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FUZZY PRE β-OPEN SET AND ITS APPLICATIONS

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Abstract: This paper deals with a new type of fuzzy open-like set, viz., fuzzy pre β-open set, the class of which is strictly larger than that of fuzzy preopen set [7]. Using fuzzy pre β-open set as a basic tool, here we introduce fuzzy pre β-regular space in which fuzzy open set and fuzzy pre β-open set coincide. Here we introduce two new types of functions, viz., fuzzy pre β-continuous function, fuzzy pre β-irresolute function. The applications of these two functions on fuzzy pre β-regular space are discussed here.

Keywords and Phrases: Fuzzy β-open set, fuzzy pre β-open set, fuzzy pre β-regular space, fuzzy pre β-continuous function, fuzzy pre β-irresolute function, fuzzy almost compact space.

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1. Introduction

After introducing fuzzy open set by Chang [4], different types of fuzzy open-like sets are introduced and studied. In [6], fuzzy β-open set is introduced. With the help of this set here we introduce fuzzy pre β-open set, the class of which is strictly larger than that of fuzzy preopen set [7]. In [4], fuzzy continuous function is introduced and in [3], fuzzy β-irresolute function is introduced. Here we introduce fuzzy pre β-continuous function, the class of which is strictly larger than that of fuzzy continuous function and fuzzy β-irresolute function.

2. Preliminary

Throughout this paper, (X, τ) or simply by X we shall mean a fuzzy topological space. A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$ [9]. The support [9] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [9] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [9] while AqB means A is quasi-coincident (q-coincident, for short) [8] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [4] and fuzzy interior [4] of A respectively. A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [8] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy quasi neighbourhood (q-nbd for short) of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t q U \leq A$. If, in addition, A is fuzzy open, then A is called a fuzzy open q-nbd of x_t [8].

A fuzzy set A in an fts (X, τ) is called fuzzy regular open [1] (resp., fuzzy β-open [6], fuzzy preopen [7]) if $A = intclA$ (resp., $A \leq cl(int(clA))$, $A \leq int(clA)$). The complement of a fuzzy β-open set is called fuzzy β-closed [6]. The union (intersection) of all fuzzy β-open (resp., fuzzy β-closed) sets contained in (resp., containing) a fuzzy set A is called fuzzy β-interior [6] (resp., fuzzy β-closure [6]) of A, denoted by $\beta intA$ (resp., βclA). A fuzzy set A in an fts X is called a fuzzy β-neighbourhood (fuzzy β-nbd, for short) [6] of a fuzzy point x_α in X if there exists a fuzzy β-open set U in X such that $x_\alpha \in U \leq A$. The collection of all fuzzy β-open (resp., fuzzy β-closed) sets in X is denoted by $F\beta O(X)$ (resp., $F\beta C(X)$) and that of fuzzy preopen (resp. fuzzy preclosed) sets is denoted by $FPO(X)$ (resp. $FPC(X)$).

3. Fuzzy Pre β-open Set : Some Properties

In this section, we introduce and study fuzzy pre β-open set. Also here we introduce fuzzy pre β-closure operator which is an idempotent operator.

Definition 3.1. *A fuzzy set A in an fts (X, τ) is called fuzzy pre β-open if $A \leq \beta int(clA)$. The complement of this set is called fuzzy pre β-closed set.*

The collection of all fuzzy pre β-open (resp., fuzzy pre β-closed) sets in (X, τ) is

denoted by $FP\beta O(X)$ (resp., $FP\beta C(X)$).
The union (resp., intersection) of all fuzzy pre β-open (resp., fuzzy pre β-closed) sets contained in (containing) a fuzzy set A is called fuzzy pre β-interior (resp., fuzzy pre β-closure) of A, denoted by $p\beta intA$ (resp., $p\beta clA$).

Result 3.2. *Union of two fuzzy pre β-open sets in an fts X is also so.*

Proof. Let $A, B \in FP\beta O(X)$. Then $A \leq \beta int(clA), B \leq \beta int(clB)$. Now $\beta int(cl(A \bigvee B)) = \beta int(clA \bigvee clB) \geq \beta int(clA) \bigvee \beta int(clB) \geq A \bigvee B$.

Remark 3.3. *Intersection of two fuzzy pre β-open sets need not be so, follows from the next example.*

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Now $F\beta O(X, \tau) = \{0_X, 1_X, U\}$ where $U \not\leq 1_X \setminus A$. Consider two fuzzy sets B and C defined by $B(a) = 0.6, B(b) = 0.3, C(a) = 0.3, C(b) = 0.6$. Then clearly $B, C \in FP\beta O(X, \tau)$. Let $D = B \bigwedge C$. Then $D(a) = D(b) = 0.3$. But $\beta int(clD) = \beta int(1_X \setminus A) = 0_X \not\geq D \Rightarrow D \notin FP\beta O(X, \tau)$.

So we can conclude that the set of all fuzzy pre β-open sets does not form a fuzzy topology.

Note 3.5. *For any fuzzy set A in an fts X, $intA \leq \beta intA$, so clearly fuzzy preopen set is fuzzy pre β-open, but not conversely follows from the next example.*

Example 3.6. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $int(clB) = A \not\geq B \Rightarrow B \notin FPO(X, \tau)$. But $\beta int(clB) = \beta int(1_X \setminus A) = 1_X \setminus A \geq B$ (as every fuzzy set in (X, τ) is fuzzy β-open in (X, τ)) $\Rightarrow B \in FP\beta O(X, \tau)$.

Definition 3.7. *A fuzzy set A in an fts (X, τ) is called fuzzy pre β-neighbourhood (fuzzy pre β-nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre β-open set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fuzzy pre β-open, then A is called fuzzy pre β-open nbd of x_α.*

Definition 3.8. *A fuzzy set A in an fts (X, τ) is called fuzzy pre β-quasi neighbourhood (fuzzy pre β-q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre β-open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy pre β-open, then A is called fuzzy pre β-open q-nbd of x_α.*

Remark 3.9. *It is clear from definitions that fuzzy nbd (resp., fuzzy q-nbd) of a fuzzy point is a fuzzy pre β-nbd (resp., fuzzy pre β-q-nbd) of that fuzzy point in an fts. But converse may not be true follows from the following example.*

Example 3.10. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Now consider the fuzzy point $a_{0.5}$ and the fuzzy set A defined by $A(a) = A(b) = 0.6$. Since every fuzzy set in (X, τ) is fuzzy pre β-open set in (X, τ), clearly A is fuzzy pre β-nbd (resp., fuzzy pre β-q-nbd) of $a_{0.5}$. But A is not a fuzzy nbd (resp., fuzzy q-nbd) of $a_{0.5}$ as there is no fuzzy open set U in X such that $a_{0.5} \in U \leq A$ (resp., $a_{0.5}qU \leq A$).

Theorem 3.11. *For any fuzzy set A in an fts (X, τ), $x_\alpha \in p\beta clA$ iff every fuzzy pre β-open q-nbd U of x_α, UqA.*

Proof. Let $x_\alpha \in p\beta clA$ for any fuzzy set A in an fts (X, τ). Let $U \in FP\beta O(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1 \Rightarrow x_\alpha \notin 1_X \setminus U \in FP\beta C(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.

Conversely, let the given condition hold. Let $U \in FP\beta C(X)$ with $A \leq U$ ... (1). We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha \Rightarrow x_\alpha q(1_X \setminus U)$ where $1_X \setminus U \in FP\beta O(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.12. *$p\beta cl(p\beta clA) = p\beta clA$ for any fuzzy set A in an fts (X, τ).*

Proof. Let $A \in I^X$. Then $A \leq p\beta clA \Rightarrow p\beta clA \leq p\beta cl(p\beta clA)$ ... (1).

Conversely, let $x_\alpha \in p\beta cl(p\beta clA)$. If possible, let $x_\alpha \notin p\beta clA$. Then there exists $U \in FP\beta O(X)$,

$$x_\alpha qU, U \not{q} A ...(2)$$

But as $x_\alpha \in p\beta cl(p\beta clA)$, $Uq(p\beta clA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (p\beta clA)(y) > 1 \Rightarrow U(y) + t > 1$ where $t = (p\beta clA)(y)$. Then $y_t \in p\beta clA$ and $y_t qU$ where $U \in FP\beta O(X)$. Then by definition, UqA, contradicts (2). So

$$p\beta cl(p\beta clA) \leq p\beta clA ...(3)$$

Combining (1) and (3), we get the result.

4. Fuzzy pre β-Continuous Function : Some Characterizations

In this section we introduce and characterize fuzzy pre β-continuous function, the class of which is strictly larger than that of fuzzy continuous function [4] and fuzzy β-irresolute function [3].

Definition 4.1. *A function $f : X \to Y$ is said to be fuzzy pre β- continuous if for each fuzzy point x_α in X and every fuzzy nbd V of $f(x_\alpha)$ in Y, $cl(f^{-1}(V))$ is a fuzzy β-nbd of x_α in X.*

Theorem 4.2. *For a function $f : X \to Y$, the following statements are equivalent :*

(a) f is fuzzy pre β-continuous,
(b) $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) $f(\beta clA) \leq cl(f(A))$, for all fuzzy open set A in X.
Proof. (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \in B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$ in Y. By (a), $cl(f^{-1}(B))$ is a fuzzy β-nbd of x_α in X. So $x_\alpha \in \beta int(cl(f^{-1}(B)))$. Since x_α is taken arbitrarily, $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \in f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$ (by (b)) $\leq cl(f^{-1}(B))$. So $cl(f^{-1}(B))$ is a fuzzy β-nbd of x_α in X.
(b) $\Rightarrow$ (c). Let A be a fuzzy open set in X. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \leq \beta int(cl(f^{-1}(1_Y \setminus cl(f(A))))) = \beta int(cl(1_X \setminus f^{-1}(cl(f(A))))) \leq \beta int(cl(1_X \setminus f^{-1}(f(A)))) \leq \beta int(cl(1_X \setminus A)) = \beta int(1_X \setminus A) = 1_X \setminus \beta clA$. Then $\beta clA \leq 1_X \setminus f^{-1}(1_Y \setminus cl(f(A))) = f^{-1}(cl(f(A)))$. So $f(\beta clA) \leq cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $int(f^{-1}(1_Y \setminus B))$ is a fuzzy open set in X. By (c), $f(\beta cl(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) = 1_Y \setminus B \Rightarrow B \leq 1_Y \setminus f(\beta cl(int(f^{-1}(1_Y \setminus B))))$. Then $f^{-1}(B) \leq f^{-1}(1_Y \setminus f(\beta cl(int(f^{-1}(1_Y \setminus B))))) = 1_X \setminus f^{-1}(f(\beta cl(int(f^{-1}(1_Y \setminus B))))) \leq 1_X \setminus \beta cl(int(f^{-1}(1_Y \setminus B))) = 1_X \setminus \beta cl(int(1_X \setminus f^{-1}(B))) = \beta int(cl(f^{-1}(B)))$.

Note 4.3. *It is clear from Theorem 4.2 that the inverse image under fuzzy pre β-continuous function of any fuzzy open set is fuzzy pre β-open.*

Theorem 4.4. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy pre β-continuous,
(b) $f^{-1}(B) \leq \beta int(cl(f^{-1}(B)))$, for all fuzzy open set B of Y,
(c) for each fuzzy point x_α in X and each fuzzy open nbd V of $f(x_\alpha)$ in Y, there exists $U \in FP\beta O(X)$ containing x_α such that $f(U) \leq V$,
(d) $f^{-1}(F) \in FP\beta C(X)$, for all fuzzy closed set F in Y,
(e) for each fuzzy point x_α in X, the inverse image under f of every fuzzy nbd of $f(x_\alpha)$ is a fuzzy pre β-nbd of x_α in X,
(f) $f(p\beta clA) \leq cl(f(A))$, for all fuzzy set A in X,
(g) $p\beta cl(f^{-1}(B)) \leq f^{-1}(clB)$, for all fuzzy set B in Y,
(h) $f^{-1}(intB) \leq p\beta int(f^{-1}(B))$, for all fuzzy set B in Y,
(i) for every basic open fuzzy set V in Y, $f^{-1}(V) \in FP\beta O(X)$.
Proof. (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open nbd of $f(x_\alpha)$ in Y. By (b), $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$ ... (1). Now $f(x_\alpha) \in V \Rightarrow x_\alpha \in f^{-1}(V)$

$(= U$, say). Then $x_\alpha \in U$ and by (1), $U(= f^{-1}(V)) \in FP\beta O(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (c), there exists $U \in FP\beta O(X)$ containing x_α such that $f(U) \leq V$. Then $x_\alpha \in U \leq f^{-1}(V)$. Now $U \leq \beta int(clU)$. Then $U \leq \beta int(clU) \leq \beta int(cl(f^{-1}(V))) \Rightarrow x_\alpha \in U \leq \beta int(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_\alpha)$ in Y. Then there exists a fuzzy open set V in Y such that $f(x_\alpha) \in V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. Then by (b), $f^{-1}(V) \in FP\beta O(X)$ and $x_\alpha \in f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy pre β-nbd of x_α in X.
(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V$. Then V is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (e), there exists $U \in FP\beta O(X)$ containing x_α such that $U \leq f^{-1}(V) \Rightarrow x_\alpha \in U \leq \beta int(clU) \leq \beta int(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq \beta int(cl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in Y. By (d), $f^{-1}(cl(f(A))) \in FP\beta C(X)$ containing A. Therefore, $p\beta clA \leq p\beta cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(p\beta clA) \leq cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(p\beta cl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB = B \Rightarrow p\beta cl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in FP\beta C(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(p\beta cl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB \Rightarrow p\beta cl(f^{-1}(B)) \leq f^{-1}(clB)$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $p\beta cl(f^{-1}(f(A))) \leq f^{-1}(cl(f(A))) \Rightarrow p\beta clA \leq f^{-1}(cl(f(A))) \Rightarrow f(p\beta clA) \leq cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \leq \beta int(cl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in FP\beta O(X) \Rightarrow f^{-1}(intB) = p\beta int(f^{-1}(intB)) \leq p\beta int(f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \leq p\beta int(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in FP\beta O(X)$.
(b) $\Rightarrow$ (i). Obvious.
(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigvee_{\alpha\in\Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigvee_{\alpha\in\Lambda} W_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(W_\alpha) \in FP\beta O(X)$ (by (i) and by Result 3.2). Hence (b) follows.

Theorem 4.5. *A function $f : X \to Y$ is fuzzy pre β-continuous if and only if for each fuzzy point x_α in X and each fuzzy open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre β-q-nbd W in X such that $f(W) \leq V$.*

Proof. Let f be fuzzy pre β-continuous function and x_α be a fuzzy point in X and V be a fuzzy open q-nbd of $f(x_\alpha)$ in Y. Then $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1 \Rightarrow V(y) > 1 - \alpha \Rightarrow V(y) > \beta > 1 - \alpha$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 4.4 (a)⇒(c), there exists $W \in FP\beta O(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - \alpha \Rightarrow x_\alpha qW$ and $f(W) \leq V$.

Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, $1/n \leq 1/m \Rightarrow 1 + 1/n \leq 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n} qV \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in FP\beta O(X)$ such that $x_{\alpha_n} qU_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in FP\beta O(X)$ (by Result 3.2) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in FP\beta O(X)$. Hence by Theorem 4.2, f is fuzzy pre β-continuous function.

Note 4.6. *Since fuzzy regular open set is fuzzy open, by Note 4.3, we can easily say that the inverse image of fuzzy regular open set under fuzzy pre β-continuous function is fuzzy pre β-open.*

Let us recall the following definition from [4, 3] for ready references.

Definition 4.7. *A function $f : X \to Y$ is called fuzzy continuous [4] (resp., fuzzy β-irresolute [3]) if the inverse image of fuzzy open (resp., fuzzy β-open) set in Y is fuzzy open set (resp., fuzzy β-open set) in X.*

Remark 4.8. *(i) It is clear from definitions that fuzzy β-irresolute function (resp. fuzzy continuous function) is fuzzy pre β-continuous, but the converse need not be so follows from the following example.*

(ii) Composition of two fuzzy pre β-continuous functions need not be so, follows

from the following example.

Example 4.9. (i) Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Then clearly i is fuzzy pre β-continuous function. Now every fuzzy set in (X, τ_2) is fuzzy β-open in (X, τ_2). Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Then $B \in F\beta O(X, \tau_2)$. Now $i^{-1}(B) = B \not\leq cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}B)) = 0_X \Rightarrow B \notin F\beta O(X, \tau_1) \Rightarrow i$ is not fuzzy β-irresolute function.
(ii) Consider above Example and the identity function $i : (X, \tau_2) \to (X, \tau_1)$. As every fuzzy set in (X, τ_2) is fuzzy pre β-open set in (X, τ_2), clearly i is fuzzy pre β-continuous function. But $A \in \tau_1, i^{-1}(A) = A \notin \tau_2$. So i is not fuzzy continuous function.
(iii). Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy pre β-continuous functions. But $B \in \tau_3$, $(i_2 \circ i_1)^{-1}(B) = B \not\leq \beta int_{\tau_1}(cl_{\tau_1}B) = 0_X \Rightarrow B \notin FP\beta O(X, \tau_1) \Rightarrow i_2 \circ i_1$ is not a fuzzy pre β-continuous function.

Lemma 4.10. [2]. *Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.*

Theorem 4.11. *Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy pre β-continuous function, so are f_1 and f_2.*
Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_\alpha)$ in X for any fuzzy point x_α in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since f is fuzzy pre β-continuous, there exists $V \in FP\beta O(Z)$ with $x_\alpha qV$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 4.10, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy pre β-continuous.
Similarly, f_2 is fuzzy pre β-continuous.

Lemma 4.12. [1]. *Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigwedge f^{-1}(B)$.*

Theorem 4.13. *Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is fuzzy pre β-continuous function, then f is so.*

Proof. Let g be fuzzy pre β-continuous function and B be a fuzzy open set in Y. Then by Lemma 4.12, $f^{-1}(B) = 1_X \bigwedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now since B is fuzzy open in Y, then $1_X \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_X \times B) = f^{-1}(B) \in FP\beta O(X)$ as g is fuzzy pre β-continuous function. Hence f is fuzzy pre β-continuous.

5. Fuzzy pre β-Irresolute Function: Some Properties

In this section we introduce fuzzy pre β-irresolute function, the class of which is strictly coarser than that of fuzzy pre β-continuous function.

Definition 5.1. *A function $f : X \to Y$ is called fuzzy pre β-irresolute if the inverse image of every fuzzy pre β-open set in Y is fuzzy pre β-open in X.*

Theorem 5.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(a) f is fuzzy pre β-irresolute,
(b) for each fuzzy point x_α in X such that each fuzzy pre β-open nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre β-open nbd U of x_α in X such that $f(U) \leq V$,
(c) $f^{-1}(F) \in FP\beta C(X)$, for all $F \in FP\beta C(Y)$,
(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy pre β-open nbd of $f(x_\alpha)$ is a fuzzy pre β-open nbd of x_α in X,
(e) $f(p\beta clA) \leq p\beta cl(f(A))$, for all $A \in I^X$,
(f) $p\beta cl(f^{-1}(B)) \leq f^{-1}(p\beta clB)$, for all $B \in I^Y$,
(g) $f^{-1}(p\beta intB) \leq p\beta int(f^{-1}(B))$, for all $B \in I^Y$.
Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function $f : X \to Y$ is fuzzy pre β-irresolute if and only if for each fuzzy point x_α in X and corresponding to any fuzzy pre β-open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre β-open q-nbd W of x_α in X such that $f(W) \leq V$.*
Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Note 5.4. *Composition of two fuzzy pre β-irresolute functions is also so.*

Theorem 5.5. *If $f : X \to Y$ is fuzzy pre β-irresolute and $g : Y \to Z$ is fuzzy pre β-continuous (resp., fuzzy continuous), then $g \circ f : X \to Z$ is fuzzy pre β-continuous.*
Proof. Obvious.

Remark 5.6. *Every fuzzy pre β-irresolute function is fuzzy pre β-continuous, but the converse is not true, in general, follows from the following example.*

Example 5.7. Fuzzy pre β-continuous function $\not\Rightarrow$ fuzzy pre β-irresolute function. Consider Example 4.9(i). Here i is fuzzy pre β-continuous function. Also here

$B \in FP\beta O(X, \tau_2)$ as every fuzzy set in (X, τ_2) is fuzzy pre β-open set in (X, τ_2). But $i^{-1}(B) = B \notin FP\beta O(X, \tau_1) \Rightarrow i$ is not fuzzy pre β-irresolute function.

6. Fuzzy pre β-Regular Space

In this section fuzzy pre β-regular space is introduced in which space fuzzy closed set and fuzzy pre β-closed set coincide. Also some applications of fuzzy pre β-continuous and fuzzy pre β-irresolute functions are shown here.

Definition 6.1. *An fts (X, τ) is said to be fuzzy pre β-regular space if for each fuzzy pre β-closed set F in X and each fuzzy point x_α in X with $x_\alpha \notin F$, there exist a fuzzy open set U in X and a fuzzy pre β-open set V in X such that $x_\alpha qU$, $F \leq V$ and $U \not{q} V$.*

Theorem 6.2. *For an fts (X, τ), the following statements are equivalent:*
(a) X is fuzzy pre β-regular,
(b) for each fuzzy point x_α in X and each fuzzy pre β-open set U in X with $x_\alpha qU$, there exists a fuzzy open set V in X such that $x_\alpha qV \leq p\beta clV \leq U$,
(c) for each fuzzy pre β-closed set F in X, $\bigwedge\{clV : F \leq V, V \in FP\beta O(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy pre β-open set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $p\beta clV \leq U$.

Proof. (a)⇒(b). Let x_α be a fuzzy point in X and U, a fuzzy pre β-open set in X with $x_\alpha qU$. Then $x_\alpha \notin 1_X \setminus U \in FP\beta C(X)$. By (a), there exist a fuzzy open set V and a fuzzy pre β-open set W in X such that $x_\alpha qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_\alpha qV \leq 1_X \setminus W \leq U \Rightarrow x_\alpha qV \leq p\beta clV \leq p\beta cl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)⇒(a). Let F be a fuzzy pre β-closed set in X and x_α be a fuzzy point in X with $x_\alpha \notin F$. Then $x_\alpha q(1_X \setminus F) \in FP\beta O(X)$. By (b), there exists a fuzzy open set V in X such that $x_\alpha qV \leq p\beta clV \leq 1_X \setminus F$. Put $U = 1_X \setminus p\beta clV$. Then $U \in FP\beta O(X)$ and $x_\alpha qV$, $F \leq U$ and $U \not{q} V$.
(b)⇒(c). Let F be fuzzy pre β-closed set in X. Then $F \leq \bigwedge\{clV : F \leq V, V \in FP\beta O(X)\}$.
Conversely, let $x_\alpha \notin F \in FP\beta C(X)$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FP\beta O(X)$. By (b), there exists a fuzzy open set U in X such that $x_\alpha qU \leq p\beta clU \leq 1_X \setminus F$. Put $V = 1_X \setminus p\beta clU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_\alpha \notin clV \Rightarrow \bigwedge\{clV : F \leq V, V \in FP\beta O(X)\} \leq F$.
(c)⇒(b). Let V be any fuzzy pre β-open set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x) + \alpha > 1 \Rightarrow x_\alpha \notin (1_X \setminus V)$ where $1_X \setminus V \in FP\beta C(X)$. By (c), there exists $G \in FP\beta O(X)$ such that $1_X \setminus V \leq G$ and $x_\alpha \notin clG$. Then there exists a fuzzy open set U in X with $x_\alpha qU$, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V \Rightarrow x_\alpha qU \leq p\beta clU \leq p\beta cl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)⇒(d). Let G be any fuzzy set in X and U be any fuzzy pre β-open set in X

with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_\alpha qU \Rightarrow x_\alpha \notin 1_X \setminus U$ where $1_X \setminus U \in FP\beta C(X)$. By (c), there exists $W \in FP\beta O(X)$ such that $1_X \setminus U \leq W$ and $x_\alpha \notin clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x) + \alpha > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $p\beta clV = p\beta cl(1_X \setminus clW) \leq p\beta cl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)⇒(b). Obvious.

Note 6.3. *It is clear from Theorem 6.2 that in a fuzzy pre β-regular space, every fuzzy pre β-closed set is fuzzy closed and hence every fuzzy pre β-open set is fuzzy open. As a result, in a fuzzy pre β-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy pre β-closed (resp., fuzzy pre β-open) sets coincide.*

Theorem 6.4. *If $f : X \to Y$ is fuzzy pre β-continuous function and Y is fuzzy pre β-regular space, then f is fuzzy pre β-irresolute function.*
Proof. Let x_α be a fuzzy point in X and V be any fuzzy pre β-open q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy pre β-regular space. By Theorem 6.2 (a)⇒(b), there exists a fuzzy open set W in Y such that $f(x_\alpha)qW \leq p\beta clW \leq V$. Since f is fuzzy pre β-continuous function, by Theorem 4.5, there exists $U \in FP\beta O(X)$ with $x_\alpha qU$ and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy pre β-irresolute function.
Let us now recall following definitions from [4, 5] for ready references.

Definition 6.5. [4]. *A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.*

Definition 6.6. [4]. *A fuzzy cover $\mathcal{U}$ of an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.*

Definition 6.7. [5]. *An fts (X, τ) is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.*

Theorem 6.8. *If $f : X \to Y$ is a fuzzy pre β-continuous, surjective function and X is fuzzy pre β-regular and almost compact space, then Y is fuzzy almost compact space.*
Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy pre β-continuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy pre β-open and hence fuzzy open cover of X as X is fuzzy pre β-regular space (by Note 6.3). Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$

of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy pre β-regular space, by Note 6.3, $clA = p\beta clA$ for all $A \in I^X$ and so $1_X = \bigcup_{i=1}^{n} p\beta cl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^{n} p\beta cl(f^{-1}(U_i))) = \bigcup_{i=1}^{n} f(p\beta cl(f^{-1}(U_i))) \leq \bigcup_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)⇒(f)) $\leq \bigcup_{i=1}^{n} cl(U_i) \Rightarrow \bigcup_{i=1}^{n} cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.

Since every fuzzy open set is fuzzy pre β-open, we can easily state the following theorem the proof of which is similar to that of Theorem 6.8.

Theorem 6.9. *If a bijective function $f : X \to Y$ is fuzzy pre β-irresolute where X is fuzzy pre β-regular, almost compact space, then Y is fuzzy almost compact.*

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