J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 10, No. 2 (2023), pp. 143-160

DOI: 10.56827/JRSMMS.2023.1002.11 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

EXPANSIVE FIXED POINT RESULTS IN SUPER METRIC SPACES

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(**Received:** *May 12, 2023* **Accepted:** *Jun. 11, 2023* **Published:** *Jun. 30, 2023)*

Abstract: Super Metric Spaces are a ground-breaking generalization of metric spaces that were recently developed by Karapinar and Khojasteh (Filomat, in press). In this paper, we initiate the study of expansive fixed points in the context of the supermetric spaces. Our results may open the door to more expansive fixed point results in a different direction.

Keywords and Phrases: Expansive mapping, metric space, fixed point, common fixed point.

2020 Mathematics Subject Classification: 54H25, 47H10, 54E50.

1. Introduction and Definitions

With the formulation of the metric fixed point by renowned mathematician Stefan Banach [6], fixed point theory has gained prominence as a research area. The fixed point theory has been the subject of numerous theoretical and practical study. Essentially, there are two widely accepted theories about how to advance the metric fixed point: the first is changing (weakening) the constraints on the mapping of contraction, and the second is altering the abstract structure. Metric spaces have already seen a number of generalisations and extensions.

Due to its importance, various mathematicians studied many interesting extensions and generalizations, (see for example [1], [2], [3], [4], [9], [11], [12], [13], [16], [17], [18], [19], [22], [23], [24], [25], [26], [30], [31], [34], [36] and references therein). One of them is the α-ψ-contractive mapping, which Samet et al. [31] first presented using α-admissible mappings. By using the notion of α-admissible mapping, the authors introduced contractive mappings and investigated the existence and uniqueness of a fixed point of such mappings in the context of metric space. Their results have attracted several authors since they are very interesting and that several existing fixed point theorems listed as consequences of the main result of this paper [31]. The approaches used in this paper have been extended and improved by a number of authors to get similar results in different settings; (see, for example, [5], [8], [18], [20]) . In this paper, the authors established several fixed point results for such mappings in complete metric spaces. In addition, Samet [31] claimed that some existing results can be inferred from their primary results.

In 1984, Wang et al. [35] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings in [27]. Further, Khan et al. [21] generalized the result of [35] by using functions. Also, Rhoades [28] and Taniguchi [33] generalized the results of Wang [35] for pair of mappings. Kang [10] generalized the result of Khan et al. [21], Rhoades [28] and Taniguchi [33] for expansion mappings. On the other hand, Shahi et al. [32] defined a new category of expansive mappings called (ξ, α)- expansive mappings as a complement of the concept of $\alpha - \psi$-contractive type mappings [31]. The authors in [32] also studied many fixed point results for these new type of expansive mappings in the context of complete metric spaces. Thereafter, in order to generalize (ξ, α)-expansive mappings, Karapinar et al. [19] introduced a new class of expansive mappings called generalized (ξ, α)-expansive mappings and studied the existence of a fixed point for these type of expansive mappings.

Very recently, Karapinar and Khojasteh [15] proposed a new structure known as supermetric space in order to remove the congestion with regard to the fixed point theory. Thereafter, Karapinar and Fulga [14] studied contractions in a rational form in the context of the supermetric space.

Following this direction of research, we aim to investigate some expansive fixed point results in the context of the supermetric space, which is a very interesting generalization of the metric space. Also, motivated by the above idea of (ξ, α)-expansive mappings and generalized (ξ, α)-expansive mappings, we introduce (ξ, α)-expansive mappings and generalized (ξ, α)-expansive mappings in the setting of supermetric spaces and establish various fixed point theorems for such mappings in complete supermetric spaces. The presented theorems extend, gener-

alize and improve many existing results in the literature. Recall some definitions which are needed in our subsequent discussions.
Now, we give the following definition of super metric space which is introduced recently by [15].

Definition 1.1. [15] *Let X be a nonempty set. We say that $m : X \times X \to [0, +\infty)$ is super-metric or super metric if*
(i) if $m(x, y) = 0$, then $x = y$ for all $x, y \in X$,
(ii) $m(x, y) = m(y, x)$, for all $x, y \in X$,
(iii) there exists $s \geq 1$ such that for all $y \in X$ there exists distinct sequences $(x_n), (y_n) \subset X$, with $m(x_n, y_n) \to 0$ when n tends to infinity, such that

$$\limsup_{n\to\infty} m(y_n, y) \leq s \limsup_{n\to\infty} m(x_n, y)$$

Then, we call (X, m) a super metric space.
The notions of convergence and the Cauchy sequence with respect to completeness of a supermetric space are defined as follows:

Definition 1.2. [15] *On a supermetric space (X, m, s), a sequence $\{x_n\}$:*
(i) converges to x in X if and only if $\lim_{n\to\infty} m(x_n, x) = 0$;
(ii) is a Cauchy sequence in X if and only if $\lim_{n\to\infty} \sup\{m(x_n, x_p) : p > n\} = 0$;

Proposition 1.1. [14] *On a supermetric space, the limit of a convergent sequence is unique.*

Definition 1.3. [15] *We say that a supermetric space (X, m, s) is complete if and only if every Cauchy sequence is convergent in X.*

Proposition 1.2. [14] *Let $T : X \to X$ be an asymptotically regular mapping on a complete supermetric space (X, m, s). Then, the Picard iteration $\{T^n x\}$ for the initial point $x \in X$ is a convergent sequence on X.*

2. Preliminary Theorems

We need the following results, consistent with [35, 31].
Wang et al. [35] defined expansion mappings in the form of the following theorem:

Theorem 2.1. [35] *Let (X, d) be a complete metric space. If f is a mapping of X into itself and if there exists a constant $q > 1$ such that*

$$d(f(x), f(y)) \geq q d(x, y)$$

for each $x, y \in X$ and f is onto, then f has a unique fixed point in X.

Recently, Samet et al. [31] considered the following family of functions and presented the new notions of α-ψ -contractive and α-admissible mappings.

Definition 2.1. [31] *Let φ denote the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy*
(i) $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.
(ii) ψ is non-decreasing.

Definition 2.2. [31] *Let (X, d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exists two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \varphi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 2.3. [31] *Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. T is said to be α-admissible if*

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Now, we present some examples of α-admissible mappings.

Example 2.1. [32] Let X be the set of all non-negative real numbers. Let us define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if x \geq y, \\ 0 & if x < y. \end{cases}$$

and define the mapping $T : X \to X$ by $Tx = x^2$ for all $x \in X$. Then T is α-admissible.

Example 2.2. [32] Let X be the set of all non-negative real numbers. Let us define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} e^{x-y} & if x \geq y, \\ 0 & if x < y. \end{cases}$$

and define the mapping $T : X \to X$ by $Tx = e^x$ for all $x \in X$. Then T is α-admissible.

In what follows, we present the main results of Samet et al. [31].

Theorem 2.2. [31] *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*
(i) T is α-admissible;

(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(iii) T *is continuous.*
Then, T *has a fixed point, that is, there exists* $x^* \in X$ *such that* $Tx^* = x^*$*.*

Theorem 2.3. [31] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an* α*-*ψ*-contractive mapping satisfying the following conditions:*
(i) T *is* α*-admissible;*
(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(iii) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to +\infty$*, then* $\alpha(x_n, x) \geq 1$ *for all* n*.*
Then, T *has a fixed point.*
Samet et al. [31] added the following condition (H) to the hypotheses of Theorem 2.4 and Theorem 2.5 to assure the uniqueness of the fixed point:
(H) : For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.
Further, Samet et al. [31] derived coupled fixed point theorems in complete metric spaces using the previous obtained results.

Theorem 2.4. [31] *Let* (X, d) *be a complete metric space and* $F : X \times X \to X$ *be a given mapping. Suppose that there exists* $\psi \in \varphi$ *and a function* $\alpha : X^2 \times X^2 \to [0, +\infty)$ *such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(y, v)),$$

for all $(x, y), (u, v) \in X \times X$*. Suppose also that*
(i) For all $(x, y), (u, v) \in X \times X$*, we have*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)) \geq 1;$$

(ii) there exists $(x_0, y_0) \in X \times X$ *such that*
$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)) \geq 1$ *and* $\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$*.*
(iii) F *is continuous.*
Then, F *has a coupled fixed point, that is, there exists* $(x^*, y^*) \in X \times X$ *such that* $x^* = F(x^*, y^*)$ *and* $y^* = F(y^*, x^*)$*.*

Theorem 2.5. [31] *Let* (X, d) *be a complete metric space and* $F : X \times X \to X$ *be a given mapping. Suppose that there exists* $\psi \in \varphi$ *and a function* $\alpha : X^2 \times X^2 \to [0, +\infty)$ *such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(y, v)),$$

for all $(x, y), (u, v) \in X \times X$*. Suppose also that*
(i) For all $(x, y), (u, v) \in X \times X$*, we have*

$$\alpha((x,y),(u,v)) \geq 1 \Rightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u)) \geq 1;$$

(ii) there exists $(x_0, y_0) \in X \times X$ *such that*
$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0)) \geq 1$ *and* $\alpha((F(y_0,x_0),F(x_0,y_0)),(y_0,x_0)) \geq 1$.
(iii) if $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*
$\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ *and* $\alpha((y_{n+1},x_{n+1}),(y_n,x_n)) \geq 1$,
$x_n \to x \in X$ *and* $y_n \to y \in X$ *as* $n \to +\infty$*, then*
$\alpha((x_n,y_n),(x,y)) \geq 1$ *and* $\alpha((y,x),(y_n,x_n)) \geq 1$ *for all* $n \in \mathbb{N}$.
Then, F *has a coupled fixed point.*
Samet et al. [31] added the following condition (H$'$) to the hypotheses of Theorem 2.6 and Theorem 2.7 to assure the uniqueness of the coupled fixed point:
(H') : For all $(x,y),(u,v) \in X \times X$, there exists $(z_1,z_2) \in X \times X$ such that

$$\alpha((x,y),(z_1,z_2)) \geq 1, \;\; \alpha((z_2,z_1),(y,x)) \geq 1$$

and

$$\alpha((u,v),(z_1,z_2)) \geq 1, \;\; \alpha((z_2,z_1),(v,u)) \geq 1.$$

In 2012, [32] introduced the following new notion of (ξ,α)-expansive mappings:
Let χ denote all functions $\xi : [0,+\infty) \to [0,+\infty)$ which satisfies the following properties:
(i) ξ is non-decreasing,
(ii) $\sum_{n=1}^{+\infty} \xi^n(a) < +\infty$ for each $a > 0$, where ξ^n is the n-th iterate of ξ.
(iii) $\xi(a+b) = \xi(a) + \xi(b)$, $\forall a,b \in [0,+\infty)$.

Lemma 2.1. [31] *If* $\xi : [0,+\infty) \to [0,+\infty)$ *is a nondecreasing function, then for each* $a > 0$, $\lim\limits_{n\to+\infty} \xi^n(a) = 0$ *implies* $\xi(a) < a$.

Definition 2.4. [32] *Let* (X,d) *be a metric space and* $T : X \to X$ *be a given mapping. We say that* T *is an* (ξ,α)*-expansive mapping if there exist two functions* $\xi \in \chi$ *and* $\alpha : X \times X \to [0,+\infty)$ *such that*

$$\xi(d(Tx,Ty)) \geq \alpha(x,y)d(x,y) \tag{1}$$

for all $x,y \in X$.

Remark 2.1. [32] *If* $T : X \to X$ *is an expansion mapping, then* T *is an* (ξ,α)*-expansive mapping, where* $\alpha(x,y) = 1$ *for all* $x,y \in X$ *and* $\xi(a) = ka$ *for all* $a \geq 0$ *and some* $k \in [0,1)$.

Theorem 2.6. [32] *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a*

bijective, (ξ,α)-expansive mapping satisfying the following conditions:
(i) T^{-1} is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) T is continuous.
Then, T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.
Also, authors in [32] showed that the coupled fixed point theorems in complete metric spaces can also be derived from their results. Let us recall the following definition due to Bhaskar and Lakshmikantham [7] before stating their results:

Definition 2.5. [7] *Let $F : X \times X \to X$ be a given mapping. We say that $(x,y) \in X \times X$ is a coupled fixed point of F if*

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Lemma 2.2. [31] *Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \to X \times X$ by*

$$T(x,y) = (F(x,y), F(y,x)), \text{ for all } (x,y) \in X \times X.$$

Then, (x,y) is a coupled fixed point of F if and only if (x,y) is a fixed point of T.

Theorem 2.7. *Let (X,d) be a complete metric space and $F : X \times X \to X$ be a given bijective mapping. Suppose that there exists $\xi \in \chi$ and a function $\alpha : X^2 \times X^2 \to [0,+\infty)$ such that*

$$\xi(d(F(x,y),F(u,v))) \geq \tfrac{1}{2}\alpha((x,y),(u,v))[d(x,u)+d(y,v)]$$

for all $(x,y),(u,v) \in X \times X$. Suppose also that
(i) For all $(x,y),(u,v) \in X \times X$, we have

$$\alpha((x,y),(u,v)) \geq 1 \Rightarrow \alpha(F^{-1}(x), F^{-1}(u)) \geq 1;$$

(ii) there exists $(x_0,y_0) \in X \times X$ such that

$$\alpha((x_0,y_0),(a,b)) \geq 1 \text{ and } \alpha((b,a),(y_0,x_0)) \geq 1,$$

where $F^{-1}(x_0) = (a,b)$.
(iii) F is continuous.
Then, F has a coupled fixed point, that is, there exists $(x^*,y^*) \in X \times X$ such that $x^* = F(x^*,y^*)$ and $y^* = F(y^*,x^*)$.
In 2014, Karapinar et al. [19] gave the following new concept of generalized (ξ,α)-expansive mappings which generalized many existing results in the literature:

Definition 2.6. [19] *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. The mapping T is a generalized (ξ,α)-expansive mapping if there exists two functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ such that for all $x,y \in X$, we have*

$$\xi(d(Tx,Ty)) \geq \alpha(x,y).m(x,y),$$

where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$.
Following is the main result of [19]:

Theorem 2.8. [19] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a bijective, generalized* (ξ, α)*-expansive mapping satisfying the following conditions:*
(i) T^{-1} *is* α*-admissible;*
(ii) there exists $x \in X$ *such that* $\alpha(x_0, T^{-1}_{x_0}) \geq 1$*;*
(iii) T *is continuous.*
Then T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.
In the sequel, authors in [19] proved that Theorem 2.8 still holds for T not necessarily continuous, assuming the following condition:
(P) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha(T^{-1}x_n, T^{-1}x) \geq 1,$$

for all n.

Theorem 2.9. [19] *If in Theorem 2.8, we replace the continuity of* T *by the condition* (P)*, then the result holds true.*
Karapinar et al. [19] also derived the following fixed-point result on a metric space endowed with a partial order:

Corollary 2.1. [19] *Let* $(X, \leqslant)$ *be a partially ordered set and* d *be a metric on* X *such that* (X, d) *is complete. Let* $T : X \to X$ *be a bijective mapping such that* T *is a non-decreasing mapping with respect to* $\leqslant$ *satisfying the following condition for all* $x, y \in X$ *with* $x \geqslant y$*:*

$$\xi(d(Tx, Ty)) \geq \alpha(x, y).m(x, y), \tag{2}$$

where $\xi \in \chi$ *and*

$$m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Suppose also that
(i) there exists $x_0 \in X$ *such that* $x_0 \leqslant T^{-1}x_0$*;*
(ii) T *is continuous or* $(X, \leqslant, d)$ *is regular.*
Then T *has a fixed point, that is, there exists* $u \in X$ *such that* $Tu = u$.
In 2003, Kirk et al. [22] introduced cyclic representations and cyclic contractions in order to generalize the Banach contraction mapping principle.

Definition 2.7. *A mapping* $T : A \cup B \to A \cup B$ *is called cyclic if* $T(A) \subset B$ *and*

$T(B) \subset A$, where A, B are nonempty subsets of a metric space (X, d).

Definition 2.8. *A mapping T is called a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$.*

Therefore, cyclic contractions do not necessarily have to be continuous, despite the fact that a contraction is.continuous. This is one of the important benefits of this theorem. Recently, various researchers have derived several fixed-point results by utilizing the cyclic representations and cyclic contractions. See for example [2], [11], [17], [24], [25], [29].

Following this direction of research, Karapinar et al. [19] derived the following fixed-point result for cyclic contractive mappings:

Corollary 2.2. [19] *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ be a given bijective mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*
(I) $T^{-1}(A_1) \subset A_2$ and $T^{-1}(A_2) \subset A_1$;
(II) there exists a function $\xi \in \chi$ such that

$$\xi(d(Tx, Ty)) \geq m(x, y), \forall (x, y) \in A_1 \times A_2. \tag{3}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

3. Main Theorems

In this section, we prove the Wang expansion mapping principle [35] in super metric spaces.

Theorem 3.1. *Let (X, m) be a complete super metric space and let $T : X \to X$ be a surjective mapping. Suppose that $\alpha > 1$ such that*

$$m(Tx, Ty) \geq \alpha m(x, y), \tag{4}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1},$$

for all $n \in \mathbb{N}$, where $x_0 \in X$. If $x_0 = x_1$, then x_1 is the fixed point and the proof is completed. So suppose that $x_0 \neq x_1$. Thus, $m(x_0, x_1) > 0$. Thus, without loss of generality, we can assume that $x_n \neq x_{n+1}$. So, $m(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

Therefore,

$$\begin{aligned} m(x_n, x_{n+1}) \leq \frac{1}{\alpha} m(Tx_n, Tx_{n+1}) \qquad &\leq \frac{1}{\alpha} m(x_{n-1}, x_n) \\ &\leq \frac{1}{\alpha^2} m(Tx_{n-1}, Tx_n) \\ &= \frac{1}{\alpha^2} m(x_{n-2}, x_{n-1}) \\ &\leq \frac{1}{\alpha^n} m(x_0, x_1) \end{aligned} \tag{5}$$

Taking limit from both side of (5) implies that

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0. \tag{6}$$

Following the line of proof of Theorem 2.1 in [15], we get that $\{x_n\}$ is a Cauchy sequence. Due to the fact that (X, m) is a complete supermetric space, the sequence $\{x_n\}$ converges to $z \in X$. We assert that z is the fixed point of T. On the contrary, assume that $m(z, Tz) > 0$. Note that as $n \to \infty$

$$m(x_{n+1}, T^{-1}z) = m(T^{-1}x_n, T^{-1}z) \leq \frac{1}{\alpha} m(x_n, z) \to 0 \tag{7}$$

Thus, $\lim_{n\to\infty} m(x_{n+1}, T^{-1}z) = 0$. If there $N > 0$ such that for all $n > N$, $x_{N+1} = z$, (7) implies that $m(z, T^{-1}z) = 0$ and so we have z is the fixed point for T. Otherwise, assume that for all $n \in \mathbb{N}$, $x_n \neq z$. Thus, we get

$$m(z, Tz) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, T^{-1}z), \tag{8}$$

and one concludes that $m(z, T^{-1}z) = 0$, which is a contradiction. Thus, $z = Tz$ is the fixed point of T in X. The uniqueness of the fixed point is clear from (4).

Example 3.1. Let $X = [1, 3]$ and define

$$m(x, y) = \begin{cases} xy & if x \neq y, \\ 0 & if x = y. \end{cases}$$

Following [15], assuming the two distinct sequences (x_n), (y_n) satisfying the condition $m(x_n, y_n) \to 0$ as $n \to \infty$. Therefore, we will have $m(x_n, y_n) = x_n y_n \to 0$, and let $y_n \to 0$ and $x_n \to u$ as $n \to infty$, where $u \in X$. Moreover, for any $y \in X$,

$$\lim_{n\to\infty} \sup m(y_n, y) = \lim_{n\to\infty} \sup y_n y = 0 \leq s \lim_{n\to\infty} \sup m(x_n, y) = \lim_{n\to\infty} \sup x_n y = u.y,$$

and it follows that (X, m) is a super-metric space.
Now, consider $T : X \to X$ as follows

$$T(x) = \begin{cases} 3 & if x \neq 1, \\ \frac{5}{2} & if x = 1. \end{cases}$$

Considering $s = \frac{9}{4}, \alpha = 2, x \neq 1$ and $y = 1$, we have

$$m(Tx, Ty) = m(3, \frac{5}{2}) = \frac{15}{2} \geq 2 \times x = \alpha m(x, y),$$

as $1 < x \leq 3$. The other cases being straightforward implies that T has a unique fixed point $x = 3$ by previous theorem for $\alpha = 2$.

We introduce here a new notion of (ξ, α)-expansive mappings in the setting of super metric spaces as follows:

Definition 3.1. *Let (X, d) be a super metric space and $T : X \to X$ be a given mapping. We say that T is an (ξ, α)-expansive mapping if there exist two functions $\xi \in \chi$ and $\alpha : X \times X \to [0, +\infty)$ such that*

$$\xi(m(Tx, Ty)) \geq \alpha(x, y) m(x, y) \tag{9}$$

for all $x, y \in X$.

Now, we prove some fixed point theorems for (ξ, α)-expansive mappings in the setting of super metric spaces.

Theorem 3.2. *Let (X, d) be a complete super metric space and $T : X \to X$ be a bijective, (ξ, α)-expansive mapping satisfying the following conditions:*
(i) T^{-1} is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha(T^{x_n}, T^{-1}x) \geq 1 \tag{10}$$

for all n. Then, T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Proof. Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \text{ for all } n \in \mathbb{N},$$

where $x_0 \in X$ be such that $\alpha(x_0, T^{-1}x_0) \geq 1$. Now, if $x_n = x_{n+1}$ for any $n \in \mathbb{N}$, one has that x_n is a fixed point of T from the definition $\{x_n\}$. Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$.

It is given that $\alpha(x_0, x_1) = \alpha(x_0, T^{-1}x_0) \geq 1$. Recalling that T^{-1} is α-admissible, therefore, we have

$$\alpha(T^{-1}x_0, T^{-1}x_1) = \alpha(x_1, x_2) \geq 1.$$

Using mathematical induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{11}$$

for all $n \in \mathbb{N}$. Using (11) and applying the inequality (9) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$m(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1})m(x_n, x_{n+1}) \leq \xi(m(Tx_n, Tx_{n+1})) = \xi(m(x_{n-1}, x_n))$$

Therefore, by repetition of the above inequality, we have that

$$m(x_n, x_{n+1}) \leq \xi^n(m(x_0, x_1)), \forall n \in \mathbb{N}. \tag{12}$$

Taking limit from both side of (12) implies that

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0. \tag{13}$$

Following the line of proof of Theorem 2.1 in [15], we get that $\{x_n\}$ is a Cauchy sequence. Due to the fact that (X, m) is a complete supermetric space, the sequence $\{x_n\}$ converges to $z \in X$. We assert that z is the fixed point of T. On the contrary, assume that $m(z, Tz) > 0$. Note that as $n \to \infty$

$$m(x_{n+1}, T^{-1}z) = m(T^{-1}x_n, T^{-1}z) \leq \alpha(T^{-1}x_n, T^{-1}z)m(T^{-1}x_n, T^{-1}z) \leq \xi(m(x_n, z)) \tag{14}$$

Thus, $\lim_{n\to\infty} m(x_{n+1}, T^{-1}z) = 0$. If there $N > 0$ such that for all $n > N$, $x_{N+1} = z$, (14) implies that $m(z, T^{-1}z) = 0$ and so we have z is the fixed point for T. Otherwise, assume that for all $n \in \mathbb{N}$, $x_n \neq z$. Thus, we get

$$m(z, Tz) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, T^{-1}z), \tag{15}$$

and one concludes that $m(z, T^{-1}z) = 0$, which is a contradiction. Thus, $z = Tz$ is the fixed point of T in X. The uniqueness of the fixed point is clear from (4).
To ensure the uniqueness of the fixed point in Theorem 3.2, we consider the condition:
(U): For all $u, v \in X$, there exists $w \in X$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$.

Theorem 3.3. *Adding condition (U) to the hypotheses of Theorem 3.2, we obtain uniqueness of the fixed point of T.*

Proof. From Theorem 3.2, the set of fixed points is non-empty. We shall show that if u and v are two fixed points of T, that is, $T(u) = u$ and $T(v) = v$, then $u = v$. From condition (U), there exists $w \in X$ such that

$$\alpha(u, w) \geq 1 \text{ and } \alpha(v, w) \geq 1.$$

Recalling the α-admissible property of T^{-1}, we obtain

$$\alpha(u, T^{-1}w) \geq 1 \text{ and } \alpha(v, T^{-1}w) \geq 1, \text{ for all } n \in \mathbb{N}.$$

Therefore, by repeatedly applying the α-admissible property of T^{-1}, we get

$$\alpha(u, T^{-n}w) \geq 1 \text{ and } \alpha(v, T^{-n}w) \geq 1, \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} m(u, T^{-n}w) &\leq \alpha(u, T^{-n}w)m(u, T^{-n}w) \\ &\leq \xi(m(u, T^{-n+1}w))) \end{aligned}$$

Repetition of the above inequality implies that

$$m(u, T^{-n}w) \leq \xi^n(m(u, w)), \text{ for all } n \in \mathbb{N}.$$

Thus, we have $T^{-n}w \to u$ as $n \to +\infty$. Using the similar steps as above, we obtain $T^{-n}w \to v$ as $n \to +\infty$. Now, uniqueness of the limit of $T^{-n}w$ gives us $u = v$. This completes the proof.

Now, we present the concept of generalized (ξ, α)-expansive mappings in the setting of super metric spaces as follows:

Definition 3.2. *Let (X, m) be a super metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (ξ, α)-expansive mapping if there exists two functions $\xi \in \chi$ and $\alpha : X \times X \to [0, +\infty)$ such that for all $x, y \in X$, we have*

$$\xi(m(Tx, Ty)) \geq \alpha(x, y).m(x, y), \tag{16}$$

where $m(x, y) = \min\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Theorem 3.4. *Let (X, m) be a complete super metric space and $T : X \to X$ be a bijective, generalized (ξ, α)-expansive mapping satisfying the following conditions:*
(i) T^{-1} is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha(T^{x_n}, T^{-1}x) \geq 1 \tag{17}$$

for all n. Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Proof. Suppose $x_0 \in X$ be such that $\alpha(x_0, T^{-1}x_0) \geq 1$. Suppose the sequence $\{x_n\}$ in X is defined by

$$x_n = Tx_{n+1}$$

for all $n \in \mathbb{N}$. If for any $n \in \mathbb{N}$ we have $x_n = x_{n+1}$, then x_n is a fixed point of T in view of the definition. So, we suppose that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$ without loss of generality. Due to the fact that T^{-1} is an α-admissible mapping and $\alpha(x_0, T^{-1}x_0) \geq 1$, we get that

$$\alpha(T^{-1}x_0, T^{-1}x_1) = \alpha(x_1, x_2) \geq 1.$$

Continuing this process, we obtain for all $n \in \mathbb{N} \cup \{0\}$

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{18}$$

Applying inequality (16) with $x = x_n$, $y = x_{n+1}$, we get

$$m(x_{n-1}, x_n) > \xi(m(Tx_n, Tx_{n+1})) \geq \alpha(x_n, x_{n+1})m(x_n, x_{n+1}).$$

Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, we have

$$m(x_{n-1}, x_n) > \xi(m(Tx_n, Tx_{n+1})) \geq \min\{m(x_n, x_{n+1}), m(x_{n-1}, x_n)\}.$$

Now, if $\min\{m(x_n, x_{n+1}), m(x_{n-1}, x_n)\} = m(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then

$$m(x_{n-1}, x_n) > \xi(m(Tx_n, Tx_{n+1})) \geq m(x_{n-1}, x_n),$$

which is a contradiction. Therefore, for all $n \in \mathbb{N}$, we get

$$\xi(m(x_{n-1}, x_n)) \geq m(x_n, x_{n+1}) \geq m(x_n, x_{n+1}).$$

By induction, we obtain

$$\xi^n(m(x_0, x_1)) \geq m(x_n, x_{n+1}). \tag{19}$$

Taking limit from both side of (19) implies that

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0. \tag{20}$$

Following the line of proof of Theorem 2.1 in [15], we get that $\{x_n\}$ is a Cauchy sequence. Due to the fact that (X, m) is a complete supermetric space, the sequence $\{x_n\}$ converges to $z \in X$. We assert that z is the fixed point of T. On the contrary, assume that $m(z, Tz) > 0$. Note that as $n \to \infty$

$$m(x_{n+1}, T^{-1}z) = m(T^{-1}x_n, T^{-1}z) \leq \alpha(T^{-1}x_n, T^{-1}z)m(T^{-1}x_n, T^{-1}z) \leq \xi(m(x_n, z)) \tag{21}$$

Thus, $\lim_{n\to\infty} m(x_{n+1}, T^{-1}z) = 0$. If there $N > 0$ such that for all $n > N$, $x_{N+1} = z$, (21) implies that $m(z, T^{-1}z) = 0$ and so we have z is the fixed point for T. Otherwise, assume that for all $n \in \mathbb{N}$, $x_n \neq z$. Thus, we get

$$m(z, Tz) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, T^{-1}z), \tag{22}$$

and one concludes that $m(z, T^{-1}z) = 0$, which is a contradiction. Thus, $z = Tz$ is the fixed point of T in X.

Acknowledgement

Thankful to the referee for the valuable inputs.

References

[1] Abdeljawad, T., Karapinar, E. and Tas, K., A generalized contraction principle with control functions on partial metric spaces, Comput. Math. Appl., 63 (2012), 716-719.

[2] Agarwal, R. P., Alghamdi, M. A. and Shahzad, N., Fixed point theory for cyclic generalized contractions in partial metric spaces, Fixed Point Theory Appl., 2012 Article ID 40 (2012).

[3] Agarwal, R. P., El-Gebeily, M. A., O' Regan, D., Generalized contractions in partially ordered metric spaces, Appl. Anal., 87 (2008), 1-8.

[4] Alqahtani, B., Fulga, A. and Karapinar, E., Sehgal Type Contractions on b-Metric Space, Symmetry, 10 (2018), 560.

[5] Amiri, P., Rezapour, S. and Shahzad, N., Fixed points of generalized α-ψ-contractions, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., 108 (2014), 519-526.

[6] Banach, S., Surles operations dans les ensembles abstraits et leur application aux equations itegrales, Fundamenta Mathematicae, 3 (1922), 133-181.

[7] Bhaskar, T. G. and Lakshmikantham, V., Fixed Point Theory in partially ordered metric spaces and applications, Nonlinear Analysis, 65 (2006), 1379-1393.

[8] Bota, M. F., Karapinar, E. and Mlesnite, O., Ulam-Hyersstability results for fixed point problems via α-ψ-contractive mapping in (b)-metric space, Abstr. Appl. Anal., 2013 Article ID 825293, (2013).

[9] Jleli, M. and Samet, B., A new generalization of the Banach contraction principle, J. Inequal. Appl., 2014 Article ID 38, (2014).

[10] Kang, S. M., Fixed points for expansion mappings, Math. Japonica, 38 (1993), 713-717.

[11] Karapinar, E., Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett. 24 (2011), 822-825.

[12] Karapinar, E., A note on a rational form contractions with discontinuities at fixed points, Fixed Point Theory, 21 (2020), 211-220.

[13] Karapinar, E., Recent advances on metric fixed point theory: A review, Appl. Comput. Math. in press.

[14] Karapinar, E. and Fulga, A., Contraction in Rational Forms in the Framework of Super Metric Spaces, Mathematics, 10 (2022), 3077.

[15] Karapinar, E. and Khojasteh, F. Super Metric Spaces, Filomat in press.

[16] Karapinar, E., Regan, D. O' and Samet, B., On the existence of fixed points that belong to the zero set of a certain function, Fixed Point Theory Appl., 2015 (2015), 14 pages.

[17] Karapinar, E and Sadaranagni, K., Fixed point theory for cyclic $(\phi - \psi)$-contractions, Fixed Point Theory Appl., 2011 Article ID 69, (2011).

[18] Karapinar, E. and Samet, B., Generalized $\alpha - \psi$-contractive type mappings and related fixed point theorems with applications, Abstract and Applied Analysis, 2012 Article ID 793486, (2012).

[19] Karapinar, E., Shahi, P., Kaur, J. and S. S. Bhatia, Generalized (ξ, α)-expansive mappings and related fixed-point theorems, J. Inequal Appl., 2014, 22 (2014).

[20] Karapinar, E., Shahi, P. and Tas, K., Generalized α-ψ-contractive type mappings of integral type and related fixed point theorems, J. Inequal. Appl., 2014, 16 (2014).

[21] Khan, M. A., Khan, M. S. and Sessa, S., Some theorems on expansion mappings and their fixed points, Demonstr. Math., 19 (1986), 673-683.

[22] Kirk, W. A., Srinivasan, P. S., Veeramani, P., Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, 4 (2003), 79-89.

[23] Nieto, J. J. and Lopez, R. R., Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sin. Engl. Ser., 23 (2007), 2205-2212.

[24] Pacurar, M. and Rus, I. A., Fixed point theory for cyclic ϕ-contractions, Nonlinear Anal., 72 (2010), 1181-1187.

[25] Petric, M. A., Some results concerning cyclic contractive mappings, Gen. Math., 18 (2010), 213-226.

[26] Ran, A. C. M., Reurings, M. C. B., A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Am. Math. Soc., 132 (2004), 1435-1443.

[27] Rhoades, B. E., A comparison of various definitions of contractive mappings, Trans. Am. Math. Soc., 226 (1977), 257-290.

[28] Rhoades, B. E., Some fixed point theorems for pairs of mappings, Jnanabha, 15 (1985), 151-156.

[29] Rus, I. A., Cyclic representations and fixed points, Ann. "TiberiuPopoviciu" Sem. Funct. Equ. Approx. Convexity, 3 (2005), 171-178.

[30] Samet, B., Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. TMA, (2010).

[31] Samet, B., Vetro, C. and Vetro, P., Fixed point theorem for α-ψ contractive type mappings, Nonlinear Anal., 75 (2012), 2154-2165.

[32] Shahi, P., Kaur, J. and Bhatia, S. S., Fixed point theorems for (ξ, α)-expansive mappings in complete metric spaces, Fixed Point Theory Appl., 2012, Article ID 157, (2012).

[33] Taniguchi, T., Common fixed point theorems on expansion type mappings on complete metric spaces, Math. Jpn., 34 (1989), 139-142.

[34] Turinici, M., Abstract comparison principles and multivariable Gronwall-Bellman inequalities, J. Math. Anal. Appl., 117 (1986), 100-127.

[35] Wang, S. Z., Li, B. Y., Gao, Z. M. and Iseki, K., Some fixed point theorems on expansion mappings, Math. Jpn., 29 (1984), 631-636.

[36] Wardowski, D., Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., 2012, 94 (2012).