathfrak{R}}(r*s))) \\ &\geq 1 - (\upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s)) \\ &= (1 - \upsilon_{\mathfrak{R}}(r)) \wedge (1 - \upsilon_{\mathfrak{R}}(s)) \\ &= \overline{\upsilon}_{\mathfrak{R}}(r) \wedge \overline{\upsilon}_{\mathfrak{R}}(s) \end{aligned}$$

Thus, $\overline{\upsilon}_{\mathfrak{R}}$ is also a fuzzy subincline.
Assuming that $\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}$ and $\overline{\upsilon}_{\mathfrak{R}}$ are fuzzy subinclines of $\mathfrak{T}$, then we've

$$\begin{aligned} \upsilon_{\mathfrak{R}}(r+s) \vee \upsilon_{\mathfrak{R}}(r*s) &= (1 - \overline{\upsilon}_{\mathfrak{R}}(r+s)) \vee (1 - \overline{\upsilon}_{\mathfrak{R}}(r*s)) \\ &= 1 - (\overline{\upsilon}_{\mathfrak{R}}(r+s) \wedge \overline{\upsilon}_{\mathfrak{R}}(r*s)) \\ &\geq 1 - (\overline{\upsilon}_{\mathfrak{R}}(r) \wedge \overline{\upsilon}_{\mathfrak{R}}(s)) \\ &= (1 - \overline{\upsilon}_{\mathfrak{R}}(r)) \vee (1 - \overline{\upsilon}_{\mathfrak{R}}(s)) \\ &= \upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s) \end{aligned}$$

Therefore, $\mathfrak{R}$ is a neutrosophic subincline of $\mathfrak{T}$.

Theorem 3.4. *If $\mathfrak{R} = \{\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}, \upsilon_{\mathfrak{R}}\}$ is a neutrosophic subincline of $\mathfrak{T}$ then $\zeta_{\mathfrak{R}}(r+s) = \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)$;$\tau_{\mathfrak{R}}(r+s) = \tau_{\mathfrak{R}}(r) \wedge \tau_{\mathfrak{R}}(s)$ and $\upsilon_{\mathfrak{R}}(r+s) = \upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s)$ $\forall\ r, s\ \in \mathfrak{T}$.*

Theorem 3.5. *The intersection of two neutrosophic subincline is again a neutrosophic subincline.*

Definition 3.6. *Let $\mathfrak{R} = (\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}, \upsilon_{\mathfrak{R}})$ be an neutorosophic subset in $\mathfrak{T}$, a mapping $f : \mathfrak{T} \to \mathfrak{S}$ [$\mathfrak{T}, \mathfrak{S}$ are inclines] then the image of $\mathfrak{R}$, $f(\mathfrak{R})$ is defined as*

$f(\mathfrak{R}) = \{r, f_{sup}(\zeta_{\mathfrak{R}}), f_{sup}(\tau_{\mathfrak{R}}), f_{inf}(\upsilon_{\mathfrak{R}})/r \in \mathfrak{S}\}$, *where*

$$f_{sup}(\zeta_{\mathfrak{R}})(s) = \begin{cases} sup_{f(r)=s}(\zeta_{\mathfrak{R}})(r) & : if \ f(r) = s \\ 0 & : Otherwise \end{cases}$$

$$f_{sup}(\tau_{\mathfrak{R}})(s) = \begin{cases} sup_{f(r)=s}(\tau_{\mathfrak{R}})(r) & : if \ f(r) = s \\ 0 & : Otherwise \end{cases}$$

$$f_{inf}(\upsilon_{\mathfrak{R}})(s) = \begin{cases} inf_{f(r)=s}(\upsilon_{\mathfrak{R}})(r) & : if \ f(r) = s \\ 1 & : Otherwise \end{cases}$$

Theorem 3.7. *Let $\mathfrak{R} = (\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}, \upsilon_{\mathfrak{R}})$ be an neutorosophic set in $\mathfrak{T}$ where $f : \mathfrak{T} \to \mathfrak{S}$ be a mapping. If $\mathfrak{R}$ is a neutrosophic subincline in $\mathfrak{T}$, then $f(\mathfrak{R})$ is also a neutrosophic subincline of $\mathfrak{S}$.*

Proof. Let $f : \mathfrak{T} \to \mathfrak{S}$ be a mapping, $\mathfrak{R}$ is a neutrosophic subincline subincline in $\mathfrak{T}$, then to prove $f(\mathfrak{R})$ is a neutrosophic subincline.
i) $\zeta_f(s_1 + s_2) \wedge \zeta_f(s_1 * s_2) \geq \zeta_f(s_1) \wedge \zeta_f(s_2)$
Since $r_1, r_2 \in \mathfrak{T}$ such that $f(r_1) = s_1$ and $f(r_2) = s_2$ for $s_1, s_2 \in \mathfrak{S}$

$$\begin{aligned} \zeta_f(s_1 + s_2) \wedge \zeta_f(s_1 * s_2) &= (sup\{\zeta_{\mathfrak{R}}(r_1 + r_2)/f(r_1 + r_2) = s_1 + s_2\})\wedge \\ &\quad (sup\{\zeta_{\mathfrak{R}}(r_1 * r_2)/f(r_1 * r_2) = s_1 * s_2\}) \\ &= (sup\{\zeta_{\mathfrak{R}}(r_1 + r_2)/f(r_1) + f(r_2) = s_1 + s_2\})\wedge \\ &\quad (sup\{\zeta_{\mathfrak{R}}(r_1 * r_2)/f(r_1) * f(r_2) = s_1 * s_2\} \\ &= sup\{\zeta_{\mathfrak{R}}(r_1 + r_2) \wedge \zeta_{\mathfrak{R}}(r_1 * r_2)/f(r_1) + f(r_2) = s_1 + s_2, \\ &\quad f(r_1) * f(r_2) = s_1 * s_2\} \\ &\geq sup\{\zeta_{\mathfrak{R}}(r_1) \wedge \zeta_{\mathfrak{R}}(r_2)/f(r_1) = s_1; f(r_2) = s_2\} \\ &= (sup\{\zeta_{\mathfrak{R}}(r_1)/f(r_1) = s_1\}) \wedge (sup\{\zeta_{\mathfrak{R}}(r_2)/f(r_2) = s_2\}) \\ &= \zeta_f(s_1) \wedge \zeta_f(s_2) \end{aligned}$$

Similarly,$\tau_f(s_1 + s_2) \wedge \tau_f(s_1 * s_2) \geq \tau_f(s_1) \wedge \Psi_f(s_2)$ and
$\upsilon_f(s_1 + s_2) \vee \upsilon_f(s_1 * s_2) \leq \upsilon_f(s_1) \vee \upsilon_f(s_2)$.

Definition 3.8. *A mapping f on $\mathfrak{T}$ and if $\mathfrak{R}_2 = (\zeta_{\mathfrak{R}_2}, \tau_{\mathfrak{R}_2}, \upsilon_{\mathfrak{R}_2})$ is a neutrosophic set in $f(\mathfrak{T})$, $\mathfrak{R}_1$ is a neutrosophic set in $\mathfrak{T}$ then the neutrosophic set $\mathfrak{R}_1 = (\zeta_{\mathfrak{R}_1}, \tau_{\mathfrak{R}_1}, \upsilon_{\mathfrak{R}_1})$ is defined as $\mathfrak{R}_2 \circ f$ such that $\zeta_{\mathfrak{R}_1}(r) = (\zeta_{\mathfrak{R}_2} \circ f)(r) = \zeta_{\mathfrak{R}_2}(f(r))$; $\tau_{\mathfrak{R}_1}(r) = (\tau_{\mathfrak{R}_2} \circ f)(r) = \tau_{\mathfrak{R}_2}(f(r))$ and $\upsilon_{\mathfrak{R}_1}(r) = (\upsilon_{\mathfrak{R}_2} \circ f)(r) = \upsilon_{\mathfrak{R}_2}(f(r))$ in $\mathfrak{T}$, this is called preimage of $\mathfrak{R}_2$ under f.*

Theorem 3.9. *An epimorphic inverse image of neutrosophic subincline of $\mathfrak{T}$ is* *also a neutrosophic subincline.*

Proof. Let $\mathfrak{T}, \mathfrak{S}$ be an incline algebra and a mapping $f : \mathfrak{T} \to \mathfrak{S}$ be an epimorphic. $\mathfrak{R}_2$ is a neutrosophic set in $f(\mathfrak{T})$ & $\mathfrak{R}_1$ be the preimage of $\mathfrak{R}_2$ under f. Now, for any $r_1, r_2 \in \mathfrak{T}$

$$\begin{aligned} \zeta_{\mathfrak{R}_1}(r_1 + r_2) \wedge \zeta_{\mathfrak{R}_1}(r_1 * r_2) &= (\zeta_{\mathfrak{R}_2} \circ f)(r_1 + r_2) \wedge (\zeta_{\mathfrak{R}_2} \circ f)(r_1 * r_2) \\ &= \zeta_{\mathfrak{R}_2}(f(r_1 + r_2)) \wedge \zeta_{\mathfrak{R}_2}(f(r_1 * r_2)) \\ &= \zeta_{\mathfrak{R}_2}(f(r_1) + f(r_2)) \wedge \zeta_{\mathfrak{R}_2}(f(r_1) * f(r_2)) \\ &\geq \zeta_{\mathfrak{R}_2}(f(r_1)) \wedge \zeta_{\mathfrak{R}_2}(f(r_2)) \\ &= \zeta_{\mathfrak{R}_1}(r_1) \wedge \zeta_{\mathfrak{R}_1}(r_2) \\ \zeta_{\mathfrak{R}_1}(r_1 + r_2) \wedge \zeta_{\mathfrak{R}_1}(r_1 * r_2) &\geq \zeta_{\mathfrak{R}_1}(r_1) \wedge \zeta_{\mathfrak{R}_1}(r_2) \end{aligned}$$

Similarly for the other membership functions.

4. Neutrosophic Level Set and Product of Ideal

Here, this segment introduces the neutrosophic ideal and discusses the level set and product of on it.

Definition 4.1. *A neutrosophic set* $\mathfrak{R} = \{r, \zeta_{\mathfrak{R}}(r), \tau_{\mathfrak{R}}(r), \upsilon_{\mathfrak{R}}(r)/r \in \mathfrak{T}\}$ *in an incline algebra* $\mathfrak{T}$ *is said to be a neutrosophic ideal of a incline algebra* $\mathfrak{T}$ *if*
(i) $\zeta_{\mathfrak{R}}(r+s) \wedge \Phi_R(r*s) \geq \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)$
(ii) $\tau_{\mathfrak{R}}(r+s) \wedge \tau_{\mathfrak{R}}(r*s) \geq \tau_{\mathfrak{R}}(r) \wedge \tau_R(s)$
(iii) $\upsilon_{\mathfrak{R}}(r+s) \vee \upsilon_{\mathfrak{R}}(r*s) \leq \upsilon_{\mathfrak{R}}(r) \vee \upsilon_{\mathfrak{R}}(s)$
(iv) $\zeta_{\mathfrak{R}}(r) \geq \zeta_{\mathfrak{R}}(s)$ *whenever* $r \leq s$
(v) $\tau_{\mathfrak{R}}(r) \geq \tau_{\mathfrak{R}}(s)$
(vi) $\upsilon_{\mathfrak{R}}(r) \leq \upsilon_{\mathfrak{R}}(s)\ \forall\ r, s \in \mathfrak{T}$.

Example 4.2. A incline algebra $\mathfrak{T} = \{e_1, b_1, c_1, d_1\}$ with two operations $+$ and $*$ defined on $\mathfrak{T}$ with the Cayley's table.

$+$	e_1	b_1	c_1	d_1
e_1	e_1	b_1	c_1	d_1
b_1	e_1	e_1	e_1	e_1
c_1	b_1	e_1	b_1	e_1
d_1	c_1	e_1	e_1	c_1

$*$	e_1	b_1	c_1	d_1
e_1	e_1	e_1	e_1	e_1
b_1	e_1	b_1	c_1	d_1
c_1	e_1	b_1	b_1	e_1
d_1	e_1	c_1	e_1	c_1

Table 2: Neutrosophic ideal

Now $\mathfrak{R} = \{e_1, b_1, c_1, d_1\}$ is ideal on $\mathfrak{T}$ and a subset $\mathscr{H} = \{e_1, b_1\}$ of $\mathfrak{R}$ is a fuzzy ideal is defined by the membership functions,

$$\zeta_{\mathscr{H}}(r) = \begin{cases} 0.7: & r = e_1 \\ 0.4: & r = b_1 \end{cases} \quad \tau_{\mathscr{H}}(r) = \begin{cases} 0.5: & r = e_1 \\ 0.25: & r = b_1 \end{cases} \quad \upsilon_{\mathscr{H}}(r) = \begin{cases} 0.3: & r = e_1 \\ 0.6: & r = b_1 \end{cases}$$

Thus, $\mathscr{H}$ is a neutrosophic ideal of $\mathfrak{T}$.

Definition 4.3. *Let $\mathfrak{T}$ be an universe and $\mathfrak{R}$ be a neutrosophic ideal of $\mathfrak{T}$ then the level subset is defined as $\mathfrak{R}_{u,v,w} = \{\zeta_{\mathfrak{R}}(r) \geq u, \tau_{\mathfrak{R}}(r) \geq v; \upsilon_{\mathfrak{R}}(r) \leq w \in \mathfrak{T}\}$ where $u, v, w \in [0,1]$.*

Theorem 4.4. *$\mathfrak{R}$ of $\mathfrak{T}$ is a neutrosophic ideal if and only if the level susbet $\mathfrak{R}_{u,v,w}$ of $\mathfrak{T}$ are ideal.*

Proof. Let $\mathfrak{R}$ be a neutrosophic ideal of $\mathfrak{T}$ and $r, s \in \mathfrak{R}_u$ & $u \in [0,1]$ then

$$\begin{aligned} \zeta_{\mathfrak{R}}(r+s) \wedge \zeta_{\mathfrak{R}}(r*s) &\geq \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s) \\ &\geq u \wedge u \\ &\geq u \end{aligned}$$

$\implies \zeta_{\mathfrak{R}}(r+s) \geq u$ and $\zeta_{\mathfrak{R}}(r*s) \geq e$
i.e., $(r+s) \in \mathfrak{R}_u$ & $(r*s) \in \mathfrak{R}_u$
Now, let $r \in \mathfrak{R}_u$ & $s \leq r$, then $\zeta_{\mathfrak{R}}(s) \geq \zeta_{\mathfrak{R}}(r) \geq u$ and so $s \in \zeta_{\mathfrak{R}_u}$
$\therefore \zeta_{\mathfrak{R}_u}$ of $\mathfrak{T}$ is an ideal. Similarly, $\zeta_{\mathfrak{R}_f}$ & $\upsilon_{\mathfrak{R}_g}$ of $\mathfrak{T}$ is an ideal.
Conversely, assume that $\mathfrak{R}_{u,v,w}$ are ideals of $\mathfrak{T}$
First to show that $\zeta_{\mathfrak{R}}(r+s) \geq \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)\ \forall r, s \in \mathfrak{T}$
If not, then there exists $r_\circ, s_\circ \in \mathfrak{T}$ such that $\zeta_{\mathfrak{R}}(r_\circ + s_\circ) < \zeta_{\mathfrak{R}}(r_\circ) \wedge \zeta_{\mathfrak{R}}(s_\circ)$
Take $u_\circ = \frac{1}{2}\ \{\zeta_{\mathfrak{R}}(r_\circ + s_\circ) \wedge \zeta_{\mathfrak{R}}(r_\circ * s_\circ) + \zeta_{\mathfrak{R}}(r_\circ) \wedge \zeta_{\mathfrak{R}}(s_\circ))\}$, we have
$\zeta_{\mathfrak{R}}(r_\circ + s_\circ) \wedge \zeta_{\mathfrak{R}}(r_\circ * s_\circ) < u_\circ < \zeta_{\mathfrak{R}}(r_\circ) \wedge \zeta_{\mathfrak{R}}(s_\circ)$
Therefore, $r_\circ, s_\circ \in \zeta_{\mathfrak{R}_{u_\circ}}$ implies $r_\circ + s_\circ \in \zeta_{\mathfrak{R}_{u_\circ}}, r_\circ * s_\circ \in \zeta_{\mathfrak{R}_{u_\circ}}$
This contradicts. Therefore, if $r_\circ, s_\circ \in \mathfrak{T}$ such that $r_\circ \leq s_\circ$ and $\zeta_{\mathfrak{R}}(r_\circ) < \zeta_{\mathfrak{R}}(s_\circ)$
Now take $u_1 = \frac{1}{2}\{\zeta_{\mathfrak{R}}(r_\circ) + \zeta_{\mathfrak{R}}(s_\circ)\}$, then $\zeta_{\mathfrak{R}}(r_\circ) < u_1 < \zeta_{\mathfrak{R}}(s_\circ)$& so on $s_\circ \in \mathfrak{R}_{u_1}$
It follows that the hypothesis $s_\circ \in \mathfrak{R}_{u_1}$ again a contradiction.
Also, conclude the other two functions in the same way.
Thus, $\mathfrak{R}$ is a neutrosophic ideal of $\mathfrak{T}$.

Theorem 4.5. *Let $\mathscr{H}$ be a ideal of an incline algebra $\mathfrak{T}$ and let $\mathfrak{R}$ be defined as $\zeta_{\mathfrak{R}}(r) = \begin{cases} u_1 & if \quad r \in \mathscr{H} \\ u_2 & if \quad r \notin \mathscr{H} \end{cases}$; $\tau_{\mathfrak{R}}(r) = \begin{cases} v1 & if \quad r \in \mathscr{H} \\ v_2 & if \quad r \notin \mathscr{R} \end{cases}$ and $\upsilon_{\mathfrak{R}}(r) = \begin{cases} w_1 & if \quad r \in \mathscr{H} \\ w_2 & if \quad r \notin \mathscr{H} \end{cases}$ where u_1, u_2, v_1, v_2, w_1 and $w_2 \in [0,1]$ also $u_1 > u_2; v_1 > v_2$; $w_1 > w_2$. Then $\mathfrak{R}$ is a neutrosophic ideal of $\mathfrak{T}$ and $\mathfrak{R}_{u_1,v_1,w_1} = \mathscr{H}$.*

Proof. Let $r, s \in \mathscr{H}$. Now, if r or $s \notin \mathscr{H}$, then its clear that

$$\begin{aligned}\zeta_{\mathfrak{R}}(r+s) \wedge \zeta_{\mathfrak{R}}(r*s) &\geq u_2 \\ &= \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)\end{aligned}$$

If $r \in \mathscr{H}$ & $s \in \mathscr{H}$,then $r+s \in \mathscr{H}$ & $r*s \in \mathscr{H}$

$$\begin{aligned}\implies \zeta_{\mathfrak{R}}(r+s) \wedge \zeta_{\mathfrak{R}}(r*s) &\geq u_1 \\ &= \zeta_{\mathfrak{R}}(r) \wedge \zeta_{\mathfrak{R}}(s)\end{aligned}$$

Now, assume that if $r \in \mathscr{H}$ & $s \leq r$ then $s \in \mathscr{H}$
$\mathscr{H}$ is an ideal of $\mathfrak{T}$, $i.e., \zeta_{\mathfrak{R}}(r) = \zeta_{\mathfrak{R}}(s)$
Suppose if $r \notin \mathscr{H}$, then $\zeta_{\mathfrak{R}}(r) = u_2 \geq \zeta_{\mathfrak{R}}(s)$.
$\therefore \zeta_{\mathfrak{R}}$ is a neutrosophic ideal of $\mathfrak{T}$.
And the proof for $\tau_{\mathfrak{R}}$ & $\upsilon_{\mathfrak{R}}$ is also in the same procedure.
Also, $\mathfrak{R}_{u_1,v_1,w_1} = H$.
Hence the proof.

Corollary 4.6. *Any ideal of an incline algebra $\mathfrak{T}$ can be realized as the level ideal for some neutrosophic ideal of $\mathfrak{T}$.*

Theorem 4.7. *Let $\{\mathscr{H}_{u,v,w}/u,v,w \in \Psi\}$ is a collections of ideals of $\mathfrak{T}$ such that $\mathfrak{T} = \cup_{u,v,w\in\Psi} H_{u,v,w}$ and $u,v,w,u_\circ,v_\circ,w_\circ \in \Psi$ iff $\mathscr{H}_{u_\circ,v_\circ,w_\circ} \subset \mathscr{H}_{u,v,w}$ then $\mathfrak{R} = \{\zeta_{\mathfrak{R}}, \tau_{\mathfrak{R}}, \upsilon_{\mathfrak{R}}\}$ in $\mathfrak{T}$ is defined by $\zeta_{\mathfrak{R}}(r) = sup\{u/r \in \mathscr{H}_u\}, \tau_{\mathfrak{R}}(r) = sup\{v/r \in \mathscr{H}_v\}; \upsilon_{\mathfrak{R}}(r) = inf\{w/r \in \mathscr{H}_w\}\ \forall\ r \in \mathfrak{T}$ is a neutrosophic ideal of $\mathfrak{T}$.*
Proof. To prove $\mathfrak{R}_{u,v,w}$ are ideals of $\mathfrak{T}$. Let us prove this by segregrating into the following cases.
$(i)\ u = \{u_1 \in \Psi/u_1 < u\}$ and $(ii) u_\circ \neq \{u_1 \in \Psi/u_1 < u\}$
$(i)^{st}$ implies that $r \in \zeta_{\mathfrak{R}_u} \Leftrightarrow \mathscr{H}_{u_1}\ \forall\ u_1 < u$
$\Leftrightarrow r \in \cap_{u_1<u}\mathscr{H}_{u_1}, i.e., \zeta_{\mathfrak{R}_u} = \cap_{u_1<u}\mathscr{H}_{u_1}$
This is an ideal of $\mathfrak{T}$
(ii) To claim $\zeta_{\mathfrak{R}_u} = \cup_{u_1\geq u}\mathscr{H}_{u_1}$
If $r \in \cup_{u_1\geq u}\mathscr{H}_{u_1}$ then $r \in \mathscr{R}_{u_1}$ for $u_1 \geq u$.
$i.e., \zeta_{\mathfrak{R}}(r) \geq u_1 \geq u$ so that $r \in \zeta_{\mathfrak{R}_u}$
$\therefore \cup_{u_1\geq u}\mathscr{H}_{u_1} \subseteq \zeta_{\mathfrak{R}_u}$
Let us now take $r \notin \cup_{u_1\geq u}\mathscr{H}_{u_1}$, then $r \notin \mathscr{H}_{u_1}$ for $u_1 \geq u$
Since $u \neq sup\{u_1 \in \Psi/u_1 < u\}$, $\ni \epsilon > 0$ such that $(u-\epsilon, u) \cup \Psi = \phi$.
Hence $r \notin \mathscr{H}_{u_1}$ for $u_1 > u - \epsilon$ that is if $r \in \mathscr{H}_{u_1}$ then $u \geq u - \epsilon$ thus
$\zeta_{\mathfrak{R}}(r) \leq u - \epsilon < u$ and $r \notin \zeta_{\mathfrak{R}_u}$
$\therefore \zeta_{\mathfrak{R}_u} \subseteq \cup_{u_1\geq u}\mathscr{H}_{u_1}$

$\zeta_{\mathfrak{R}_u} = \cup_{u_1 \geq u} \mathscr{H}_{u_1}$ is an ideal of $\mathfrak{T}$.
Similarly, for $\tau_{\mathfrak{R}_v}$ & $\upsilon_{\mathfrak{R}_w}$
Hence $\mathfrak{R}$ is a neutrosophic ideal of $\mathfrak{T}$.

Definition 4.8. *Let $\mathfrak{R}_1 \in \mathfrak{T}_1$ and $\mathfrak{R}_2 \in \mathfrak{T}_2$ then the product of $\mathfrak{R}_1$ & $\mathfrak{R}_2$ is the element of $\mathfrak{T}_1 \times \mathfrak{T}_2$ and is defined by*
$(\mathfrak{R}_1 \times \mathfrak{R}_2)(r,s) = \{(r,s), \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r,s), \tau_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r,s); \upsilon_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r,s)/r,s \in \mathfrak{T}_1 \times \mathfrak{T}_2\}$, *where* $\zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2} = \zeta_{\mathfrak{R}_1}(r) \wedge \zeta_{\mathfrak{R}_2}(s); \tau_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r,s) = \tau_{\mathfrak{R}_1}(u) \wedge \tau_{\mathfrak{R}_2}(v)$ *and* $\upsilon_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r,s) = \upsilon_{\mathfrak{R}_1}(r) \vee \upsilon_{\mathfrak{R}_1}(s)\ \forall\ r,s \in \mathfrak{T}_1 \times \mathfrak{T}_2$.

Theorem 4.9. *If $\mathfrak{R}_i$ is a neutrosophic ideals of $\mathfrak{T}_i$, then $\mathfrak{R}_1 \times \mathfrak{R}_2$ is a neutrosophic ideal of $\mathfrak{T}_1 \times \mathfrak{T}_2$.*
Proof. Let $(r_1,r_2),(s_1,s_2) \in \mathfrak{T}_1 \times \mathfrak{T}_2$ then

$$\begin{aligned}
&(i)(\zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}((r_1,r_2)+(s_1,s_2)) \wedge \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}((r_1,r_2)*(s_1,s_2)) \\
&\quad = \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}((r_1+s_1),(r_2+s_2)) \wedge \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}((r_1*s_1),(r_2*s_2)) \\
&\quad = (\zeta_{\mathfrak{R}_1}(r_1+s_1) \wedge \zeta_{\mathfrak{R}_2}(r_2+s_2)) \wedge (\zeta_{\mathfrak{R}_1}(r_1*s_1) \wedge \zeta_{\mathfrak{R}_2}(r_2*s_2)) \\
&\quad = (\zeta_{\mathfrak{R}_1}(r_1+s_1) \wedge (\zeta_{\mathfrak{R}_1}(r_1*s_1)) \wedge (\zeta_{\mathfrak{R}_2}(r_2+s_2) \wedge \zeta_{\mathfrak{R}_2}(r_2*s_2)) \\
&\quad = (\zeta_{\mathfrak{R}_1}(r_1) \wedge \zeta_{\mathfrak{R}_1}(s_1)) \wedge (\zeta_{\mathfrak{R}_2}(r_2) \wedge \zeta_{\mathfrak{R}_2}(s_2)) \\
&\quad = (\zeta_{\mathfrak{R}_1}(r_1) \wedge \zeta_{\mathfrak{R}_2}(r_2)) \wedge (\zeta_{\mathfrak{R}_1}(s_1) \wedge \zeta_{\mathfrak{R}_2}(s_2)) \\
&\quad = (\zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r_1,r_2) \wedge (\zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2})(s_1,s_2)
\end{aligned}$$

(ii) By using the above lemma "If $\mathfrak{R}_1 \in \mathfrak{T}_1$ and $\mathfrak{R}_2 \in \mathfrak{T}_2$ satisfy order reversing property then so is $\mathfrak{R}_1 \times \mathfrak{R}_2 \in \mathfrak{T}_1 \times \mathfrak{T}_2$, it concludes that when $r \leq s$"
$\implies \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}(r) \geq \zeta_{\mathfrak{R}_1 \times \mathfrak{R}_2}(s)$.
Hence, the same process is carried out for indeterminate and falsity function.
$\therefore \mathfrak{R}_1 \times \mathfrak{R}_2$ is a neutrosophic ideal of $\mathfrak{T}_1 \times \mathfrak{T}_2$.

Definition 4.10. *Let $\mathfrak{R}_1 \in \mathfrak{T}_1$, the projection of $\mathfrak{R}_1$ on $\mathfrak{T}_1$ is the neutrosophic subset and its represented as $proj(\mathfrak{R}_1) \in \mathfrak{T}_1$ is defined by $proj_1(\zeta_{\mathfrak{R}_1})(r) = sup\{\zeta_{\mathfrak{R}_1}(r,s)/s \in \mathfrak{T}_2\}, proj_1(\tau_{\mathfrak{R}_1})(r) = sup\{\tau_{\mathfrak{R}_1}(r,s)/s \in \mathfrak{T}_2\}$ and $proj_1(\upsilon_{\mathfrak{R}_1})(r) = inf\{\upsilon_{\mathfrak{R}_1}(r,s)/s \in \mathfrak{T}_2\}\ \ \forall r \in \mathfrak{T}_1$ respectively, $proj_2(\zeta_{\mathfrak{R}_2})(s) = sup\{\zeta_{\mathfrak{R}_2}(r,s)/r \in \mathfrak{T}_1\}$ and similarly for $proj_2(\tau_{\mathfrak{R}_2})(s), proj_1(\upsilon_{\mathfrak{R}_2})(s)\ \ \forall\ s \in \mathfrak{T}_2$.*

Theorem 4.11. *Let $\mathfrak{T}_2$ be an idempotent incline and $\mathfrak{R}_1$ in $\mathfrak{T}_1 \times \mathfrak{T}_2$ is a neutrosophic ideal, then the projection $proj_i(\mathfrak{R}_1)$ are neutrosophic ideal of $\mathfrak{T}_i, i = 1,2,....$ respectively.*
Proof. For $r,s \in \mathfrak{T}_1$

$$\begin{aligned}
&proj_1(\zeta_{\mathfrak{R}_1})(r+s) \wedge proj_1(\zeta_{\mathfrak{R}_1})(r*s) \\
&= sup\{\zeta_{\mathfrak{R}_1}(r+s,t)/t \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}(r*s,w)/w \in \mathfrak{T}_2\}
\end{aligned}$$

$= sup\{\zeta_{\mathfrak{R}_1}(r+s, t+t)/t \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}(r*s, w*w)/w \in \mathfrak{T}_2\}$
$= sup\{\zeta_{\mathfrak{R}_1}((r+t),(s+t))/t \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}((r*w),(s*w))/w \in \mathfrak{T}_2\}$
$\geq sup\{\zeta_{\mathfrak{R}_1}(r,t) \wedge \zeta_{\mathfrak{R}_1}(s,t)/t \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}(r,s) \wedge \zeta_{\mathfrak{R}_1}(s,w)/w \in \mathfrak{T}_2\}$
$= (sup\{\zeta_{\mathfrak{R}_1}(r,t)/t \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}(s,t)/t \in \mathfrak{T}_2\})\wedge$
$(sup\{\zeta_{\mathfrak{R}_1}(r,w)/w \in \mathfrak{T}_2\} \wedge sup\{\zeta_{\mathfrak{R}_1}(t,w)/w \in \mathfrak{T}_2\})$
$= proj_1(\zeta_{\mathfrak{R}_1})(r) \wedge proj_1(\zeta_{\mathfrak{R}_1})(t)$

Now assume $\mathfrak{R}_1 \in \mathfrak{T}_1 \times \mathfrak{T}_2$ is order reversing and $r \leq s$ in $\mathfrak{T}_1$ then

$$\begin{aligned} proj_1(\zeta_{\mathfrak{R}_1}(r)) &= sup\{\zeta_{\mathfrak{R}_1}(r,t)/t \in \mathfrak{T}_2\} \\ &\leq sup\{\zeta_{\mathfrak{R}_1}(s,t)/t \in \mathfrak{T}_2\} \\ &= proj_1(\zeta_{\mathfrak{R}_1})(s) \end{aligned}$$

Therefore, $proj_1(\zeta_{\mathfrak{R}_1}))$ is a neutrosophic ideal of $\mathfrak{T}_1$.
Similarly, for the other two components.
Hence, $proj_i(\zeta_{\mathfrak{R}_1})$ is a neutrosophic ideal of $\mathfrak{T}_i (i = 1, 2,)$

5. Conclusion

In this paper, the concept of neutrosophic subincline and ideal of incline algebra is proposed also studies its properties such as homomorphic image, preimage, level and cartesian product are discussed. This also help to promote the fusion between the fuzzy and incline theory. This can also be referred to other algebraic structures.

References

[1] Ahn, S. S., Jun, S. S., Kim, H. S., Ideals and Quotients of Incline Algebras, Comm. Koren Math. Soc., 16 (4), (2001), 573-583.

[2] Cao, Z., A new algebraic system - slope, B usefal 6(1), (1981), 22-27.

[3] Jun, Y. B. S. S. Ahn, H. S. Kim, Fuzzy Subincline (Ideals) of Incline Algebras, Fuzzy Sets and Systems, 123 (2001), 217-255.

[4] Kim, K. H., Roush, F. W., Inclines of algebraic structures, Fuzzy Sets and Systems, 72 (1995), 189-196.

[5] Kim, K. H., Roush F. W., Markowsky, G., Representation of Incline Algebras, Algebra Coll, 4 (1997), 461-470.

[6] Smarandache, F., Neutrosophic Set, A generalization of Intuitionistic fuzzy sets, International Journal of Pure and Applied Mathematics, 24 (5), (2005), 287-297.

[7] Venkata Arvinda Raju, A Note on Incline Algebras, International Journal of Mathematical Archive, 8 (9), (2017) 154-157.

[8] Volety, V. S., Ramachandram, On some properties of Inclines, Journal of Science & Arts, 1 (18), (2012), 13-16.

[9] Zadeh, L. A., Fuzzy Sets, Information and Control, 8 (3), (1965), 338-353.