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CONGRUENCES FOR BIPARTITIONS WITH ODD DESIGNATED SUMMANDS

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Abstract: Andrews, Lewis and Lovejoy investigated a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. Let $B_2(n)$ count the number of bipartitions of n with designated summands in which all parts are odd. In this work, we establish many infinite families of congruences modulo powers of 2 and 3 for $B_2(n)$. For example, for each $n \geq 0$ and $\alpha \geq 0$,

$$B_2\left(48 \cdot 5^{2\alpha+2}n + a_1 \cdot 5^{2\alpha+1}\right) \equiv 0 \pmod 9,$$

where $a_1 \in \{88, 136, 184, 232\}$.

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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. A partition is a ℓ-regular partition of n if none of the part is divisible by ℓ.

Andrews, Lewis and Lovejoy [1] investigated a new class of partitions with designated summands by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of n with designated summands is denoted by $PD(n)$. The generating function for $PD(n)$ is given by

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}, \tag{1.1}$$

where

$$f_n := \prod_{j=1}^{\infty}(1 - q^{nj}), n \geq 1. \tag{1.2}$$

For example $PD(4) = 10$, namely

$$4', \quad 3' + 1', \quad 2' + 2, \quad 2 + 2', \quad 2' + 1' + 1, \; 2' + 1 + 1', \quad 1' + 1 + 1 + 1,$$
$$1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'.$$

Mahadeva Naika and Gireesh [10] studied $PD_3(n)$, the number of partitions of n with designated summands whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{1.3}$$

Andrews et al. [1], Baruah and Ojah [2] have also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. The generating function for $PDO(n)$ is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{1.4}$$

Mahadeva Naika and Shivashankar [16] established many congruences for $BPD(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} BPD(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \tag{1.5}$$

For more details, one can see [5, 11, 12, 13, 14, 15, 18].

Motivated by the above works, in this paper, we define $B_2(n)$, the number of bipartitions of n with odd designated summands. The generating function for $B_2(n)$ is given by

$$\sum_{n=0}^{\infty} B_2(n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_3^2 f_{12}^2}. \tag{1.6}$$

In this paper, we list few dissection formulas which are helps to prove our main results in section 2. In section 3, we obtain many infinite families of congruences for $B_2(n)$ modulo powers of 2 and congruences modulo powers of 3 in section 4.

2. Preliminary Results

In this section, we list few dissection formulas which are helps to prove our main results.

Lemma 2.1. *The following* 2*-dissections hold:*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.1}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{2.2}$$

The equation (2.1) is essentially (1.9.4) in [6]. The equation (2.2) can be obtained from (2.1) by replacing q by $-q$. Also, see [4, p. 40, Entry 25].

Lemma 2.2. *The following* 3*-dissection holds:*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{2.3}$$

Identity (2.3) is nothing but Lemma 2.6 in [2].

Lemma 2.3. *The following* 3*-dissections hold:*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.4}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9} \tag{2.5}$$

and

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.6}$$

Lemma 2.3 was proved by Hirschhorn and Sellers [9].

Lemma 2.4. *The following* 2*-dissections hold:*

$$\frac{1}{f_1f_3} = \frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}, \tag{2.7}$$

$$f_1f_3 = \frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}. \tag{2.8}$$

The equation (2.7) was proved by Baruah and Ojah [2]. Replacing q by $-q$ in (2.7) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we get (2.8).

Lemma 2.5. *The following* 2*-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.9}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}, \tag{2.10}$$

$$\frac{f_1}{f_3^3} = \frac{f_2f_4^2f_{12}^2}{f_6^7} - q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}, \tag{2.11}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}. \tag{2.12}$$

Hirschhorn et al. [7] proved (2.9). For a proof of (2.10), we can see [3]. The proof of the equations (2.11) and (2.12) follows by replacing $-q$ by q in the equations (2.9) and (2.10) respectively with the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$.

Lemma 2.6. *The following* 3*-dissection holds:*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}. \tag{2.13}$$

For a proof, we can see [8].

Lemma 2.7. *The following* 3*-dissection holds:*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4f_9^5}{f_3^8f_{18}} + 6q^2\frac{f_6^3f_9^2f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2f_{18}^5}{f_3^6f_9}. \tag{2.14}$$

The identity (2.14) was proved by Toh [17].

Lemma 2.8. [6, p. 85, 8.1.1] *We have the following 5-dissection formula*

$$f_1 = f_{25}\left(a(q^5) - q - q^2/a(q^5)\right), \tag{2.15}$$

where

$$a := a(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{2.16}$$

Lemma 2.9. *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.17}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.
Lemma 2.9 is an exercise in [6], see [6, 10.5]. Also, we can see [4, p.303, Entry 17(v)].

Lemma 2.10. *For positive integers k and m, we have*

$$f_k^{3m} \equiv f_{3k}^m \pmod 3, \tag{2.18}$$

$$f_k^{9m} \equiv f_{3k}^{3m} \pmod 9, \tag{2.19}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod 4 \tag{2.20}$$

and

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod 8. \tag{2.21}$$

3. Congruences modulo powers of 2

Theorem 3.1. *Let $c_1 \in \{23, 47, 71, 119\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have*

$$B_2(24n+7) \equiv 0 \pmod{16}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 5^{2\beta}n + 19\cdot 5^{2\beta}\right) q^n \equiv 8 f_1 f_3^6 \pmod{16}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 5^{2\beta+1}n + 23\cdot 5^{2\beta+1}\right) q^n \equiv 8 q^3 f_5 f_{15}^6 \pmod{16}, \tag{3.3}$$

$$B_2\left(24\cdot 5^{2\beta+2}n + c_1\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}, \tag{3.4}$$

$$B_2\left(12\cdot 5^{2\beta+2}n + 5^{2\beta+2}\right) \equiv 3^{\beta+1}\cdot B_2(12n+1) \pmod 8, \tag{3.5}$$

$$B_2\left(60(5n+i)+25\right) \equiv 0 \pmod 8, \tag{3.6}$$

where $i = 1, 2, 3, 4$.

Proof. From the equation (1.6), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(n)q^n &= \frac{f_4^2 f_6^4}{f_{12}^2}\left(\frac{1}{f_1 f_3}\right)^2 \\ &= \frac{f_4^2 f_6^4}{f_{12}^2}\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right)^2, \end{aligned} \tag{3.7}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(2n)q^n = \frac{f_4^4 f_6^8}{f_1^4 f_3^4 f_{12}^4} + q\frac{f_2^{12} f_{12}^4}{f_1^8 f_4^4 f_6^4} \tag{3.8}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(2n+1)q^n &= 2\frac{f_2^6 f_6^2}{f_1^6 f_3^2} \\ &= 2\frac{f_6^2}{f_3^2}\left(\frac{f_2^3}{f_1^3}\right)^2 \qquad (3.9) \\ &= 2\frac{f_6^2}{f_3^2}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right)^2, \qquad (3.10)\end{aligned}$$

which implies

$$\sum_{n=0}^{\infty} B_2(6n+1)q^n = 2\frac{f_2^4}{f_1^4} + 72\frac{qf_2^9 f_3^7 f_6}{f_1^{17}} + 48\frac{qf_2^5 f_6^5}{f_1^9 f_3} + 288\frac{q^2 f_2^6 f_6^{10}}{f_1^{14} f_3^2}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} B_2(6n+3)q^n = 12\frac{f_2^7 f_3^5}{f_1^{11} f_6} + 216\frac{qf_2^8 f_3^4 f_6^4}{f_1^{16}} \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} B_2(6n+5)q^n = 18\frac{f_2^{10} f_3^{10}}{f_1^{18} f_6^2} + 24\frac{f_2^6 f_3^2 f_6^2}{f_1^{10}} + 288\frac{qf_2^7 f_3 f_6^7}{f_1^{15}}. \tag{3.13}$$

The equation (3.11) reduce to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(6n+1)q^n &\equiv 2\frac{f_2^4}{f_1^4} + 8qf_2 f_6^3\left(\frac{f_3^3}{f_1}\right) \\ &\equiv 2f_2^4\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) + 8qf_2 f_6^3\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{16}, \end{aligned} \tag{3.14}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(12n+1)q^n \equiv 2\frac{f_2^2}{f_1^2} + 8qf_6^3 \cdot \frac{f_3^3}{f_1} \pmod{16} \tag{3.15}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(12n+7)q^n &\equiv 8f_2^7 + 8f_2^3 \cdot \frac{f_3^3}{f_1} \\ &\equiv 8f_2^7 + 8f_2^3\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{16},\end{aligned} \tag{3.16}$$

which implies

$$\sum_{n=0}^{\infty} B_2(24n+7)q^n \equiv 8f_1^7 + 8f_1 f_2^3 \pmod{16} \tag{3.17}$$

and

$$\sum_{n=0}^{\infty} B_2(24n+19)q^n \equiv 8f_1 f_6^3 \pmod{16}. \tag{3.18}$$

From the equation (3.17), we arrive at (3.1).

The equation (3.18) is $\beta = 0$ case of (3.2). Suppose that the congruence (3.2) is true for $\beta > 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta} n + 19 \cdot 5^{2\beta}\right) q^n \\ &\equiv 8f_1 f_3^6 \\ &\equiv 8f_{25}(a(q^5) - q - q^2/a(q^5)) \times f_{75}^6 \left(a(q^{15}) - q^3 - q^6/a(q^{15})\right)^6 \pmod{16},\end{aligned} \tag{3.19}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta+1} n + 23 \cdot 5^{2\beta+1}\right) q^n \equiv 8q^3 f_5 f_{15}^6 \pmod{16}, \tag{3.20}$$

which proves (3.3) and which implies

$$\sum_{n=0}^{\infty} B_2\left(24 \cdot 5^{2\beta+2} n + 19 \cdot 5^{2\beta+2}\right) q^n \equiv 8f_1 f_3^6 \pmod{16}, \tag{3.21}$$

which implies that the congruence (3.2) is true for $\beta+1$. So, by induction, the congruence (3.2) holds for all $\beta \geq 0$.

Extracting the terms involving q^{5n+i} for $i = 0, 1, 2, 4$ from (3.3), we obtain (3.4).

The congruence (3.15) reduces to

$$\sum_{n=0}^{\infty} B_2(12n+1)q^n \equiv 2f_1^2 \equiv 2f_{25}^2 \left(a(q^5) - q - q^2/a(q^5)\right)^2 \pmod{8}, \tag{3.22}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(60n+25)q^n \equiv 6f_5^2 \pmod{8}, \tag{3.23}$$

which implies

$$\sum_{n=0}^{\infty} B_2(300n+25)q^n \equiv 6f_1^2 \pmod{8}. \tag{3.24}$$

From the congruences (3.22) and (3.24), we find that

$$B_2(300n+25) \equiv 3 \cdot B_2(12n+1) \pmod{8}. \tag{3.25}$$

By using the above relation and by induction on β, we arrive at (3.5).

Extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from (3.23), we get (3.6).

Theorem 3.2. *Let $c_2 \in \{21, 93, 237, 309\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have*

$$B_2(72n+69) \equiv 0 \pmod{32}, \tag{3.26}$$

$$B_2\left(2 \cdot 3^{2\beta+3}n + 3^{2\beta+3}\right) \equiv B_2(6n+3) \pmod{32}, \tag{3.27}$$

$$\sum_{n=0}^{\infty} B_2\left(72 \cdot 5^{2\beta}n + 33 \cdot 5^{2\beta}\right) q^n \equiv 16f_2f_3^3 \pmod{32}, \tag{3.28}$$

$$\sum_{n=0}^{\infty} B_2\left(72 \cdot 5^{2\beta+1}n + 21 \cdot 5^{2\beta+1}\right) q^n \equiv 16q^2 f_{10}f_{15}^3 \pmod{32}, \tag{3.29}$$

$$B_2\left(72 \cdot 5^{2\beta+2}n + c_2 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{3.30}$$

$$\sum_{n=0}^{\infty} B_2\left(72 \cdot 7^{2\beta}n + 15 \cdot 7^{2\beta}\right) q^n \equiv 8f_1^5 \pmod{16}, \tag{3.31}$$

$$\sum_{n=0}^{\infty} B_2\left(72 \cdot 7^{2\beta+1} n + 33 \cdot 7^{2\beta+1}\right) q^n \equiv 8q f_7^5 \pmod{16}, \tag{3.32}$$

$$B_2\left(18 \cdot 5^{2\beta+2} n + 3 \cdot 5^{2\beta+2}\right) \equiv 3^{\beta+1} \cdot B_2(18n+3) \pmod{16}, \tag{3.33}$$

$$B_2(90(5n+i)+75) \equiv 0 \pmod{16}, \tag{3.34}$$

where $i = 1, 2, 3, 4$.

Proof. The equation (3.12) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(6n+3)q^n &\equiv 12\frac{f_2^3 f_3^5}{f_1^3 f_6} + 24q f_3^4 f_6^4 \pmod{32} \qquad (3.35)\\ &\equiv 12\frac{f_3^5}{f_6}\left(\frac{f_2^3}{f_1^3}\right) + 24q f_3^4 f_6^4\\ &\equiv 12\frac{f_3^5}{f_6}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right)\\ &+ 24q f_3^4 f_6^4 \pmod{32}, \qquad (3.36)\end{aligned}$$

which implies

$$\sum_{n=0}^{\infty} B_2(18n+3)q^n \equiv 12 f_1^4 + 16q f_1 f_3^9 \pmod{32}, \tag{3.37}$$

$$\sum_{n=0}^{\infty} B_2(18n+9)q^n \equiv 4\frac{f_2^3 f_3^5}{f_1^3 f_6} + 24 f_1^4 f_2^4 \pmod{32} \tag{3.38}$$

and

$$\sum_{n=0}^{\infty} B_2(18n+15)q^n \equiv 8 f_1^2 f_3^2 f_6^2 \pmod{32}. \tag{3.39}$$

From the equation (3.38), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(18n+9)q^n &\equiv 4\frac{f_3^5}{f_6}\left(\frac{f_2^3}{f_1^3}\right) + 24\left(f_1 f_2\right)^4 \qquad (3.40)\\ &\equiv 4\frac{f_3^5}{f_6}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right)\\ &+ 24\left(\frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}\right)^4 \pmod{32}, \qquad (3.41)\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(54n+27)q^n \equiv 12\frac{f_2^3 f_3^5}{f_1^3 f_6} + 24q f_3^4 f_6^4 \pmod{32}. \tag{3.42}$$

In the view of congruences (3.35) and (3.42), we see that

$$B_2(54n+27) \equiv B_2(6n+3) \pmod{32}. \tag{3.43}$$

Using the above relation and by induction on β, we get (3.27).

From the equation (3.39), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(18n+15)q^n &\equiv 8f_6^2(f_1f_3)^2 \\ &\equiv 8f_6^2\left(\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}\right)^2 \pmod{32},\end{aligned} \tag{3.44}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(36n+15)q^n \equiv 8f_1^2f_4^2 + 8q\frac{f_3^4f_6^4}{f_1^2} \pmod{32} \tag{3.45}$$

and

$$\sum_{n=0}^{\infty} B_2(36n+33)q^n \equiv 16f_4f_6^3 \pmod{32}. \tag{3.46}$$

Comparing the coefficients of q^{2n+1} on both sides of the equation (3.46), we obtain (3.26).

The congruence (3.46) implies that

$$\sum_{n=0}^{\infty} B_2(72n+33)q^n \equiv 16f_2f_3^3 \pmod{32}, \tag{3.47}$$

which is $\beta = 0$ case of (3.28). Suppose that the congruence (3.28) is true for $\beta > 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\beta}n + 33\cdot 5^{2\beta}\right)q^n \\ &\equiv 16f_2f_3^3 \\ &\equiv 16f_{50}\left(a(q^{10}) - q^2 - q^4/a(q^{10})\right) \times f_{75}^3\left(a(q^{15}) - q^3 - q^6/a(q^{15})\right)^3 \pmod{32},\end{aligned} \tag{3.48}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\beta+1}n+21\cdot 5^{2\beta+1}\right)q^n \equiv 16q^2 f_{10}f_{15}^3 \pmod{32}, \tag{3.49}$$

which proves (3.29) and which implies that

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\beta+2}n+33\cdot 5^{2\beta+2}\right)q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{3.50}$$

which implies that the congruence (3.28) is true for $\beta+1$. Hence, by induction, the congruence (3.28) holds for all $\beta \geq 0$.

Extracting the terms involving q^{5n+i} for $i=0,1,3,4$ from (3.29), we arrive at (3.30).

The congruence (3.45) reduces to

$$\sum_{n=0}^{\infty} B_2(36n+15)q^n \equiv 8f_2^5 + 8q\frac{f_6^6}{f_2} \pmod{16}, \tag{3.51}$$

which implies

$$\sum_{n=0}^{\infty} B_2(72n+15)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.52}$$

which is $\beta=0$ case of (3.31). Suppose that the congruence (3.31) is true for $\beta>0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} B_2\left(72\cdot 7^{2\beta}n+15\cdot 7^{2\beta}\right)q^n \\ &\equiv 8f_1^5 \equiv 8f_{49}^5\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^5 \pmod{16},\end{aligned} \tag{3.53}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 7^{2\beta+1}n+33\cdot 7^{2\beta+1}\right)q^n \equiv 8qf_7^5 \pmod{16}, \tag{3.54}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 7^{2\beta+2}n+15\cdot 7^{2\beta+2}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.55}$$

which implies that the congruence (3.31) is true for $\beta+1$. So, by induction, the congruence (3.31) holds for integers $\beta \geq 0$.

Employing (2.17) in (3.31) and then collecting the terms involving q^{7n+3}, we arrive at (3.32).

The equation (3.37) reduces to

$$\sum_{n=0}^{\infty} B_2(18n+3)q^n \equiv 12f_1^4 \equiv 12f_{25}^4\left(a(q^5)-q-q^2/a(q^5)\right)^4 \pmod{16}, \tag{3.56}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(90n+75)q^n \equiv 4f_5^4 \pmod{16}, \tag{3.57}$$

which implies

$$\sum_{n=0}^{\infty} B_2(450n+75)q^n \equiv 4f_1^4 \pmod{16}. \tag{3.58}$$

From the congruences (3.56) and (3.58), we see that

$$B_2(450n+75) \equiv 3\cdot B_2(18n+3) \pmod{16}. \tag{3.59}$$

Using the above relation and by induction on β, we arrive at (3.33).

Extracting the terms involving q^{5n+i} for $i=1,2,3,4$ from the equation (3.57), we obtain (3.34).

Theorem 3.3. *Let $c_3 \in \{7,31,79,103\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have*

$$B_2(24n+23) \equiv 0 \pmod{8}, \tag{3.60}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 5^{2\beta}n+11\cdot 5^{2\beta}\right)q^n \equiv 4f_2f_3^3 \pmod{8}, \tag{3.61}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 5^{2\beta+1}n+7\cdot 5^{2\beta+1}\right)q^n \equiv 4q^2f_{10}f_{15}^3 \pmod{8}, \tag{3.62}$$

$$B_2\left(24\cdot 5^{2\beta+2}n+c_3\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{8}, \tag{3.63}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 7^{2\beta}n+5\cdot 7^{2\beta}\right)q^n \equiv 2f_1^5 \pmod{4}, \tag{3.64}$$

$$\sum_{n=0}^{\infty} B_2\left(24\cdot 7^{2\beta+1}n+11\cdot 7^{2\beta+1}\right)q^n \equiv 2qf_7^5 \pmod 4. \tag{3.65}$$

Proof. The equation (3.13) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2\left(6n+5\right)q^n &\equiv 2\frac{f_2^2 f_3^6}{f_1^2} \pmod 8\\ &\equiv 2f_2^2\left(\frac{f_3^3}{f_1}\right)^2\\ &\equiv 2f_2^2\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}}+q\frac{f_{12}^3}{f_4}\right)^2 \pmod 8,\end{aligned} \tag{3.66}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(12n+5\right)q^n \equiv 2\frac{f_2^6 f_3^4}{f_1^2 f_6^2}+2q\frac{f_1^2 f_6^6}{f_2^2} \pmod 8 \tag{3.67}$$

and

$$\sum_{n=0}^{\infty} B_2\left(12n+11\right)q^n \equiv 4f_4 f_6^3 \pmod 8. \tag{3.68}$$

Extracting the terms involving q^{2n+1} from both sides of the equation (3.68), we get (3.60).

The congruence (3.68) implies that

$$\sum_{n=0}^{\infty} B_2\left(24n+11\right)q^n \equiv 4f_2 f_3^3 \pmod 8, \tag{3.69}$$

which is $\beta = 0$ case of (3.61). The rest of the proofs of the identities (3.61)-(3.63) are similar to the proofs of the identities (3.28)-(3.30). So, we omit the details.

The congruence (3.67) reduces to

$$\sum_{n=0}^{\infty} B_2\left(12n+5\right)q^n \equiv 2f_2^5+2q\frac{f_6^6}{f_2} \pmod 4, \tag{3.70}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(24n+5\right)q^n \equiv 2f_1^5 \pmod 4, \tag{3.71}$$

which is $\beta = 0$ case of (3.64). The remaining proofs of the identities (3.64) and (3.63) are similar to the proofs of the identities (3.31) and (3.32). So, we omit the details.

4. Congruences modulo powers of 3

Theorem 4.1. *Let* $c_4 \in \{88, 136, 184, 232\}$, $c_5 \in \{22, 34, 46, 58\}$, *then for all* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$B_2(6n+5) \equiv 0 \pmod 6, \tag{4.1}$$

$$B_2(24n+20) \equiv 0 \pmod 9, \tag{4.2}$$

$$B_2\left(3 \cdot 2^{2\alpha+4}n + 2^{2\alpha+5}\right) \equiv B_2\left(12n+8\right) \pmod 9, \tag{4.3}$$

$$\sum_{n=0}^{\infty} B_2\left(48 \cdot 5^{2\alpha}n + 8 \cdot 5^{2\alpha}\right) q^n \equiv 6 f_1 f_3 \pmod 9, \tag{4.4}$$

$$\sum_{n=0}^{\infty} B_2\left(48 \cdot 5^{2\alpha+1}n + 8 \cdot 5^{2\alpha+2}\right) q^n \equiv 6 f_5 f_{15} \pmod 9, \tag{4.5}$$

$$B_2\left(48 \cdot 5^{2\alpha+2}n + c_4 \cdot 5^{2\alpha+1}\right) \equiv 0 \pmod 9, \tag{4.6}$$

$$\sum_{n=0}^{\infty} B_2\left(12 \cdot 5^{2\alpha}n + 2 \cdot 5^{2\alpha}\right) q^n \equiv 2 f_1^4 \pmod 3, \tag{4.7}$$

$$\sum_{n=0}^{\infty} B_2\left(12 \cdot 5^{2\alpha+1}n + 2 \cdot 5^{2\alpha+2}\right) q^n \equiv 2 f_5^4 \pmod 3, \tag{4.8}$$

$$B_2\left(12 \cdot 5^{2\alpha+2}n + c_5 \cdot 5^{2\alpha+1}\right) \equiv 0 \pmod 3. \tag{4.9}$$

Proof. From the equation (1.6), we have

$$\sum_{n=0}^{\infty} B_2(n) q^n = \frac{f_6^4}{f_3^2 f_{12}^2} \times \frac{f_4^2}{f_1^2}.$$

$$= \frac{f_6^4}{f_3^2 f_{12}^2} \left(\frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3} \right)^2, \tag{4.10}$$

which implies

$$\sum_{n=0}^{\infty} B_2(3n) q^n = \frac{f_2^4 f_6^8}{f_1^8 f_{12}^4} + 4q \frac{f_2^7 f_3^3 f_{12}^2}{f_1^9 f_4^2 f_6}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} B_2(3n+1) q^n = 2 \frac{f_2^6 f_3^3 f_6^2}{f_1^9 f_4 f_{12}} + 4q \frac{f_2^6 f_6^2 f_{12}^2}{f_1^8 f_4^2} \tag{4.12}$$

and

$$\sum_{n=0}^{\infty} B_2(3n+2)q^n = \frac{f_2^8 f_3^6 f_{12}^2}{f_1^{10} f_4^2 f_6^4} + 4\frac{f_2^5 f_6^5}{f_1^8 f_4 f_{12}}. \tag{4.13}$$

The equation (4.13) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(3n+2)q^n &\equiv \frac{f_{12}^2}{f_2 f_4^2 f_6}\left(\frac{f_3^3}{f_1}\right) + 4\frac{f_2^5 f_6^5}{f_4 f_{12}}\left(\frac{f_1}{f_3^3}\right) \\ &\equiv \frac{f_{12}^2}{f_2 f_4^2 f_6}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) + 4\frac{f_2^3 f_6^5}{f_4 f_{12}}\left(\frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}\right) \pmod 9,\end{aligned} \tag{4.14}$$

which implies

$$\sum_{n=0}^{\infty} B_2(6n+2)q^n \equiv \frac{f_2 f_3 f_6}{f_1^3} + 4\frac{f_1^6 f_2 f_6}{f_3^2} \pmod 9 \tag{4.15}$$

and

$$\sum_{n=0}^{\infty} B_2(6n+5)q^n \equiv 6\frac{f_6^4}{f_1 f_3} \pmod 9. \tag{4.16}$$

From the equation (4.16), we arrive at (4.1).

The equation (4.15) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(6n+2)q^n &\equiv 5 f_2 f_6 \left(\frac{f_3}{f_1^3}\right) \\ &\equiv 5 f_2 f_6 \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right) \pmod 9,\end{aligned} \tag{4.17}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(12n+2)q^n \equiv 5\frac{f_2^6 f_3^4}{f_1^8 f_6^2} \pmod 9 \tag{4.18}$$

and

$$\sum_{n=0}^{\infty} B_2(12n+8)q^n \equiv 6\frac{f_2^2 f_3^2 f_6^2}{f_1^6} \pmod 9. \tag{4.19}$$

The equation (4.19) reduces to

$$\sum_{n=0}^{\infty} B_2(12n+8)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.20}$$

Extracting the terms involving q^{2n+1} from both sides of the above equation, we get (4.2).

The equation (4.20) implies that

$$\sum_{n=0}^{\infty} B_2(24n+8)q^n \equiv 6\frac{f_3^3}{f_1} \equiv 6\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 9, \tag{4.21}$$

which implies

$$\sum_{n=0}^{\infty} B_2(48n+8)q^n \equiv 6\frac{f_3^2}{f_1^2} \pmod 9 \tag{4.22}$$

and

$$\sum_{n=0}^{\infty} B_2(48n+32)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.23}$$

In the view of congruences (4.20) and (4.23), we see that

$$B_2(48n+32) \equiv B_2(12n+8) \pmod 9. \tag{4.24}$$

Using the above relation and by induction on α, we arrive at (4.3).

The equation (4.22) becomes

$$\sum_{n=0}^{\infty} B_2(48n+8)q^n \equiv 6f_1f_3 \pmod 9, \tag{4.25}$$

which is $\alpha = 0$ case of (4.4). Suppose that the congruence (4.4) is true for $\alpha > 0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_2\left(48\cdot 5^{2\alpha}n + 8\cdot 5^{2\alpha}\right) q^n &\equiv 6f_1f_3 \\ &\equiv 6f_{25}\left(a(q^5) - q - q^2/a(q^5)\right) \\ &\quad \times f_{75}(a(q^{15}) - q^3 - q^6/a(q^{15})) \pmod 9, \end{aligned} \tag{4.26}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2\left(48\cdot 5^{2\alpha+1}n + 8\cdot 5^{2\alpha+2}\right) q^n \equiv 6f_5f_{15} \pmod 9, \tag{4.27}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(48\cdot 5^{2\alpha+2}n+8\cdot 5^{2\alpha+2}\right)q^n \equiv 6f_1f_3 \pmod 9, \tag{4.28}$$

which implies that the congruence (4.4) is true for $\alpha+1$. So, by induction, the congruence (4.4) holds for all $\alpha \geq 0$.

Using (2.15) in (4.4) and then collecting the coefficients of q^{5n+4} from both sides, we get (4.5).

From the congruence (4.5), we get (4.6).

The equation (4.18) reduce to

$$\sum_{n=0}^{\infty} B_2(12n+2)q^n \equiv 2f_1^4 \pmod 3, \tag{4.29}$$

which is $\alpha=0$ case of (4.7). Suppose the congruence (4.7) is true for $\alpha>0$, we have

$$\sum_{n=0}^{\infty} B_2\left(12\cdot 5^{2\alpha}n+2\cdot 5^{2\alpha}\right)q^n \equiv 2f_1^4 \equiv 2f_{25}^4\left(a(q^5)-q-q^2/(q^5)\right)^4 \pmod 3, \tag{4.30}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2\left(12\cdot 5^{2\alpha+1}n+2\cdot 5^{2\alpha+2}\right)q^n \equiv 2f_5^4 \pmod 3, \tag{4.31}$$

which implies

$$\sum_{n=0}^{\infty} B_2\left(12\cdot 5^{2\alpha+2}n+2\cdot 5^{2\alpha+2}\right)q^n \equiv 2f_1^4 \pmod 3, \tag{4.32}$$

which implies that the congruence (4.7) is true for $\alpha+1$. Hence, by induction, the congruence (4.7) holds for all integers $\alpha \geq 0$.

Using (2.15) in (4.7) and then comparing the coefficients of q^{5n+4} on both sides, we get (4.8).

Extracting the terms involving q^{5n+i} for $i=1,2,3,4$ from the congruence (4.8), we arrive at (4.9).

Theorem 4.2. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$B_2(144n+120) \equiv 0 \pmod 9, \tag{4.33}$$

$$B_2\left(2\cdot 3^{\alpha+2}n+3^{\alpha+2}\right)\equiv 2^{\alpha+1}\cdot B_2(6n+3) \pmod 9, \tag{4.34}$$

$$B_2\left(9\cdot 2^{2\alpha+4}n+3\cdot 2^{2\alpha+4}\right)\equiv B_2(36n+12) \pmod 9, \tag{4.35}$$

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\alpha}n+12\cdot 5^{2\alpha}\right)q^n\equiv 3f_1f_3 \pmod 9, \tag{4.36}$$

$$\sum_{n=0}^{\infty} B_2\left(72\cdot 5^{2\alpha+1}n+12\cdot 5^{2\alpha+2}\right)q^n\equiv 3f_5f_{15} \pmod 9, \tag{4.37}$$

$$B_2\left(72\cdot 5^{2\alpha+2}n+c_6\cdot 5^{2\alpha+1}\right)\equiv 0 \pmod 9, \tag{4.38}$$

where $c_6\in\{132,204,276,348\}$.

Proof. The congruence (4.11) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(3n)q^n &\equiv \frac{f_2^4f_6^8}{f_{12}^4}\left(\frac{f_1}{f_3^3}\right)+4q\frac{f_6^2f_{12}^2}{f_2^2f_4^2}\\ &\equiv \frac{f_2^4f_6^8}{f_{12}^4}\left(\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}\right)+4q\frac{f_6^2f_{12}^2}{f_2^2f_4^2} \pmod 9,\end{aligned} \tag{4.39}$$

which implies

$$\sum_{n=0}^{\infty} B_2(6n)q^n\equiv \frac{f_1^5f_3f_2^2}{f_6^2} \pmod 9 \tag{4.40}$$

and

$$\sum_{n=0}^{\infty} B_2(6n+3)q^n\equiv 4\frac{f_3^2f_6^2}{f_1^2f_2^2}+8\frac{f_1^7f_6^2}{f_2^2f_3} \pmod 9. \tag{4.41}$$

The congruence (4.41) reduces to

$$\sum_{n=0}^{\infty} B_2(6n+3)q^n\equiv 3f_1f_2f_3f_6 \tag{4.42}$$

$$\equiv 3f_3f_6\left(\frac{f_6f_9^4}{f_3f_{18}^2}-qf_9f_{18}-2q^2\frac{f_3f_{18}^4}{f_6f_9^2}\right) \pmod 9, \tag{4.43}$$

we extract

$$\sum_{n=0}^{\infty} B_2(18n+9)q^n\equiv 6f_1f_2f_3f_6 \pmod 9. \tag{4.44}$$

In the view of congruences (4.42) and (4.44), we see that

$$B_2(18n+9)\equiv 2\cdot B_2(6n+3) \pmod 9. \tag{4.45}$$

Using the above relation and induction on α, we obtain (4.34).

The congruence (4.40) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(6n)q^n &\equiv \frac{f_3^4}{f_6^2}\left(\frac{f_2}{f_1^2}\right)^2 \\ &\equiv \frac{f_3^4}{f_6^2}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2 \pmod 9, \end{aligned} \tag{4.46}$$

which implies

$$\sum_{n=0}^{\infty} B_2(18n)q^n \equiv \frac{f_2^6 f_3^9}{f_1^3 f_6^6} + 7qf_2^3 f_6^3 \pmod 9, \tag{4.47}$$

$$\sum_{n=0}^{\infty} B_2(18n+6)q^n \equiv 4\frac{f_2^5 f_3^6}{f_1^2 f_6^3} + 7q\frac{f_1 f_2^2 f_6^6}{f_3^3} \pmod 9 \tag{4.48}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(18n+12)q^n &\equiv 3\frac{f_2^4 f_3^3}{f_1} \\ &\equiv 3f_2^4\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 9, \end{aligned} \tag{4.49}$$

we extract

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(36n+12)q^n &\equiv 3\frac{f_1^2 f_2^3 f_3^2}{f_6} \\ &\equiv 3\frac{f_3^3}{f_1} \\ &\equiv 3\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 9, \end{aligned} \tag{4.50}$$

which implies

$$\sum_{n=0}^{\infty} B_2(72n+12)q^n \equiv 3f_1 f_3 \pmod 9 \tag{4.51}$$

and

$$\sum_{n=0}^{\infty} B_2(72n+48)q^n \equiv 3\frac{f_6^3}{f_2} \pmod 9. \tag{4.52}$$

Extracting the terms involving q^{2n+1} from the equation (4.52), we get (4.33).

The equation (4.52) implies

$$\sum_{n=0}^{\infty} B_2(144n+48)q^n \equiv 3\frac{f_3^3}{f_1} \pmod 9. \tag{4.53}$$

From the congruences (4.50) and (4.53), we see that

$$B_2(144n+48) \equiv B_2(36n+12) \pmod 9. \tag{4.54}$$

Using the above relation and by induction on α, we arrive at (4.35).

The congruence (4.51) is $\alpha = 0$ case of (4.36). The rest of the proofs of the identities (4.36)-(4.38) are similar to the proofs of the identities (4.4)-(4.6). So, we omit the details.

Theorem 4.3. *Let $c_7 \in \{264, 408, 552, 696\}$, $c_8 \in \{66, 102, 138, 174\}$, then for all $n \geq 0$ and $\alpha \geq 0$, we have*

$$B_2(72n+70) \equiv 0 \pmod 9, \tag{4.55}$$

$$B_2\left(9 \cdot 2^{2\alpha+4}n + 6 \cdot 2^{2\alpha+4}\right) \equiv B_2(36n+24) \pmod 9, \tag{4.56}$$

$$\sum_{n=0}^{\infty} B_2\left(144 \cdot 5^{2\alpha}n + 24 \cdot 5^{2\alpha}\right) q^n \equiv 6f_1f_3 \pmod 9, \tag{4.57}$$

$$\sum_{n=0}^{\infty} B_2\left(144 \cdot 5^{2\alpha+1}n + 24 \cdot 5^{2\alpha+2}\right) q^n \equiv 3f_5f_{15} \pmod 9, \tag{4.58}$$

$$B_2\left(144 \cdot 5^{2\alpha+2}n + c_7 \cdot 5^{2\alpha+1}\right) \equiv 0 \pmod 9, \tag{4.59}$$

$$\sum_{n=0}^{\infty} B_2\left(36 \cdot 5^{2\alpha}n + 6 \cdot 5^{2\alpha}\right) q^n \equiv f_1^4 \pmod 3, \tag{4.60}$$

$$\sum_{n=0}^{\infty} B_2\left(36 \cdot 5^{2\alpha+1}n + 6 \cdot 5^{2\alpha+2}\right) q^n \equiv f_5^4 \pmod 3, \tag{4.61}$$

$$B_2\left(36 \cdot 5^{2\alpha+2}n + c_8 \cdot 5^{2\alpha+1}\right) \equiv 0 \pmod 3. \tag{4.62}$$

Proof. The equation (4.48) becomes

$$\sum_{n=0}^{\infty} B_2(18n+6)q^n \equiv 4\frac{f_2^5}{f_6^3}\left(\frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}\right)^2 + 7qf_2^2f_6^6\left(\frac{f_2f_4^2f_{12}^2}{f_6^7} - q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}\right) \pmod 9, \tag{4.63}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(36n+6)q^n \equiv 4\frac{f_1 f_3 f_2^6}{f_6^2} + 6q\frac{f_1^5 f_6^6}{f_3^3 f_2^2} \pmod 9 \tag{4.64}$$

and

$$\sum_{n=0}^{\infty} B_2(36n+24)q^n \equiv 6\frac{f_1^3 f_2^2 f_6^2}{f_3} \pmod 9. \tag{4.65}$$

The congruence (4.65) reduces to

$$\sum_{n=0}^{\infty} B_2(36n+24)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.66}$$

Extracting the terms involving q^{2n+1} form the above equation, we get (4.55).

The congruence (4.66) implies that

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(72n+24)q^n &\equiv 6\frac{f_3^3}{f_1} \\ &\equiv 6\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 9,\end{aligned} \tag{4.67}$$

which implies

$$\sum_{n=0}^{\infty} B_2(144n+24)q^n \equiv 6f_1 f_3 \pmod 9 \tag{4.68}$$

and

$$\sum_{n=0}^{\infty} B_2(144n+96)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.69}$$

In the view of congruences (4.66) and (4.69), we see that

$$B_2(144n+96) \equiv B_2(36n+24) \pmod 9. \tag{4.70}$$

Using the above relation and by induction on α, we arrive at (4.56).

The rest of the proofs of the identities (4.57)-(4.59) are similar to the proofs of the identities (4.4)-(4.6). So, we omit the details.

The congruence (4.64) reduces to

$$\sum_{n=0}^{\infty} B_2(36n+6)q^n \equiv f_1^4 \pmod 3. \tag{4.71}$$

The rest of the proofs of the identities (4.60)-(4.62) are similar to the proofs of the identities (4.7)-(4.9). So, we omit the details.

Theorem 4.4. *For all $n \geq 0$ and $\alpha \geq 0$, we gave*

$$B_2\left(4 \cdot 3^{\alpha+3} n + 2 \cdot 3^{\alpha+3}\right) \equiv B_2(36n+18) \pmod 9. \tag{4.72}$$

Proof. The congruence (4.47) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(18n) q^n &\equiv \frac{f_2^6}{f_6^6}\left(\frac{f_3^3}{f_1}\right)^3 + 7q f_2^3 f_6^3 \\ &\equiv \frac{f_2^6}{f_6^6}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)^3 + 7q f_2^3 f_6^3 \pmod 9,\end{aligned} \tag{4.73}$$

from which we extract

$$\sum_{n=0}^{\infty} B_2(36n) q^n \equiv \frac{f_2^9}{f_6^3} + 3q\frac{f_1 f_2 f_6^5}{f_3^3} \pmod 9 \tag{4.74}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_2(36n+18) q^n &\equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + q\frac{f_1^6 f_6^9}{f_2^3 f_3^6} + 7 f_1^3 f_3^3 \\ &\equiv 3\frac{f_6^2}{f_3}\left(\frac{f_2^2}{f_1}\right) + +q\frac{f_6^9}{f_3^6}\left(\frac{f_1^2}{f_2}\right)^3 + 7 f_1^3 f_3^3 \pmod 9.\end{aligned} \tag{4.75}$$

Employing (2.4), (2.5) and (2.8) in the above equation and then collecting the coefficients of q^{3n+1}, we get

$$\sum_{n=0}^{\infty} B_2(108n+54) q^n \equiv 3\frac{f_2^2 f_6^2}{f_1 f_3} + q\frac{f_1^6 f_6^9}{f_2^3 f_3^6} + 7 f_1^3 f_3^3 \pmod 9. \tag{4.76}$$

In the view of congruences (4.75) and (4.76), we see that

$$B_2(108n+54) \equiv B_2(36n+18) \pmod 9. \tag{4.77}$$

Using the above relation and by induction on α, we arrive at (4.72).

Theorem 4.5. *For all $n \geq 1$ and $\alpha \geq 0$, we have*

$$B_2(216n+180) \equiv 0 \pmod 9, \tag{4.78}$$

$$B_2(4\cdot 3^{\alpha+3}n)\equiv B_2(36n) \pmod 9, \tag{4.79}$$

$$B_2\left(27\cdot 2^{2\alpha+4}n+9\cdot 2^{2\alpha+4}\right)\equiv B_2(108n+36) \pmod 9, \tag{4.80}$$

$$B_2\left(27\cdot 2^{2\alpha+4}n+18\cdot 2^{2\alpha+4}\right)\equiv B_2(108n+72) \pmod 9. \tag{4.81}$$

Proof. The congruence (4.74) reduces to

$$\sum_{n=0}^{\infty} B_2(36n)q^n \equiv 1+3q\frac{f_1f_2f_6^5}{f_3^3} \pmod 9, \tag{4.82}$$

which implies

$$\sum_{n=1}^{\infty} B_2(36n)q^n \equiv 3q\frac{f_1f_2f_6^5}{f_3^3} \tag{4.83}$$

$$\equiv 3q\frac{f_6^5}{f_3^3}\left(\frac{f_6f_9^4}{f_3f_{18}^2}-qf_9f_{18}-2q^2\frac{f_3f_{18}^4}{f_6f_9^2}\right) \pmod 9, \tag{4.84}$$

which implies

$$\sum_{n=1}^{\infty} B_2(108n)q^n \equiv 3q\frac{f_1f_2f_6^5}{f_3^3} \pmod 9, \tag{4.85}$$

$$\sum_{n=1}^{\infty} B_2(108n+36)q^n \equiv 3\frac{f_3^3}{f_1} \pmod 9 \tag{4.86}$$

and

$$\sum_{n=1}^{\infty} B_2(108n+72)q^n \equiv 6\frac{f_6^3}{f_2} \pmod 9. \tag{4.87}$$

Extracting the terms involving q^{2n+1} from both sides of the equation (4.87), we get (4.78).

In the view of congruences (4.83) and (4.85), we see that

$$B_2(108n)\equiv B_2(36n) \pmod 9. \tag{4.88}$$

Using the above relation and by induction on α, we arrive at (4.79).

The proofs of the identities (4.80) and (4.81) are similar to the proof of the identity (4.3). So, we omit the details.

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