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LINEAR AND NON-LINEAR WAVELET APPROXIMATIONS OF FUNCTIONS OF LIPSCHITZ CLASS AND RELATED CLASSES USING THE HAAR WAVELET SERIES

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Abstract: In this paper, linear wavelet approximation of a functions f belonging to Lipξ, and $Lip(\xi, p), 1 \leq p < \infty$, have been determined by using Haar scaling function and Haar wavelet series. A non-linear wavelet approximation of function belonging $L^2(\mathbb{R})$ and Lipα class has been obtained. The comparison of the order of linear and non-linear approximations have been studied. It is observed that a non-linear approximation error is better and sharper than linear approximation error.

Keywords and Phrases: Lipα class of functions, Lip(α, p) class of functions, Lipξ class of functions, Lip(ξ, p) class of functions, Multiresolution analysis, Scaling function ϕ, Linear approximation, Non-linear approximation.

2020 Mathematics Subject Classification: 40A30, 42C15, 42A16, 42C40, 65T60, 65L10, 65L60, 65R20.

1. Introduction

Wavelet analysis plays a vital role in Functional Analysis, Numerical Analysis, Signal Processing, Engineering and Modern Technology. The roles of target functions and approximating functions are often observed in modern analysis. The fundamental approach applicable in the approximation theory is to resolve the target function into the suitable approximates. A certain function is expressed in the form of a Fourier series. Similarly, a target function may be represented in the form of the wavelet series. In wavelet analysis, if the target function is approximated by N^{th} partial sums S_N of the wavelet series then this approximation is known as the N-terms approximation. If N is taken sufficiently large, then a better approximation is obtained. One of the main goals for an approximation is to replace S_n. If the partial sum S_N is replaced by another finite sum having fewer coefficients by which the target function f is approximated then this estimator is considered as non-linear approximation.

In 1979, the method of N-term approximation, the first time, has been utilized for multivariate splines by Oskolkov [19]. The idea of non-linear approximation is given by DeVore [5], Traub et al. [26] and Novak [18]. The wavelet approximation has been studied by several researchers like Daubechies [3], Chui [2], Morlet et al. [16], Meyer [15], Strang [24], Natanson [17], Lal et al. [7, 8, 9, 10, 11, 12, 13, 14], Keshavarz et al. [6], Walter [27, 28] and, so forth. The relationship between irregular fractals and various phenomena such as Brownian trajectories, fractional Brownian motion, typical Feynmann path, turbulent fluid motion and complex Bernoulli spiral (*see* [21]) are established. These irregular fractals exhibit a locally Lipschitz condition within specific finite intervals at each point. Rehman & Siddiqi [20] and Shiri & Azadi's [22] works provide the basis for the current study by obtaining approximations of these fractals in different functional spaces using different norms.

The purpose of this paper is to determine the linear wavelet approximation of the function f belonging to Lipξ, Lip$(\xi, p), 1 \leq p \leq \infty$ and non-linear approximation of $f \in L^2(\mathbb{R})$ and Lipα class using Haar scaling function and wavelet series. By the comparison of linear as well as non-linear wavelet approximation, it is found that the non-linear wavelet approximation error decays more quickly than a linear approximation.

2. Definitions and Preliminaries

2.1. Function of Lipα class and function of Lip(ξ, p) class

A function $f \in$ Lipα if,

$$|f(x) - f(y)| = O(|x-y|^{\alpha}), \text{for } \ 0 < \alpha \leq 1, \quad [25].$$

Let ξ be a monotonic increasing function of t. Then $f \in$ Lip(ξ, p) if

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx\right\}^{\frac{1}{p}} = O(\xi(t)), 1 \leq p < \infty, \quad [23].$$

2.2. Remarks

(i) It is important to note that $\mathrm{Lip}(\xi, p)$ class coincides with $\mathrm{Lip}\alpha$ if

$$\xi(t) = t^{\alpha},\ 0 < \alpha \leq 1 \ \text{ and } \ p \to \infty.$$

(ii) Define

$$f(x) = |x| \quad \forall \ \ x \in [0,1], \quad \text{Then } f \in \mathrm{Lip}\alpha.$$

(iii) Define a function $f : (0,1] \to \mathbb{R}$

$$f(x) = \frac{1}{x} \quad \forall \ \ x \in (0,1]. \ \ \text{Then } f \text{ is continuous \ but } f \notin \mathrm{Lip}\alpha.$$

(iv) If $f \in \mathrm{Lip}\alpha \ \ \alpha \geq 0$, then f is continuous, indeed uniformly continuous.

(v) $\mathrm{Lip}\alpha$ class is a linear space over $\mathbb{R}$ or $\mathbb{C}$

(vi) If $f \in \mathrm{Lip}\alpha \ \ \alpha > 1$, then f is constant function.

2.3. Multiresolution Analysis and Haar Scaling Function

A multiresolution analysis of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$, with the following properties:

1. $V_j \subset V_{j+1}$,
2. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,
3. $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$,
4. $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$,
5. there exists a function $\phi \in V_0$, such that the collection $\{\phi(x-k); k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Let $\psi \in L^2(\mathbb{R}), \ \ and \ \ \psi_{j,k} := 2^{\frac{j}{2}}\psi(2^j - k)$ and

$$W_j := clos \left\langle \psi_{j,k} : k \in \mathbb{Z} \right\rangle.$$

Then this family of subspaces of $L^2(\mathbb{R})$ gives a direct sum decomposition of $L^2(\mathbb{R})$ is the same that every $f \in L^2(\mathbb{R})$ has a unique decomposition

$$f(x) = \cdots + g_{-2}(x) + g_{-1}(x) + g_0(x) + g_1(x) + g_2(x) + \cdots$$

where $g_j \in W_j$ for all $j \in \mathbb{Z}$ and we describe this by writing

$$L^2(\mathbb{R}) = V_j \oplus_{i=j}^{\infty} W_i$$

Where

$$V_j := \oplus_{k=-\infty}^{j-1} W_k.$$

$\left\{\psi_{j,k}; k \in \mathbb{Z} \text{ where } \psi j, k = 2^{\frac{j}{2}}\psi(2^j x - k)\right\}$, is a Riesz basis of W_j.

A function $\phi \in L^2(\mathbb{R})$ is called a scaling function, if the subspace V_j, defined by

$$V_j := clos_{L^2\mathbb{R}} \left\{\phi_{j,k} : k \in \mathbb{Z}\right\}, j \in \mathbb{Z}$$

satisfy the properties (1) to (5) stated above in this section. It is important to note that the scaling function ϕ generates a Multiresolution analysis $\{V_j\}$ of $L^2(\mathbb{R})$, [4]. Haar scaling function, denoted by ϕ, is defined by

$$\phi(t) = \chi_{[0,1)} = \begin{cases} 1, & 0 \le x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

The family of functions

$$\left\{\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j. - k) \;\; where \;\; j, k \in \mathbb{Z}\right\},$$

is called the system of Haar scaling functions.
An orthogonal wavelet is the Haar function is denoted by ψ_H and defined by

$$\psi_H(t) := \begin{cases} 1 & \text{for } \; 0 \le t < \frac{1}{2}; \\ -1 & \text{for } \; \frac{1}{2} \le t < 1; \\ 0 & \text{otherwise, [1].} \end{cases}$$

Consider $f \in L^2([0,1),)$ has an expansion in terms of Haar functions as follows. For any integer $n \ge 0$,

$$\begin{aligned} f(t) &= \sum_{k=0}^{2^j-1} \langle f, \phi_{n,k}\rangle \phi_{n,k}(t) + \sum_{j=n}^{\infty}\sum_{k=0}^{2^j-1} \langle f, \psi_{n,k}\rangle \psi_{n,k}(t) \\ &= \sum_{k=0}^{2^j-1} c_{n,k}\phi_{n,k}(t) + \sum_{j=n}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{n,k}(t) \end{aligned}$$

which is known as Haar series and $d_{j,k}$ and $c_{j,k}$ are the Haar coefficients for wavelet and Haar scaling coefficients, respectively.

2.4. Linear Wavelet Approximation

Let f be a target function on $\Omega = [0,1)$. Let N be a positive integer and $T =: \{0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots < t_N = 1\}$ be an ordered sets of points in Ω. These points determine a partition $T = \{I_k\}_{k=1}^{N}$ of Ω in to N, disjoint intervals $I_k = [t_{k-1}, t_k], 1 \leq k \leq N$. Let $S^{'}(T)$ denote the space of piecewise constant functions relative to the partition $P(T)$. The characteristic function $\{\chi_I; I \in P\}$ form a basis for $S^{'}(T)$; each function $s \in S^{'}(T)$ is uniquely represented by

$$s = \sum_{I \in P} C_I \chi_I$$

Thus $S^{'}(T)$ is a linear space of dimension N.
For, $0 < p < \infty$, the error in approximating a function $f \in L^p[0,1)$ by the elements of $S^{'}(T)$ is given by

$$E(f)_p = \inf_{s \in S^{'}(T)} \|f - s\|_{L^p[0,1)} = \inf_{s \in S^{'}(T)} \left\{ \frac{1}{2\pi} \int_0^1 |f(t) - s(t)|^p \, dt \right\}^{\frac{1}{p}}$$

For $p = \infty$,

$$E(f)_\infty = \sup_{x \in I_k, 1 \leq k \leq N} |f(x) - s(x)| = \|f - s\|_{L^\infty[0,1)} = \sup_{0 \leq t < 1} |f(t) - s(t)| .$$

3. Main Results Related to Linear Wavelet Approximation

In this section, three new theorems have been established in the following forms:

Theorem 3.1. *A function $f \in Lip\xi$ class i.e.,*

$$|f(x) - f(y)| = O\left(\xi\left(x - y\right)\right),$$

where ξ is a non negative monotonic increasing function of t such that $\xi(t) \to 0$ as $t \to 0^+$ iff

$$E(f)_\infty = O\left(\xi\left(\frac{\delta}{2}\right)\right), \text{ where } \delta = \max_{0 \leq k < N} |t_{k+1} - t_k| .$$

Proof. We define the piecewise constant function $s \in S^{'}(T)$ by

$$s(x) = f(x_I), x \in I, I \in P_n$$

with x_I the mid point of I. Then $|x - x_I| \leq \frac{\delta}{2}$
Now

$$\begin{aligned} |f(x) - s(x)| &= |f(x) - f(x_I)| \\ &\leq M\ \xi(x - x_I), where\ M \text{ is a suitable positive constant} \\ &\leq M\ \xi\left(\frac{\delta}{2}\right), |x - x_I| \leq \frac{\delta}{2}. \end{aligned}$$

Then

$$\begin{aligned} E(f)_\infty &= \sup_{s \in S'(T)} |f - s| \\ &= \|f - s\|_{L^\infty[0,1)} \\ &= O\left(\xi\left(\frac{\delta}{2}\right)\right). \end{aligned}$$

Conversely, let δ be the best approximation to f from $S'(T)$ in the $L^\infty[0,1)$ norm. If x and y are two points from $\Omega = [0,1)$ that are in the same interval $I \in P(T)$. Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - S_T(x) + S_T(x) - S_T(y) + S_T(y) - f(y)| \\ &\leq |f(x) - S_T(x)| + |S_T(x) - S_T(y)| + |S_T(y) - f(y)| \\ &= |f(x) - S_T(x)| + |S_T(y) - f(y)| \quad S_T \text{ is constant on I} \\ &\leq E(f)_\infty + E(f)_\infty \\ &= 2E(f)_\infty \\ &= 2M\xi\left(\frac{\delta}{2}\right) \\ &\leq 2M\xi\left(x - y\right). \end{aligned}$$

So $f \in \text{Lip}\xi$ class.
Thus, Theorem **3.1** is completely established.

Theorem 3.2. *A function $f \in Lip(\xi, p)$ class i.e.,*

$$\|f(. + h) - f(.)\|_{L^p[0,1)} = O\left(\xi\left(h\right)\right); \quad 0 < p \leq \infty$$

iff

$E(f)_p \leq \|f - S_T\| \leq C_p \|f\|_p \xi(\delta)$ *where* C_p *depends on* p *and* $\delta = \max\limits_{0 \leq k < N} |t_{k+1} - t_k|$.

Proof. Let us define the piecewise constant function $S_T \in S^{'}(T)$ by

$$S_T(x) = f(x_I),\ x_I \ \&\ x \in I,\ I \in P.$$

Then $|x - x_I| \leq \delta,\ x \in I$.
Thus

$$\begin{aligned} E(f)_p &= \min_{S_T} \left\{ \frac{1}{2\pi} \int_0^1 |f(x) - S_T(x)|^p \, dx \right\}^{\frac{1}{p}} \\ &= \min_{S_T} \left\{ \frac{1}{2\pi} \int_0^1 |f(x) - f(x_I)|^p \, dx \right\}^{\frac{1}{p}} \\ &= M\xi(x - x_I) \leq M\xi(\delta) = O\left(\xi\left(\delta\right)\right). \end{aligned}$$

Conversely, let S_T be the best approximation to f from $S^{'}(T)$ in the space Lip(ξ, p) and $x + h$ & x be two points from $\Omega = [0, 1)$ in the same interval $I \in P(T)$, we have

$$\begin{aligned} |f(x+h) - f(x)| &= |f(x+h) - S_T(x+h) + S_T(x+h) - S_T(x) + S_T(x) - f(x)| \\ &\leq |f(x+h) - S_T(x+h)| + |S_T(x+h) - S_T(x)| + |S_T(x) - f(x)| \\ &= |f(x+h) - S_T(x+h)| + |S_T(x) - f(x)|\,,\ S_T \text{ is constant on } I. \end{aligned}$$

So

$$\begin{aligned} \left\{ \frac{1}{2\pi} \int_0^1 |f(x+h) - f(x)|^p \, dx \right\}^{\frac{1}{p}} &\leq \left\{ \frac{1}{2\pi} \int_0^1 |f(x+h) - S_T(x+h)|^p \, dx \right\}^{\frac{1}{p}} \\ &+ \left\{ \frac{1}{2\pi} \int_0^1 |S_T(x) - f(x)|^p \, dx \right\}^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\begin{aligned} |f(.+h) - f(.)|_p &\leq ME(f)_p + ME(f)_p \\ &= 2M\xi(\delta) \\ &= 2M\xi(h) \\ &= O\left(\xi\left(h\right)\right). \end{aligned}$$

So $f \in$ Lip(ξ, p).
Thus, Theorem **3.2** is completely established.

Theorem 3.3. *If $f \in Lip(\xi, p), 1 \leq p \leq \infty$ and $g(t) = \sum_{j=0}^{N} \sum_{k=0}^{2^j - 1} d_{j,k}\psi_{j,k}$ is the Haar*

Wavelet series of f for some positive integer N, then the error of the approximation in Lip(ξ, p) is given by

$$\|f-g\|_{Lip(\xi,p)} = O\left(\xi\left(\frac{1}{2^N}\right)\right)$$

Proof. By Theorem **3.2**, $f \in \text{Lip}(\xi, p),\quad 1 \le p \le \infty$, we have

$$\begin{aligned} E_n(f, S_N)_p &= \inf \|f-g\|_p \le C_p \|f\|_{Lip(\xi,p)} \xi(\delta) \\ &= O\left(\xi\left(\delta\right)\right) \text{ where } \delta = \max_{0\le t\le N} |t_{k-1} - t_k| . \end{aligned}$$

So the error of the approximation in $\text{Lip}(\xi, p)$ class is

$$\begin{aligned} \|f-g\|_p &= \left\| f - \sum_{j=0}^{N-1} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k} \right\|_p = \left\| \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k} \right\|_p \\ &\le C_p \|f\|_{Lip(\xi,p)} \xi\left(\frac{1}{2^N}\right) = O\left(\xi\left(\frac{1}{2^N}\right)\right). \end{aligned}$$

Thus, Theorem **3.3** is completely established.

4. Corollaries

In this section, two new corollaries related to Theorems **3.1** and **3.2** have been established in the following forms:

Corollary 4.1. *If a function $f \in Lip\alpha$ i.e.,*

$$|f(x) - f(y)| = O\left(|x-y|^\alpha\right), \quad 0 < \alpha \le 1,$$

then

$$E_n(f)_\infty = O\left(\frac{1}{2^{(n+1)\alpha}}\right)$$

The proof of this Corollary can be developed on the lines of Theorem **3.1** by taking

$$\xi(t) = t^\alpha,\ o < \alpha \le 1, \quad \delta = \max_{0\le k<N} |t_{k+1} - t_k| ,\ N = 2^n.$$

Corollary 4.2. *If $f \in Lip(\alpha, p)$ class i.e.*

$$\|f(.+h) - f(.)\|_{L^p[0,1)} = O\left(|h|^\alpha\right),\ 0 < \alpha \le 1,\ 1 \le p < \infty$$

then

$$E_n(f)_p = \left(\frac{1}{n^{\alpha p}}\right)$$

The proof of Corollary **4.2** can be obtained by the Theorem **3.2** and taking

$$\xi(t) = t^{\alpha}, \quad \delta = \frac{1}{2^n}.$$

5. Non-linear Approximation by Haar Wavelet

At first, let us consider, a function

$$f(t) = \begin{cases} t, & 0 \leq t < 1; \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\psi_{j,k}(t) = 2^{\frac{j}{2}} \psi(2^j t - k).$$

$$\psi_{j,k}(t) = 2^{\frac{j}{2}} \begin{cases} 1, & 0 \leq 2^j t - k < \frac{1}{2}; \\ -1, & \frac{1}{2} \leq 2^j t - k < 1; \\ 0, & \text{otherwise}. \end{cases}$$

$$\psi_{j,k}(t) = \begin{cases} 2^{\frac{j}{2}}, & \frac{k}{2^j} \leq t < \frac{1}{2^{j+1}} + \frac{k}{2^j}; \\ -2^{\frac{j}{2}}, & \frac{1}{2^{j+1}} + \frac{k}{2^j} \leq t < \frac{k+1}{2^j}; \\ 0, & \text{otherwise}. \end{cases}$$

Now, we calculate $\langle f, \psi j, k \rangle$ for fixed j.
Since

$$\langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(t) \psi_{j,k}(t),$$

therefore, for $j = 0$ we have

$$\langle f, \psi_{0,k} \rangle = \int_{-\infty}^{\infty} f(t) \psi_{0,k}(t) dt = \int_{k}^{k+\frac{1}{2}} f(t) dt - \int_{k+\frac{1}{2}}^{k+1} f(t) dt.$$

Next,

$$\begin{aligned} \langle f, \psi_{0,0} \rangle &= \int_0^{\frac{1}{2}} f(t) dt - \int_{\frac{1}{2}}^{1} f(t) dt = \int_0^{\frac{1}{2}} t dt - \int_{\frac{1}{2}}^{1} t dt \\ &= -\frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} \langle f, \psi_{0,k} \rangle &= \int_{k}^{k+\frac{1}{2}} f(t) dt - \int_{k+\frac{1}{2}}^{1} f(t) dt = \int_{k}^{k+\frac{1}{2}} 0 dt - \int_{k+\frac{1}{2}}^{1} 0 dt \\ &= 0 \ \text{ for } \ k \in (\mathbb{Z} - \{0\}). \end{aligned}$$

So

$$\sum_{k=0}^{2^{j-1}} |\langle f, \psi_{0,k}\rangle| = \frac{1}{4}(1)$$

For $j = 1$ we have,

$$\langle f, \psi_{1,k}\rangle = \int_{-\infty}^{\infty} f(t)\psi_{1,k}(t)dt = 2^{\frac{1}{2}}\left(\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{4}} f(t)dt - \int_{\frac{k}{2}+\frac{1}{4}}^{\frac{k+1}{2}} f(t)dt\right).$$

Now, for $k = 0$ we have,

$$\begin{aligned}\langle f, \psi_{1,0}\rangle &= \int_{-\infty}^{\infty} f(t)\psi_{1,0}(t)dt = 2^{\frac{1}{2}}\left(\int_{0}^{\frac{1}{4}} f(t)dt - \int_{\frac{1}{4}}^{\frac{1}{2}} f(t)dt\right)\\ &= 2^{\frac{1}{2}}\left(\int_{0}^{\frac{1}{4}} tdt - \int_{\frac{1}{4}}^{\frac{1}{2}} tdt\right)\\ &= \frac{1}{2^{\frac{1}{2}}}\left(-\frac{1}{8}\right),\end{aligned}$$

And, for $k = 1$

$$\begin{aligned}\langle f, \psi_{1,1}\rangle &= 2^{\frac{1}{2}}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{4}} f(t)dt - \int_{\frac{1}{2}+\frac{1}{4}}^{1} f(t)dt\right) = 2^{\frac{1}{2}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} tdt - \int_{\frac{3}{4}}^{1} tdt\right)\\ &= \frac{1}{2^{\frac{1}{2}}}\left(-\frac{1}{8}\right).\end{aligned}$$

Also

$$\begin{aligned}\langle f, \psi_{1,k}\rangle &= 2^{\frac{1}{2}}\left(\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{4}} f(t)dt - \int_{\frac{k}{2}+\frac{1}{4}}^{\frac{k+1}{2}} f(t)dt\right) = 2^{\frac{1}{2}}\left(\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{4}} 0dt - \int_{\frac{k}{2}+\frac{1}{4}}^{\frac{k+1}{2}} 0dt\right),\\ &= 0 \quad \text{for} \quad k \in (\mathbb{Z} - \{0, 1\}).\end{aligned}$$

So

$$\begin{aligned}\sum_{k=0}^{2^{1}-1} |f, \psi_{1,k}| &= \left(\frac{1}{8} + \frac{1}{8}\right)\left(\frac{1}{2^{\frac{1}{2}}}\right)\\ &= \frac{1}{4}\left(\frac{1}{2^{\frac{1}{2}}}\right).\end{aligned}$$

For, $j = 2$ we have,

$$\langle f, \psi_{2,k} \rangle = \int_{-\infty}^{\infty} f(t)\psi_{2,k}(t)dt = 2\left(\int_{\frac{k}{2^2}}^{\frac{1}{2^3}+\frac{k}{2^2}} f(t)dt - \int_{\frac{1}{2^3}+\frac{k}{2^2}}^{\frac{k+1}{2^2}} f(t)dt\right).$$

Next,

$$\begin{aligned}\langle f, \psi_{2,0} \rangle &= 2\left(\int_0^{\frac{1}{8}} f(t)dt - \int_{\frac{1}{8}}^{\frac{1}{4}} f(t)dt\right) = 2\left(\int_0^{\frac{1}{8}} tdt - \int_{\frac{1}{8}}^{\frac{1}{4}} tdt\right) \\ &= \frac{1}{2}\left(-\frac{1}{16}\right)\end{aligned}$$

and

$$\begin{aligned}\langle f, \psi_{2,1} \rangle &= 2\left(\int_{\frac{1}{4}}^{\frac{3}{8}} f(t)dt - \int_{\frac{3}{8}}^{\frac{1}{2}} f(t)dt\right) = 2\left(\int_{\frac{1}{4}}^{\frac{3}{8}} tdt - \int_{\frac{3}{8}}^{\frac{1}{2}} tdt\right) \\ &= \frac{1}{2}\left(-\frac{1}{16}\right) \\ \langle f, \psi_{2,2} \rangle &= 2\left(\int_{\frac{1}{2}}^{\frac{5}{8}} f(t)dt - \int_{\frac{5}{8}}^{\frac{3}{4}} f(t)dt\right) = 2\left(\int_{\frac{1}{2}}^{\frac{5}{8}} tdt - \int_{\frac{5}{8}}^{\frac{3}{4}} tdt\right) \\ &= \frac{1}{2}\left(-\frac{1}{16}\right) \\ \langle f, \psi_{2,3} \rangle &= 2\left(\int_{\frac{3}{4}}^{\frac{7}{8}} f(t)dt - \int_{\frac{7}{8}}^{1} f(t)dt\right) = 2\left(\int_{\frac{3}{4}}^{\frac{7}{8}} tdt - \int_{\frac{7}{8}}^{1} tdt\right) \\ &= \frac{1}{2}\left(-\frac{1}{16}\right).\end{aligned}$$

Also

$$\begin{aligned}\langle f, \psi_{2,k} \rangle &= \int_{-\infty}^{\infty} f(t)\psi_{2,k}(t)dt = 2\left(\int_{\frac{k}{2^2}}^{\frac{1}{2^3}+\frac{k}{2^2}} 0dt - \int_{\frac{1}{2^3}+\frac{k}{2^2}}^{\frac{k+1}{2^2}} 0dt\right) \\ &= 0 \quad \text{for} \quad k \in (\mathbb{Z} - \{0,1,2,3\}).\end{aligned}$$

So

$$\begin{aligned}\sum_{k=0}^{2^2-1} |\langle f, \psi_{2,k} \rangle| &= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{4}\left(\frac{1}{2}\right).\end{aligned}$$

$\langle f, \psi_{j,k}\rangle$	$\langle f, \psi_{j,0}\rangle$	$\langle f, \psi_{j,1}\rangle$	$\langle f, \psi_{j,2}\rangle$	$\langle f, \psi_{j,3}\rangle$	$\langle f, \psi_{j,4}\rangle$	$\sum_{k=0}^{2^{j-1}} \vert \langle f, \psi_{j,k}\rangle \vert$
j=0	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$
j=1	$-\frac{1}{8}\,\frac{1}{2^{1/2}}$	$-\frac{1}{8}\,\frac{1}{2^{1/2}}$	0	0	0	$\frac{1}{4}\,\frac{1}{2^{1/2}}$
j=2	$-\frac{1}{16}\,\frac{1}{2}$	$-\frac{1}{16}\,\frac{1}{2}$	$-\frac{1}{16}\,\frac{1}{2}$	$-\frac{1}{16}\,\frac{1}{2}$	0	$\frac{1}{4}\,\frac{1}{2}$

Table 1: $\langle f, \psi_{j,k}\rangle$ for different values of $j = 0, 1, 2,$ and $k = 0, 1, 2, \cdots, 2^{j-1}$.

Similarly, for fixed j, we have $\sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k}\rangle| = M\left(\frac{1}{2^{\frac{j}{2}}}\right)$ and

$$\sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k}\rangle|^2 \;=\; M^2\left(\frac{1}{2^j}\right),$$

where M is suitable positive constant.

Previously, finite linear approximation by Haar wavelet has been studied. Here non linear approximation via Haar wavelet is to be discussed.

For $j \geq 0$, $\langle f, \psi_{j,k}\rangle$ are non zeros, for $0 \leq k \leq 2^j - 1$ and

$$\sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k}\rangle| = O\left(\frac{1}{2^{\frac{j}{2}}}\right).$$

If we take $2^n = N$ biggest Haar coefficient in that case, the approximation error is

$$\sigma_{2^n}(f)_X = dis(f, S_N)_X = \inf_{g\in\sum_{2^n}} \|f - g\|_X$$

where $\sum_{2^n}$ and $\sigma_{2^n}(f)$ denote the set of wavelets and approximation error respectively, in the non-linear spaces.

6. Main Result Related to Non-linear Wavelet Approximation

In this section, a new theorem has been established in the following forms:

Theorem 6.1. *If $f \in L^2(\mathbb{R})$ and the partial sum of the Haar wavelet series of f is*

$$g = \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k}\rangle\, \psi_{j,k} \text{ for } n \in \mathbb{N},$$

then the error of the non-linear approximation $\sigma_{2^n}(f)$ is given by

$$\sigma_{2^n}(f) = O\left(\frac{1}{2^{2^{n-1}}}\right)$$

Proof. Consider

$$\left\| f - \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\|_{L^2}^2 = \left\| \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\|_{L^2}^2$$

$$= \int_{-\infty}^{\infty} \left| \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \overline{\sum_{l=2^n}^{\infty} \sum_{m=0}^{2^l-1} \langle f, \psi_{l,m} \rangle \psi_{l,m}(t)} dt$$

$$\leq \int_{-\infty}^{\infty} \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}(t)|^2 dt$$

$$= \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 \int_{-\infty}^{\infty} |\psi_{j,k}(t)|^2 dt.$$

$$= \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} 2^j \left| \psi(2^j t - k) \right|^2 dt. \tag{1}$$

Next, $\displaystyle\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} 2^j \left| \psi(2^j t - k) \right|^2 dt = \int_0^1 |\psi(u)|^2 du;$ put $2^j t - k = u$

$$= \int_0^1 du = 1 \tag{2}$$

Now using equation (2) in equation (1) we have

$$\left\| f - \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\|_{L^2}^2 \leq \sum_{j=2^n}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 = \sum_{j=2^n}^{\infty} \left(\frac{M^2}{2^j} \right) = M^2 \sum_{j=2^n}^{\infty} \frac{1}{2^j}$$

$$= 2M^2 \frac{1}{2^{2^n}}.$$

So,

$$\left\| f - \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k} \right\|_{L^2} \leq \left(2M^2 \frac{1}{2^{2^n}} \right)^{\frac{1}{2}} = \sqrt{2} M \frac{1}{2^{2^{n-1}}}$$

$$= O\left(\frac{1}{2^{2^{n-1}}} \right)$$

Thus, Theorem **6.1** is completely established.

7. Conclusion

(i) It is observed that in case of non-linear approximation there is a significant improvement in the order of approximation in comparison to the order of linear approximation.

(ii) We have derived two corollaries, **4.1** and **4.2**, from our Theorems **3.1** & **3.2** respectively.

(iii) Independent proofs of these corollaries can be developed for specific contributions of these estimates in wavelet analysis.

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