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ANALYTICAL APPROXIMATIONS WITH EXACT NON-INTEGRAL PART FOR VOLTERRA'S POPULATION MODEL

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Abstract: The present paper is strongly motivated by the brilliant work of Wazwaz [9, Sections 4 and 5] computing analytical approximation in the form of a series truncated at t^8 and applying [4/4] Pade approximant to the series. In this paper, we make an attempt to workout analytical approximations in the form of the series truncated at t^4 and apply suitable Pade approximations as well as asymptotic approximations with the following features: i) The solution contains exact non integral part. ii) The solution exhibits the population rapid rise along logistic curve followed by decay to zero in the long run. iii) The solution is reasonably comparable with that of Wazwaz [9] using the information from the series with terms only upto t^4.

Keywords and Phrases: Volterra Population Model, Adomian decomposition method, Pade approximants, Asympote approximations.

2020 Mathematics Subject Classification: 41A21.

1. Introduction

Following Wazwaz [5, 7, 8, 9], the Volterra population model is

$$k\frac{du}{dt} = u(t) - u^2(t) - u(t)\int_0^t u(x)dx \qquad u(0) = 0.1 \tag{1.1}$$

The following are well described [5, 8, 9]:

i) $u = u(t)$ stands for scaled population of identical individuals at time t.

ii) $k\dfrac{du}{dt} = u(t) - u^2(t)$ is well known as logistic equation where $k = \frac{c}{ab}$ is a non dimensionalized parameter in which $a > 0$ is birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is toxicity coefficient.

iii) The integral term $\int_0^t u(x)dx$ actually characterizes the accumulated toxicity produced since time zero.

In the present paper, we multiply a parameter $\epsilon > 0$ to integral part of (1.1) to obtain the following method.

$$k\frac{du}{dt} = u(t) - u^2(t) - \epsilon\ u(t)\int_0^t u(x)dx \qquad u(0) = 0.1 \tag{1.2}$$

When $\epsilon = 0$, the exact solution of (1.2) is

$$u(t) = \frac{1}{1 + 9e^{(\frac{-t}{k})}} \tag{1.3}$$

And when $\epsilon = 1$, we set back the model (1.1), we call the solution (1.3) as exact non integral part of the solution of (1.1). Following section 4 of [9].
Put $y(t) = \int_0^t u(x)dx \implies y'(t) = u(t)$.
Then (1.3) transforms into the following second order nonlinear ODE:

$$y''(t) = \frac{1}{k}[y'(t) - (y'(t))^2 - \epsilon y(t)y'(t)]$$
$$y(0) = 0, \qquad y'(0) = u(0) = 0.1 \tag{1.4}$$

Further, integration twice on both side of (1.4), we obtain [6]

$$y(t) = (0.1)t + \frac{1}{k}\int_0^t (t-x)[y'(t) - (y'(t))^2 - \epsilon y(t)y'(t)]dt \tag{1.5}$$

In the next section, we apply the Adomian decomposition method to solve (1.5) approximately keeping only first few terms. In the ensuring we make an attempt to apply not only Pade approximation technique but an assymptotic technique to construct an approximats solution with the following features:

i) The solution contains exact nonintegral part given by (1.3).

ii) The solution exhibits the population rapid rise along logistic curve followed by decay to zero in the long run.

iii) The solution is reasonably comparable with that of Wazwaz [9].

2. The Decomposition Method

Following [1, 2, 6, 9, 10], let us represent the solution of (1.5) by

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0 + y_1 + y_2 + y_3 + \cdots \tag{2.1}$$

Then,

$$y^{'}(t) = \sum_{n=0}^{\infty} A_n(t),$$

where

$$A_n(t) = y_n^{'}(t) \tag{2.2}$$

$$(y^{'}(t))^2 = \left(\sum_{n=0}^{\infty} A_n(t)\right)^2 = \sum_{n=0}^{\infty} B_n(t),$$

where

$$B_n(t) = \sum_{k=0}^{n} A_k(t) A_{n-k}(t) \tag{2.3}$$

$$y(t)y^{'}(t) = \left[\sum_{n=0}^{\infty} y_n(t)\right]\left[\sum_{n=0}^{\infty} A_n(t)\right] = \sum_{n=0}^{\infty} C_n(t),$$

where

$$C_n(t) = \sum_{k=0}^{n} y_k(t) A_{n-k}(t) \tag{2.4}$$

Substituting (2.1) – (2.4) in (1.5) we get

$$\begin{aligned} y_0(t)+y_1(t)+y_2(t)+\cdots = (0.1)\ t &+ \frac{1}{k}\int_0^t (t-x)[A_0(x)-B_0(x)-\epsilon C_0(x)]dx \\ &+ \frac{1}{k}\int_0^t (t-x)[A_1(x)-B_1(x)-\epsilon C_1(x)]dx \\ &+ \cdots \end{aligned}$$

As a result, one can equate,

$$\begin{aligned} y_0(t) &= (0.1)t \\ y_1(t) &= \frac{1}{k}\int_0^t (t-x)[A_0(x)-B_0(x)-\epsilon C_0(x)]dx \\ y_2(t) &= \frac{1}{k}\int_0^t (t-x)[A_1(x)-B_1(x)-\epsilon C_1(x)]dx \\ \vdots &= \vdots \end{aligned}$$

The first iteration:

$$\begin{aligned} y_1(t) &= \frac{1}{k}\int_o^t (t-x)\left[y_0^{'}(x)-\{y_0(x)\}^2-\epsilon y_0(x)y_0^{'}(x)\right]dx \\ &= \left(\frac{9}{200k}\right)t^2-\left(\frac{\epsilon}{600k}\right)t^3 \end{aligned}$$

Second iteration:

$$\begin{aligned} y_2(t) &= \frac{1}{k}\int_o^t (t-x)\left[y_1^{'}(x)-\{2y_0^{'}(x)y_1^{'}(x)\}-\epsilon\{y_0(x)y_1^{'}(x)+y_1(x)y_0^{'}(x)\}\right]dx \\ &= \left(\frac{3}{250k^2}\right)t^3-\left(\frac{7\epsilon}{4800k^2}\right)t^4+\left(\frac{\epsilon^2}{30000k^2}\right)t^5 \end{aligned}$$

Third iteration:

$$\begin{aligned} y_3(t) &= \frac{1}{k}\int_o^t (t-x)\Big[y_2^{'}(x)-(2y_0^{'}(x)y_2^{'}(x)+\{y_1^{'}(x)\}^2) \\ &\qquad -\epsilon\{y_0(x)y_2^{'}(x)+y_1(x)y_1^{'}(x)+y_2(x)y_0^{'}(x)\}\Big]dx \\ &= \left(\frac{69}{40000k^3}\right)t^4-\left(\frac{226\epsilon+531}{1200000k^3}\right)t^5+\left(\frac{26\epsilon^2+265\epsilon}{7200000k^3}\right)t^6-\left(\frac{17\epsilon^2}{25200000k^3}\right)t^7 \end{aligned}$$

Fourth iteration:

$$\begin{aligned}
y_4(t) &= \frac{1}{k}\int_o^t (t-x)\Big[y_3^{'}(x) - (2y_0^{'}(x)y_3^{'}(x) + (2y_1^{'}(x)y_2^{'}(x)) \\
&\qquad - \epsilon\{y_0(x)y_3^{'}(x) + y_1(x)y_2^{'}(x) + y_2(x)y_1^{'}(x) + y_3(x)y_0^{'}(x)\}\Big]\,dx \\
&= \left(\frac{-3}{62500k^4}\right)t^5 + \left(\frac{788\epsilon - 6399}{36000000k^4}\right)t^6 + \left(\frac{208\epsilon^2 + 7346\epsilon + 4421}{504000000k^4}\right)t^7 \\
&+ \left(\frac{480\epsilon^3 - 8652\epsilon^2 - 7420\epsilon - 1088}{16128000000k^4}\right)t^8 + \left(\frac{112\epsilon^3 + 136}{18144000000k^4}\right)t^9
\end{aligned}$$

For the particular cases $\epsilon = 1, \quad k = 0.1$ (given by [9])

$$\begin{aligned}
y_1(x) &= \frac{9}{20}t^2 - \frac{1}{60}t^3 = 0.45t^2 - \frac{1}{60}t^3 \\
y_2(t) &= \frac{6}{5}t^3 - \frac{7}{48}t^4 + \frac{1}{300}t^5 \\
y_3(t) &= \frac{69}{40}t^4 - \frac{757}{1200}t^5 - \frac{97}{2400}t^6 - \frac{17}{25200}t^6 \\
y_4(t) &= \frac{-300}{625}t^5 - \frac{5611}{3600}t^6 - \frac{11975}{504000}t^7 - \frac{16680}{16128000}t^8 + O(t^9) \\
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \cdots \\
&= (0.1)t + \left[\frac{9}{200k}t^2 - \frac{\epsilon}{600k}t^3\right] \\
&\quad + \left[\frac{3}{250k^2}t^3 - \frac{7\epsilon}{4800k^2}t^4 + \frac{\epsilon^2}{30000k^2}t^5\right] \\
&\quad + \left[\frac{69}{40000k^3}t^4 - \frac{(226\epsilon + 531)}{1200000k^3}t^5 + \frac{(26\epsilon^2 + 265\epsilon)}{7200000k^3}t^6 - \frac{17\epsilon^2}{25200000k^3}t^7\right] \\
&= (0.1)t + \left[\frac{9}{200k}t^2 - \frac{\epsilon}{600k}t^3\right] \\
&\quad + \left[\frac{3}{250k^2}t^3 - \frac{7\epsilon}{4800k^2}t^4 + \frac{\epsilon^2}{30000k^2}t^5\right] \\
&\quad + \left[\frac{69}{40000k^3}t^4 - \frac{(226\epsilon + 531)}{1200000k^3}t^5\right] + \left[\frac{-3}{62500k^4}t^5\right] + O(t^6) \\
&= (0.1)t + \frac{9}{200k}t^2 + \left[\frac{-\epsilon}{600k} + \frac{3}{250k^2}\right]t^3 + \left[-\frac{7\epsilon}{4800k^2} + \frac{69}{40000k^3}\right]t^4
\end{aligned}$$

$$+\left[\frac{\epsilon^2}{30000k^2}-\frac{(226\epsilon+531)}{1200000k^3}-\frac{3}{62500k^4}\right]t^5+O(t^6)$$

$$u(t)=y'(t) = \left[(0.1)+\frac{9}{100k}t+\frac{9}{250k^2}t^2\frac{69}{10000k^3}t^3-\left(\frac{531}{2400000k^3}+\frac{3}{12500k^4}\right)t^4\right]$$
$$-\epsilon\left[\frac{1}{200k}t^2+\frac{7}{1200k^2}t^3+\frac{226}{240000k^3}t^4\right]+\epsilon^2\left[\frac{1}{6000k^2}t^4\right]+O(t^5)$$

3. Analysis of the Method

$u(t)\approx u_A(t)=b_0(t)-a_1(t)\epsilon+a_2(t)\epsilon^2$, where

$$b_0(t) = \left[(0.1)+\frac{9}{100k}t+\frac{9}{250k^2}t^2\frac{69}{10000k^3}t^3-\left(\frac{531}{2400000k^3}+\frac{3}{12500k^4}\right)t^4\right]$$
$$a_1(t) = \left[\frac{1}{200k}t^2+\frac{7}{1200k^2}t^3+\frac{226}{240000k^3}t^4\right]$$
$$a_2(t) = \left[\frac{1}{6000k^2}t^4\right]$$

Let us write

$$b_0(t) = \left[\frac{1}{10}+\frac{9}{100}\left(\frac{t}{k}\right)+\frac{9}{250}\left(\frac{t}{k}\right)^2\right]+O(t^3)$$

[0/2] Pade approximant [3] to $b_0(t)$ is

$$[0/2]b_0(t) = \frac{1/10}{1-\frac{9}{10}\left(\frac{t}{k}\right)+\frac{9}{10}\left[\frac{1}{2}\left(\frac{t}{k}\right)^2\right]}=\frac{1}{1+9\left[1-\left(\frac{t}{k}\right)+\frac{1}{2}\left(\frac{t}{k}\right)^2\right]}$$

Let us apply the following asymptotic approximation [4]

$$1-\left(\frac{t}{k}\right)+\frac{1}{2}\left(\frac{t}{k}\right)^2\sim e^{\frac{-t}{k}} \quad \text{as} \quad \frac{t}{k}\to 0.$$

Hence

$$b_0(t) \approx \frac{1}{1+9e^{\frac{-t}{k}}}=a_0(t,k)$$

and

$$u_A(t) \approx \frac{1}{(1+9e^{\frac{-t}{k}})}-\epsilon\left[\frac{1}{200k}t^2+\frac{7}{1200k^2}t^3+\frac{226}{240000k^3}t^4\right]+\epsilon^2\left[\frac{1}{6000k^2}t^4\right]$$
$$= a_0(t,k)-a_1(t,k)\epsilon+a_2(t,k)\epsilon^2=u_B(t,k,\epsilon).$$

Let us apply [0/1] Pade approximation and an asymptotic approximation as $t \to 0$.

$$u_B(t,k,\epsilon) \quad \sim \quad \frac{a_0(t,k)}{1+\dfrac{a_1(t,k)}{a_0(t,k)}\epsilon} + a_2(t,k)\epsilon^2 e^{-t}, \quad t \to 0.$$

Hence the desired approximation is given by

$$u_1(t) = \frac{\dfrac{1}{1+9e^{\frac{-t}{k}}}}{1+\epsilon\left[\dfrac{1}{200k}t^2+\dfrac{7}{1200k^2}t^3+\dfrac{226}{240000k^3}t^4\right](1+9e^{\frac{-t}{k}})} + \epsilon^2\left[\frac{1}{6000k^2}t^4\right]e^{-t} \tag{3.1}$$

For the purpose of comparison, we mention the following approximation given in [9]:

$$u(t) \approx u_2(t)$$
$$= \frac{0.1+0.4687931695\ t+0.9249573236\ t^2+0.9231293234\ t^3+0.4004233108\ t^4}{1-4.312068305\ t+12.55818798\ t^2-13.88064046\ t^3+10.86830522\ t^4} \tag{3.2}$$

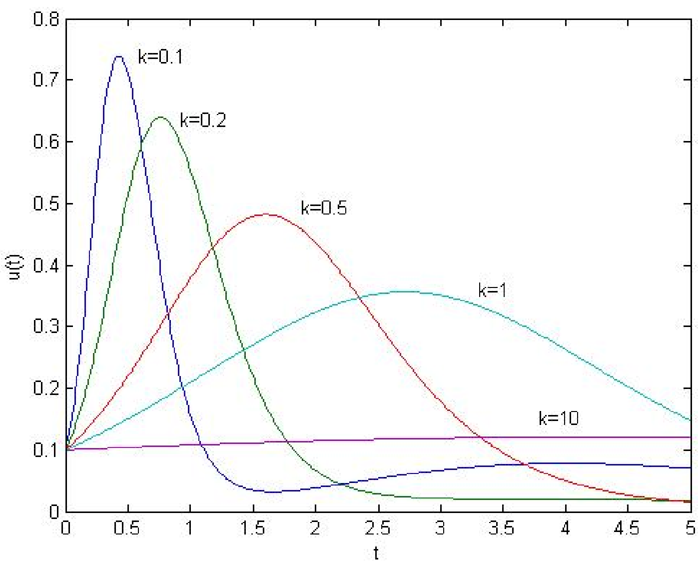

Figure 1: Approximate solution of (1.1) for $u(0) = 1$, obtained by solving (1.5) with values $k = 0.1, 0.2, 0.5, 1, 10$.

t	$u_1(t)$	$u_2(t)$
0	0.100000000000000	0.100000000000000
0.5	0.818333070656960	0.761078065585266
1.0	0.394225231596236	0.451941057102017
1.5	0.146239846466385	0.259259406173715
2.0	0.083860107770043	0.175705102389915
2.5	0.083860107770043	0.134308516684525
3.0	0.077910589902910	0.110756712739661
3.5	0.081479713514703	0.095904327346303
4.0	0.081715605297143	0.085808937664095
4.5	0.078189024232110	0.078553522581046
5.0 ‘	0.071693633825905	0.073112177807395

Table 1: Comparison of $u_1(t)$ and $u_2(t)$

The analysis given by expression for $u_1(t)$, numerical results in the Figure 1 and numerical results in the Table 1 clearly convey the following conclusions:

i) The solution contains exact non integral part.

ii) The solution exhibits the population rapid rise along logistic curve followed by decay to zero in the long run.

iii) The solution is reasonably comparable with that of Wazwaz [9] using the information from the series with terms only upto t^4.

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