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SOME PROPERTIES OF k-RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL OPERATOR

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Abstract: In this paper we will introduce some properties of k- Riemann Liouville fractional integral operator involving convolution property. The fractional derivative of k- Riemann Liouville fractional integral operator of integral transforms will be obtained. Applications of this operator will be introduced. All results of nature will be discussed as special cases.

Keywords and Phrases: Gamma function, Integral Transform, Riemann Liouville fractional integral, Fractional Singular Kernel.

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1. Introduction and Definitions

Various problems of physics and engineering are based on mathematical calculations. Some specific problems various fields based on theory of special functions and fractional calculus operators. Many mathematicians (like see [1], [5], [7], [8], [9]) have introduced various properties and applications of special functions and fractional calculus operators. For main results of this paper, we are considering following definitions.

In 2007, R. Diaz and E. Pariguan have introduced the following Pochhammer k-symbol and k-Gamma function (see [2], [3])

$$(a)_{n,k} = a.\,(a+k)\,.\,(a+2k)\,.\,(a+3k)\,.......\,(a+(n-1)\,k)\,;n\geq 1, k>0 \qquad (1.1)$$

$$(a)_{0,k} = 1; \qquad n = 0 \tag{1.2}$$

For k>0,$z \in C$ and $Re\,(z) > 0$, the k-Gamma function is defined as

$$\Gamma_k\,(z) = \int_0^\infty x^{z-1} e^{-\frac{x^k}{k}}\,dx \tag{1.3}$$

Equation (1.1) we can write as

$$\Gamma_k\,(z) = k^{\frac{x}{k}}\Gamma\left(\frac{x}{k}\right) \tag{1.4}$$

The k-Beta function $B_k\,(m, n)$ for two variable m and n is defined by (see [2], [3])

$$B_k\,(m, n) = \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}\,(1-x)^{\frac{m}{k}-1}\,dx; Re\,(m) > 0, Re\,(n) > 0 \tag{1.5}$$

Equation (1.5) in term of k-Gamma function is given by

$$B_k\,(m, n) = \frac{\Gamma_k\,(m)\,\Gamma_k\,(n)}{\Gamma_k\,(m+n)}; Re\,(m) > 0, Re\,(n) > 0 \tag{1.6}$$

Recently, many researchers ([1], [3], [4], [6], [7]) have extended the research work Riemann Liouville integral operator in term of k-fractional integral of Riemann Liouville.

In 2012, Mubeen et. al. [6] have introduced k-fractional integral of Riemann Liouville type

$$I_k^\lambda f\,(t) = \frac{1}{k\Gamma_k\,(\lambda)}\int_0^t (t-\xi)^{\frac{\lambda}{k}-1} f\,(\xi)\,d\xi, \qquad \lambda > 0, t > 0, k > 0 \tag{1.7}$$

In particular if $k \to 1$, then it is reduced to the classical Riemann Liouville fractional integrals (Mathai et. al [8]).

Let λ be a real number such that $0 < \lambda < 1, k > 0$ then k-Riemann-Liouville fractional singular kernel is given by (Mubeen et. al [6])

$$j_{\lambda,k} f\,(t) = \frac{t^{\frac{\lambda}{k}-1}}{k\Gamma_k\,(\lambda)}; \qquad t > 0 \tag{1.8}$$

Equation (1.7) is in term of (1.8) as

$$I_k^\lambda f\,(t) = j_{\lambda,k}\,(t) * f\,(t) \tag{1.9}$$

Let λ be a real number such that $0 < \lambda \leq 1, k > 0$ then k-Riemann-Liouville fractional integral is given as

$$D_k^{\lambda} f(t) = \frac{d}{dt} I_k^{1-\lambda} f(t) \tag{1.10}$$

Let λ be a real number such that $0 < \lambda \leq 1, k > 0$ and f be sufficient well-defined function then the Laplace transform of k-Riemann-Liouville fractional singular kernel as

$$L\{j_{\lambda,k}(t)\} = L\left\{\frac{t^{\frac{\lambda}{k}-1}}{k\Gamma_k(\lambda)}\right\} = \frac{1}{kk^{\frac{\lambda}{k}-1}\Gamma\left(\frac{\lambda}{k}\right)} \frac{\Gamma\left(\frac{\lambda}{k}\right)}{\omega^{\frac{\lambda}{k}}} = \frac{1}{(\omega k)^{\frac{\lambda}{k}}} \tag{1.11}$$

By the help of equations (1.9) and (1.11)

$$L\left\{I_k^{\lambda} f(t);\omega\right\} = L\left\{j_{\lambda,k}(t) * f(t);\omega\right\} = L\left\{j_{\lambda,k}(t)\right\} L\left\{f(t)\right\} = \frac{L(f(t))}{(\omega k)^{\frac{\lambda}{k}}} \tag{1.12}$$

2. Main Results

Theorem 2.1. *Let λ be a real number such that $0 < \lambda \leq 1, k > 0$ and f be sufficient well-defined function then*

$$(1)\ L\left\{D_k^{\lambda} f(v)\right\} = \omega (k\omega)^{-\frac{1-\lambda}{k}} L\{f(v)\} - I_k^{1-\lambda} f(0) \tag{2.1}$$

$$(2)\ L - S_{gen}\left[j_{k,\lambda}(v);\omega, s\right] = \Gamma_{\mu}\left(\tfrac{\lambda}{k}, \omega, s\right) L - S_{gen}\left[f(v)\right] \tag{2.2}$$

where L denotes Laplace transform operator, $L - S_{gen}$ denotes Laplace-Generalized Stieltjes Transform and $\Gamma_{\omega}\left(\frac{\lambda}{k}, p, s\right)$ denotes ultra Gamma function.

Proof. Using equation (1.10)

$$L\{D_k^{\lambda} f(v)\} = L\left\{\frac{d}{dv} I_k^{1-\lambda} f(v)\right\}$$

$$\because L\left\{\frac{d}{dv} f(v)\right\} = sL\{f(v)\} - f(0)$$

$$= \omega L\left\{I_k^{1-\lambda} f(v)\right\} - I_k^{1-\lambda} f(0)$$

$$= \omega (\omega k)^{-\frac{1-\lambda}{k}} L\{f(v)\} - I_k^{1-\lambda} f(0)$$

For $v \in (0, \infty)$ and $f(v)$, then Laplace-Generalized Stieltjes Transform $j_{\lambda,k}(v)$and $I_k^{\lambda} f(v)$ be as

$$\begin{aligned} L - S_{gen}\left[j_{k,\lambda}(v); \omega, s\right] &= \int_0^{\infty} j_{k,\lambda}(v) \frac{e^{-sv}}{(v+\omega)^{\mu}} dv \\ &= \int_0^{\infty} \frac{v^{\frac{\lambda}{k}-1} e^{-sv}}{(v+\omega)^{\mu}} dv \\ &= \Gamma_{\mu}\left(\tfrac{\lambda}{k}, \omega, s\right) \end{aligned} \tag{2.3}$$

where $= \Gamma_{\omega}\left(\frac{\lambda}{k}, p, s\right)$ denotes ultra Gamma function. It has been introduced by Banerji and Sinha (see [9])

Now using equation (1.9) and (2.3)

$$\begin{aligned} L - S_{gen}\left\{I_k^{\lambda} f(v); s, \omega, \mu\right\} &= L - S_{gen}\left\{j_{\lambda,k}(v) * f(v); s, \omega, \mu\right\} \\ &= L - S_{gen}\left[j_{k,\lambda}(v)\right] L - S_{gen}\left[f(v)\right] \\ &= L - S_{gen}\left[j_{k,\lambda}(v)\right] L - S_{gen}\left[f(v)\right] \\ &= \Gamma_{\mu}\left(\tfrac{\lambda}{k}, \omega, s\right) L - S_{gen}\left[f(v)\right] \end{aligned}$$

Theorem 2.2. *Let λ be a real number such that $0 < \lambda \leq 1, k > 0, v \geq 0$ and f be sufficient well-defined function then*

$$\begin{aligned} M\left\{D_k^{\lambda} f(v); \omega\right\} &= -(\omega - 1) M\left\{I_k^{1-\lambda} f(v)\right\} \\ &= \frac{B_k(1-\lambda, \lambda - pk)}{\Gamma_k(1-\lambda)} M\left\{f(t); \tfrac{1-\lambda}{k} + \omega\right\} \end{aligned} \tag{2.4}$$

where M denotes Millen transform operator.

Proof. Using equation (1.10)

$$M\{D_k^{\lambda} f(v)\} = M\left\{\frac{d}{dv} I_k^{1-\lambda} f(v)\right\}$$

$$\begin{aligned} \because M\left\{\frac{d}{dv} f(v)\right\} &= -(s-1) M\left\{f(v)\right\} \\ &= -(\omega - 1) M\left\{I_k^{1-\lambda} f(v)\right\} \\ &= -(\omega - 1) M\left\{I_k^{1-\lambda} f(v)\right\} \end{aligned} \tag{2.5}$$

$$\because M\left\{I_k^{\beta} f(t); p\right\} = \frac{B_k(\beta, 1-\beta - pk)}{\Gamma_k(\beta)} M\left\{f(t); \tfrac{\beta}{k} + p\right\} \tag{2.6}$$

Using equation (2.6) in (2.5) theorem 2.2 will be proved.

Theorem 2.3. *Let λ be a real number such that $0 < \lambda \leq 1, k > 0, v \geq 0$ and f be sufficient well-defined function then*

$$F\left\{D_k^{\lambda} f(v); \omega\right\} = -(i\omega)\frac{F[f(v)]}{(i\omega k)} \tag{2.7}$$

where F denotes Fourier transform operator.

Proof. Using equation (1.10)

$$F\{D_k^{\lambda} f(v)\} = F\left\{\frac{d}{dv} I_k^{1-\lambda} f(v)\right\}$$

$$\because F\left\{\frac{d}{dv} f(v)\right\} = -(si)F\{f(v)\}$$

$$= -(\omega i)F\left\{I_k^{1-\lambda} f(v)\right\}$$

$$= -(\omega i)F\left\{I_k^{1-\lambda} f(v)\right\} = -(\omega i)F\left[j_{1-\lambda,k}(v) * f(v)\right], \qquad v \geq 0$$

$$= -(i\omega)\frac{F[f(t)]}{(-i\omega k)^{\frac{\lambda}{k}}}$$

Theorem 2.4. *For $\alpha, \beta > 0, k > 0, z \geq 0$ then following integral representation holds true*

$$\int_0^t z^{\frac{\alpha}{k}-1}(t-z)^{\frac{\beta}{k}-1} dz = B_k(\alpha,\beta)(tk)^{\frac{\alpha+\beta}{k}-1} \tag{2.8}$$

Proof.

$$\int_0^t z^{\frac{\alpha}{k}-1}(t-z)^{\frac{\beta}{k}-1} dz = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta)}\int_0^t (t-z)^{\frac{\beta}{k}-1} z^{\frac{u}{k}-1} dz$$

$$\Rightarrow \int_0^t t^{\frac{\alpha}{k}-1}(t-z)^{\frac{\beta}{k}-1} dz = k\Gamma_k(\beta) I_k^{\beta} t^{\frac{\alpha}{k}-1}$$

Taking Laplace transform both side

$$L\left\{\int_0^t z^{\frac{\alpha}{k}}(t-z)^{\frac{\beta}{k}-1}\right\} = k\Gamma_k(\beta)\frac{1}{(pk)^{\frac{\beta}{k}}}\frac{\Gamma\left(\frac{\alpha}{k}\right)}{(p)^{\frac{\alpha}{k}}}$$

$$= \frac{k^{\frac{\alpha}{k}-1}\Gamma(\alpha)\Gamma_k(\beta)}{(p)^{\frac{\alpha+\beta}{k}}}$$

$$\because \Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\alpha)$$

$$L\left\{\int_0^t z^{\frac{\alpha}{k}}(t-z)^{\frac{\beta}{k}-1}\right\} = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{(p)^{\frac{\alpha+\beta}{k}}}$$

Taking inverse of Laplace transform both side

$$\int_0^t t^{\frac{\alpha}{k}-1}(t-z)^{\frac{\beta}{k}-1}\,dz = \frac{\Gamma_k(\alpha)\,\Gamma_k(\beta)}{\Gamma\left(\frac{\alpha+\beta}{k}\right)} t^{\frac{\alpha+\beta}{k}-1}$$

$$\because \Gamma_k(\alpha+\beta) = k^{\frac{\alpha+\beta}{k}-1}\Gamma\left(\frac{\alpha+\beta}{k}\right)$$

$$= k^{\frac{\alpha+\beta}{k}-1}\frac{\Gamma_k(\alpha)\,\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} t^{\frac{\alpha+\beta}{t}-1} = B_k(\alpha,\beta)\,(tk)^{\frac{\alpha+\beta}{k}-1}$$

Theorems 2.1 to 2.4 are become properties of Riemann-Liouville integral operator if $k = 1$.

3. Applications

Theorem 2.1 of (2.1) can be written in term of Fox's H-function (see [7], [9])

$$L - S_{gen}\left[j_{k,\lambda}(v);\omega,s\right]$$

$$= \Gamma_\mu\left(\frac{\lambda}{k},\omega,s\right) L - S_{gen}\left[f(v)\right] = \frac{s^{\mu-1}}{\Gamma(\lambda)} H_{1,1}^{1,2}\left[\begin{array}{c|c} \left(\frac{\lambda}{k},1\right),(\lambda,1) & \frac{1}{\omega s} \\ \left(\frac{\lambda}{k}+\mu-1\right) & \end{array}\right]$$

If apply properties of integral transform in theorems 2.1, 2.3 and 2.4, we can get various type of multiple integrals, These integrals can be studied by researchers in mathematical analysis of scientific problems.

4. Conclusion

We have got new results of k-fractional integral operator associated with Laplace transform, Fourier transform and Millen transform and discussed about nature of results at special cases. Main results of paper will be useful solving various fractional differential and integral equations.

References

[1] Bapna I. B. and Prajapat, R. S., Some Applications of Extension k-Gamma, k-Beta Functions and k-Beta Distribution, J. of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 7 (2020), 83-98.

[2] Diaz, R. and Pariguan, E., On hypergeometric functions and Pochhammer k-symbol, Divulgaciones Matemticas, Vol. 15 (2007), 179-192.

[3] Diaz, R. and Teruel, C., k-Generalized Gamma and Beta functions, Journal of Nonlinear Mathematical Physics, 12 (2005), 118-134.

[4] Iqbal, S. Mubeen, S., and Tomar, M., On hadamard k- fractional integrals, J. of Fractional calculus and Applications, Vol. 9 (2018), 255-267.

[5] Lebedev, N. N., and Silverman R. A., Special functions and their applications, Dover Publications INC New York, 1967.

[6] Mubeen S., Habibullah G. M., k- fractional integral and applications, International Journal Contemporary Mathematical Science, Vol. 7 (2012), 89-94.

[7] Mathai, A., Haubold, H., Special function for applied scientists, Springer, 2008.

[8] Rainville, E. D., Special Functions, The Macmillan Co. Inc., New York, 1960; Reprinted by Chelsea Publ. Co. Bronx, New York, 1971.

[9] Vays, D. N., Ultra Gamma function, properties and applications : prodigy, J. of Ramanujan Society of Math. Sc., Vol. 5 No. 1 (2016), 127-146.

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