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ON TWO INTERESTING SOMOS'S THETA FUNCTION IDENTITIES OF LEVEL 14

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Abstract: Michael Somos has discovered around 6300 theta function identities using computer and runs PARI/GP scripts. In this paper, we give a proof of two Somos's interesting and elegant theta function identities of level 14 and also we derive two P-Q theta function identities of level 14 due to Ramanujan.

Keywords and Phrases: Theta functions, Dedekind eta-function.

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1. Introduction

Throughout the paper, we assume $|q| < 1$ and for each positive integer n, we use the standard notation

$$(a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right) \qquad \text{and} \qquad (a;q)_\infty := \prod_{n=0}^{\infty}\left(1-aq^n\right).$$

Ramanujan's theta functions are defined by

$$\varphi(q) \;:=\; \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$
$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$

and

$$\chi(q) := (-q;q^2)_\infty.$$

Note that, if $q = e^{2\pi i\tau}$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function. The theta function identity which relates $f(-q)$ to $f(-q^n)$ is called theta function identity of level n. For convenience we set $f_n = f_n(q) = f(-q^n)$.

M. Somos discovered several theta function identites of level 14. In [6] K R Vasuki and R G Veeresha have proved 24 theta function identites of level 14. Following are two interesting theta function identities of level 14 discovered by Somos:

$$\varphi^4(-q) + 7\varphi^4(-q^7) = 8\chi(-q)\chi(-q^7)\left[f_2^4 + 7q^2 f_{14}^4\right] \tag{1.1}$$

and

$$f_1^4 + 7qf_7^4 = \chi(-q)\chi(-q^7)\left[\psi^4(q) + 7q^3\psi^4(q^7)\right]. \tag{1.2}$$

To the best of our knowledge, no proof of the above identities are constructed in the literature, also these identities are the most elegant and interesting among the existing theta function identities of level 14. The aim of this paper is to prove (1.1) and (1.2) by employing the following well-known Ramanujan's theta function identities of level 14:

$$\frac{\varphi^2(q)}{\varphi^2(q^7)} = \frac{\varphi(q)\varphi(q^7)\left[1 - 8q^2\dfrac{\chi(q)}{\chi^7(q^7)}\right]}{\varphi(-q^2)\varphi(-q^{14}) - 2q\psi(q)\psi(q^7)} \tag{1.3}$$

and

$$7\frac{\varphi^2(q^7)}{\varphi^2(q)} = \frac{\varphi(q)\varphi(q^7)\left[1 - 8\dfrac{\chi(q^7)}{\chi^7(q)}\right]}{2q\psi(q)\psi(q^7) - \varphi(-q^2)\varphi(-q^{14})}. \tag{1.4}$$

Ramanujan has not recorded (1.3) and (1.4) explicitly in the above form, instead he recorded in the form of modular equation of degree 7 [1, Entry 19(ii), p. 314]. Further in this paper we prove two existing theta function identities of level 14.

We close this section by recalling an interesting identity of Ramanujan, that we also need to prove (1.1) and (1.2).

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{1.5}$$

2. Proof of Somos's Identities

Proof of (1.1). Adding (1.3) and (1.4), we find that

$$\varphi^4(q) + 7\varphi^4(q^7) = \frac{8q\dfrac{\varphi^3(q)\varphi^3(q^7)}{\chi^3(q)\chi^3(q^7)}\left[\dfrac{\chi^4(q^7)}{q\,\chi^4(q)} - q\dfrac{\chi^4(q)}{\chi^4(q^7)}\right]}{\varphi(-q^2)\varphi(-q^{14}) - 2q\psi(q)\psi(q^7)}. \tag{2.1}$$

Now using the fact that $\varphi(q) = f_2\chi^2(q)$ in (1.3), we find that

$$\frac{f_2^2}{f_{14}^2} = \frac{q\varphi(q)\varphi(q^7)\left[\dfrac{\chi^4(q^7)}{q\,\chi^4(q)} - \dfrac{8q}{\chi^3(q)\chi^3(q^7)}\right]}{\varphi(-q^2)\varphi(-q^{14}) - 2q\psi(q)\psi(q^7)}. \tag{2.2}$$

Similarly, from (1.4), we have

$$7q^2\,\frac{f_{14}^2}{f_2^2} = \frac{q\varphi(q)\varphi(q^7)\left[\dfrac{8q}{\chi^3(q)\chi^3(q^7)} - \dfrac{q\,\chi^4(q)}{\chi^4(q^7)}\right]}{\varphi(-q^2)\varphi(-q^{14}) - 2q\psi(q)\psi(q^7)}. \tag{2.3}$$

Adding (2.2) and (2.3), we deduce that

$$f_2^4 + 7q^2 f_{14}^4 = \frac{q\varphi(q)\varphi(q^7)f_2^2 f_{14}^2\left[\dfrac{\chi^4(q^7)}{q\,\chi^4(q)} - \dfrac{q\,\chi^4(q)}{\chi^4(q^7)}\right]}{\varphi(-q^2)\varphi(-q^{14}) - 2q\psi(q)\psi(q^7)}. \tag{2.4}$$

Dividing (2.1) by (2.4), and then using $\varphi(q) = f_2\chi^2(q)$, we find that

$$\frac{\varphi^4(q) + 7\varphi^4(q^7)}{f_2^4 + 7q^2 f_{14}^4} = 8\chi(q)\chi(q^7).$$

Equivalently,

$$\varphi^4(q) + 7\varphi^4(q^7) = 8\chi(q)\chi(q^7)\left[f_2^4 + 7q^2 f_{14}^4\right]. \tag{2.5}$$

Replacing q by $-q$ in (2.5), we obtain (1.1).

Proof of (1.2). Subtracting (2.5) from (2.1), and then employing (1.5), we obtain

$$16q\psi^4(q^2) + 112\psi^4(q^{14}) = 8\left[f_2^4 + 7q^2 f_{14}^4\right]\left[\chi(q)\chi(q^7) - \chi(-q)\chi(-q^7)\right].$$

Now using the well-known identity of Ramanujan [1, Entry 19(i), p. 314]

$$\chi(q)\chi(q^7) - \chi(-q)\chi(-q^7) = \frac{2q}{\chi(-q^2)\chi(-q^{14})} \tag{2.6}$$

in the above, and then replacing q by $q^{1/2}$ and then dividing throughout by $16q^{1/2}$, we obtain (1.2).

Theorem 1. *[2, Entry 55, p. 209]* [3] *Setting*

$$P := \frac{f_1^2}{q^{1/2}\ f_7^2} \qquad \text{and} \qquad Q := \frac{f_2^2}{q\ f_{14}^2},$$

we have

$$PQ + \frac{49}{PQ} = \left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 - 8\left(\frac{P}{Q} + \frac{Q}{P}\right).$$

Proof. Using $\psi(q) = \dfrac{f_2^2}{f_1}$ in (1.2) and then dividing throughout by $q^{1/2}f_1^2f_7^2$, we find that

$$\frac{f_1^2}{q^{1/2}\ f_7^2} + 7\frac{q^{1/2}\ f_7^2}{f_1^2} = \chi(-q)\chi(-q^7)\left[\frac{f_2^8}{q^{1/2}\ f_1^6 f_7^2} + 7\frac{q^{5/2}\ f_{14}^8}{f_1^2 f_7^6}\right]. \tag{2.7}$$

Similarly using $\varphi(-q) = \dfrac{f_1^2}{f_2}$ in (1.1) and then dividing throughout by $q\ f_2^2 f_{14}^2$, we find that

$$Q + \frac{7}{Q} = \frac{1}{8\chi(-q)\chi(-q^7)}\left[\frac{f_1^8}{q\ f_2^6 f_{14}^2} + 7\frac{f_7^8}{q\ f_2^2 f_{14}^6}\right].$$

Now multiplying (2.7) with the above, we obtain the required result.

Theorem 2. [5] *If*

$$P := \frac{\varphi(-q)}{q^{1/4}\ \psi(q^2)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^7)}{q^{7/4}\ \psi(q^{14})},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = (PQ)^{3/2} + \frac{64}{(PQ)^{3/2}} + 7\left(PQ + \frac{16}{PQ}\right) + 28\left(\sqrt{PQ} + \frac{4}{\sqrt{PQ}}\right).$$

Proof. Using the facts that $\varphi(-q) = \dfrac{f_1^2}{f_2}$, $\psi(q) = \dfrac{f_2^2}{f_1}$ and $\chi(-q) = \dfrac{f_1}{f_2}$ in the definition of P and Q, we see that

$$\frac{P}{Q} = q^{3/2}\frac{\chi^2(-q)\chi^2(-q^2)}{\chi^2(-q^7)\chi^2(-q^{14})}$$

and

$$PQ = \frac{\chi^2(-q)\chi^2(-q^7)\chi^2(-q^2)\chi^2(-q^{14})}{q^2}.$$

From [1, Entry 19(ix), p. 315] and [4], we have

$$\frac{\chi^4(-q^7)}{q\,\chi^4(-q)} + \frac{q\,\chi^4(-q)}{\chi^4(-q^7)} = \frac{\chi^3(-q)\chi^3(-q^7)}{q} + \frac{8q}{\chi^3(-q)\chi^3(-q^7)} + 7. \tag{2.8}$$

Replacing q by q^2 in the above, we find that

$$\frac{\chi^4(-q^{14})}{q^2\,\chi^4(-q^2)} + \frac{q^2\,\chi^4(-q^2)}{\chi^4(-q^{14})} = \frac{\chi^3(-q^2)\chi^3(-q^{14})}{q^2} + \frac{8q^2}{\chi^3(-q^2)\chi^3(-q^{14})} + 7.$$

Multiplying the above identity by (2.8), we obtain

$$\begin{aligned}\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = (PQ)^{3/2} + \frac{64}{(PQ)^{3/2}} + \frac{7\,\chi^3(q)\chi^3(q^7)}{q} \\ + \frac{7\,\chi^3(-q)\chi^3(-q^7)}{q} + \frac{56q}{\chi^3(-q)\chi^3(-q^7)} \\ + \frac{7\,\chi^3(-q^2)\chi^3(-q^{14})}{q^2} + \frac{56\,q^2}{\chi^3(-q^2)\chi^3(-q^{14})} + 56,\end{aligned} \tag{2.9}$$

where we have used $\chi(q)\chi(-q) = \chi(-q^2)$ and (2.8) with q replaced by $-q$. Multiplying (2.6) throughout by $\dfrac{\chi^2(-q)\chi^2(-q^7)}{q}$, we obtain

$$\sqrt{PQ} = \frac{\chi^3(-q)\chi^3(-q^7)}{q} + 2\frac{\chi(-q)\chi(-q^7)}{\chi(q)\chi(q^7)}. \tag{2.10}$$

Dividing (2.6) by $\chi(q)\chi(q^7)$, we obtain

$$\frac{\chi(-q)\chi(-q^7)}{\chi(q)\chi(q^7)} = 1 - \frac{2}{\sqrt{X}},$$

where

$$X = \frac{\chi^2(q)\chi^2(q^7)\chi^2(-q^2)\chi^2(-q^{14})}{q^2}.$$

Using the above in (2.10), we find that

$$\frac{\chi^3(-q)\chi^3(-q^7)}{q} = \sqrt{PQ} + \frac{4}{\sqrt{X}} - 2. \tag{2.11}$$

Replacing q by $-q$ in the above and then using the fact that as $q \longrightarrow -q$, $\sqrt{X} = -\sqrt{PQ}$, we find that

$$\frac{\chi^3(q)\chi^3(q^7)}{q} = \sqrt{X} + \frac{4}{\sqrt{PQ}} + 2. \tag{2.12}$$

Multiplying (2.6) by $\dfrac{q}{\chi^2(-q)\chi^2(-q^7)\chi(-q^2)\chi(-q^{14})}$, we obtain

$$\frac{q}{\chi^3(-q)\chi^3(-q^7)} = \frac{1}{\sqrt{PQ}} + \frac{2}{PQ}. \tag{2.13}$$

Multiplying (2.6) by $\dfrac{q}{\chi^2(-q^2)\chi^2(-q^{14})}$, we obtain

$$\frac{2\ q^2}{\chi^3(-q^2)\chi^3(-q^{14})} = \frac{1}{\sqrt{PQ}} - \frac{1}{\sqrt{X}}. \tag{2.14}$$

Again multiplying (2.6) by $\dfrac{\chi(-q)\chi(-q^7)\chi^2(-q^2)\chi^2(-q^{14})}{q^2}$, we obtain

$$\frac{\chi^3(-q^2)\chi^3(-q^{14})}{q^2} = PQ + 2\sqrt{PQ}. \tag{2.15}$$

Now using (2.11), (2.12), (2.13), (2.14) and (2.15) in (2.9), we obtain

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = (PQ)^{3/2} + \frac{64}{(PQ)^{3/2}} + 7\left(PQ + \frac{16}{PQ}\right) \\ + 28\left(\sqrt{PQ} + \frac{4}{\sqrt{PQ}}\right) + 70 - 7\sqrt{PQ} - 14 + 7\sqrt{X}.$$

From (2.6), we see that $7\sqrt{PQ}+14-7\sqrt{X} = 0$. Using this in the above, we obtain the required result.

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