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RELATIONSHIPS BETWEEN MOCK THETA FUNCTIONS AND q-CONTINUED FRACTIONS

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Abstract: The main object of this paper is to present five new interrelationships between mock theta functions and q-continued fractions.

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1. Introduction and Definitions

Three months before his death in early 1920 Ramanujan sent a letter to Hardy of 17 functions, which he called mock theta functions, his functions being separated into three groups, four of order three, ten of order five and three of order seven. These mock theta functions are q-series which converge of $|q| < 1$ and have certain properties as the theta functions when q tends to a root of unity.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

The q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1}(1 - aq^k) & (n \in \mathbb{N}), \end{cases}$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m} \quad (m \in \mathbb{N}_0)$. We also write

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n) = \prod_{n=1}^{\infty}(1 - aq^{n-1}) \qquad (a, q \in \mathbb{C};\ |q| < 1). \tag{c}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (c) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied. The following notations are also frequently used in our investigation:

$$(a_1, a_2, a_3 \dots a_k; q)_n = (a_1; q)_n (a_2; q)_n (a_3; q)_n \dots (a_k; q)_n$$

and

$$(a_1, a_2, a_3 \dots a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty \dots (a_k; q)_\infty.$$

Ramanujan (see [5, pp. 13] and [6]) defined the general theta function $\mathfrak{f}(a, b)$ as follows:

$$f(a, b) = 1 + \sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}\,(a^n + b^n) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}} = f(b, a), \quad (|ab| < 1),$$

where a and b are two complex numbers.
Andrews et al. [1] introduces the general family $R(s, t, l, u, v, w)$ as follows

$$R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l, u, v, w : n), \tag{1}$$

where

$$r(l, u, v, w : n) := \sum_{j=0}^{[\frac{n}{u}]}(-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q; q)_{n-uj}(q^{uv}; q^{uv})_j}. \tag{2}$$

In 2020, for the general family $R(s, t, l, u, v, w)$, Srivastava *et al.* [9] gave the following notations:

Proposition 1.1. *We have*

$$R_\alpha(q) := R(2, 1, 1, 1, 2, 2) = (-q; q^2)_\infty, \tag{3}$$

$$R_\beta(q) := R(2, 2, 1, 1, 2, 2) = (-q^2; q^2)_\infty \tag{4}$$

and

$$R_m := R(m, m, 1, 1, 1, 2) = \frac{(q^{2m}; q^{2m})_\infty}{(q^m; q^{2m})_\infty},\ (m \in \mathbb{N}^*). \tag{5}$$

2. Continued fractions

We begin by recalling some results on continued fractions.
A continued fraction is an expression of the form

$$a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}. \tag{6}$$

With a finite or infinite number of steps, as an example, we have

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

Remark: *We have the following equivalence [7, pp. 33, eq (2.3.14)]*

$$a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots \approx a_0 + \frac{r_1 b_1}{r_1 a_1+}\frac{r_2 r_1 b_2}{r_2 a_2+}\frac{r_3 r_2 b_3}{r_3 a_3+}\dots. \tag{7}$$

3. Preliminary Theorems

Lemma 3.1. [8] *Let a, b, c and q be complex numbers with $|q| < 1$. We have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(b;q)_n z^n}{(c;q)_n} &= \sum_{n=0}^{\infty} \frac{(b;q)_n (bzq/c;q)_n c^n z^n (1-bzq^{2n}) q^{n^2-n}}{(c;q)_n (z;q)_{n+1}} \\ &= \frac{1}{1-}\frac{z(1-b)}{(1-c)-}\frac{z(1-q)(b-c)}{(1-cq)-}\frac{zq(1-bq)(1-c)}{(1-cq^2)-}\frac{zq^2(1-q^2)(b-cq)}{(1-cq^3)-} \\ &\quad \frac{zq^2(1-bq^2)(1-cq)}{(1-cq^4)-}\frac{zq^2(1-q^3)(b-cq^2)}{(1-cq^5)-}\frac{zq^3(1-bq^3)(1-cq^2)}{(1-cq^6)-} \\ &\quad \frac{zq^3(1-q^6)(b-cq^3)}{(1-cq^7)\dots}.\end{aligned} \tag{8}$$

Theorem 3.2. [7]

$$\begin{aligned} A(q) : &= (q^2;q^2)_\infty(-q;q)_\infty \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-}\cdots \\ &= \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}. \end{aligned} \tag{9}$$

$$\begin{aligned} B(q) : &= \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots \\ &= \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\vdots}}}}}}}. \end{aligned} \tag{10}$$

$$\begin{aligned} C(q): &= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+}\frac{q^5}{1+}\frac{q^6}{1+}\cdots \\ &= 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\vdots}}}}}}. \end{aligned} \tag{11}$$

$$\begin{aligned} X(q) &= q^{1/4}\frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty} \\ &= \cfrac{q^{1/4}(1-q^2)}{1-q^{3/2}+\cfrac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)+\cfrac{(1-q^{5/2})(1-q^{13/2})}{q^{3/2}(1-q^{3/2})(1+q^6)+\vdots}}}. \end{aligned} \tag{12}$$

$$\begin{aligned} Y(q) &= q\frac{(q,q^{11};q^{12})_\infty}{(q^5,q^7;q^{12})_\infty} \\ &= \cfrac{q(1-q)}{1-q^3+\cfrac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+\cfrac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})+\vdots}}}. \end{aligned} \tag{13}$$

$$\begin{aligned} F(a,b,q) &= \frac{(a^2q^3,b^2q^3;q^4)_\infty}{(a^2q,b^2q;q^4)_\infty} \\ &= \cfrac{1}{1-ab+\cfrac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+\cfrac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^{14})+\vdots}}}. \end{aligned} \tag{14}$$

Proposition 3.3. *Let* $m \in \mathbb{N}$ *we have,*

$$F(q^m, q^m, q^{2m}) = R_{4m}^2 = \cfrac{1}{1 - q^{2m} + \cfrac{(q^m - q^{3m})^2}{(1-q^{2m})(1+q^{4m}) + \cfrac{(q^m - q^{7m})^2}{(1-q^{2m})(1+q^{28m}) + \vdots}}}. \tag{15}$$

Proof. To show identity (15) it suffise to take $a = b = q^m$ and replace q by q^2 in (14) we have

$$F(q^m, q^m, q^{2m}) = \frac{(q^{2m}q^{6m}, q^{2m}q^{6m}; q^{8m})_\infty}{(q^{2m}q^{2m}, q^{2m}q^{2m}; q^{8m})_\infty} = \frac{(q^{8m}; q^{8m})_\infty^2}{(q^{4m}; q^{8m})_\infty^2} = R_{4m}^2.$$

4. Third Order Mock Theta Functions

Third order mock theta functions are defined as (see [2]):

$$f_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}, \tag{16}$$

$$\chi_3(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - q^m + q^{2m})}, \tag{17}$$

$$\omega_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \tag{18}$$

and

$$\rho_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=1}^{n}(1 + q^{2m+1} + q^{4m+2})}. \tag{19}$$

5. Eighth Order Mock Theta Functions

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}. \tag{20}$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}. \tag{21}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{22}$$

6. Interrelationships Between Mock Theta Functions and q-Continued Fractions

Following identities are recorded in [7]:

$$\begin{aligned}\rho_3(q)+\rho_3(-q) &+ \frac{1}{2}\left(\omega_3(q)+\omega_3(-q)\right)\\ &= \frac{3(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty}. \end{aligned}\tag{23}$$

$$(q^2;q^2)_\infty(\rho_3(q)+\frac{1}{2}\omega_3(q)) = \frac{3}{2}\frac{(q^6;q^6)^2_\infty}{(q^3;q^6)^2_\infty}. \tag{24}$$

$$\begin{aligned}B(z;q) &= \sum_{-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/4}}{1+zq^{(n+1)/2)}}\\ &= \frac{(q^{1/2},q)_\infty(q^2,zq,z^{-1}q;q^2)_\infty}{(-zq^{1/2},-z^{-1/2}q^{1/2})_\infty}.\end{aligned}\tag{25}$$

$$A(-q^2;q^8) = \sum_{-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{(8n+2)}}. \tag{26}$$

$$A(-q^2;q^8) = \frac{(q^8;q^8)^2_\infty}{(q^2;q^4)_\infty}. \tag{27}$$

$$V_1(q)-V_1(-q) = 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{(8n+2)}}. \tag{28}$$

$$U_0(q)+2U_1(q) = (-q;q^2)^3_\infty(q^2;q^2)_\infty(q^2;q^4)_\infty. \tag{29}$$

$$U_0(-q)+2U_1(-q) = 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{30}$$

7. Main Theorems

Theorem 7.1.

$$\begin{aligned}\rho_3(q)+\rho_3(-q) &+ \frac{1}{2}\left(\omega_3(q)+\omega_3(-q)\right)\\ &= 3\frac{(q^{24};q^{48})_\infty(q^{24};q^{24})_\infty(-q^{12};q^{24})_\infty}{(q^2,q^4,q^6,q^6,q^8,q^{10},q^{14},q^{18},q^{18},q^{20},q^{22},q^{24},q^{24};q^{24})_\infty}\\ &\times \cfrac{1}{1-q^6+\cfrac{(q^3-q^9)^2}{(1-q^6)(1+q^{12})+\cfrac{(q^3-q^{21})^2}{(1-q^6)(1+q^{84})+\vdots}}}.\end{aligned}\tag{31}$$

$$\rho_3(q) + \frac{1}{2}\omega_3(q) = \frac{3}{2}(-q;q)_\infty \tag{32}$$
$$\times \left[\cfrac{1}{1 - \cfrac{q}{1 + \cfrac{q(1-q)}{1 - \cfrac{q^3}{1 + \cfrac{q^2(1-q^2)}{1 - \cfrac{q^5}{1 + \cfrac{q^3(1-q^3)}{1 - \vdots}}}}}}}\right]^{-1} \times \left[\cfrac{1}{1 - \cfrac{q^3}{1 + \cfrac{q^3(1-q^3)}{1 - \cfrac{q^9}{1 + \cfrac{q^6(1-q^6)}{1 - \cfrac{q^{15}}{1 + \cfrac{q^9(1-q^9)}{1 - \vdots}}}}}}}\right].$$

$$V_1(q) - V_1(-q) = \frac{2q}{(q^2, q^6, q^8; q^8)_\infty} \tag{33}$$
$$\times \cfrac{1}{1 - q^2 + \cfrac{(q-q^3)^2}{(1-q^2)(1+q^4) + \cfrac{(q-q^7)^2}{(1-q^2)(1+q^{28}) + \vdots}}}.$$

$$U_0(q) + 2U_1(q) = (q^2;q^4)_\infty \tag{34}$$
$$\times \left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}\right]^3 \times \left[\cfrac{1}{1-\cfrac{q^2}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^6}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{10}}{1+\cfrac{q^6(1-q^6)}{1-\vdots}}}}}}}\right]^{-2}.$$

$$U_0(-q) + 2U_1(-q) = 2\left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}\right]^{-1}. \tag{35}$$

Proof. First of all we have to list some q-product identities, which are required to prove our findings in this theorem, see for details in [3]:

$$\begin{aligned}
(q^2;q^2)_\infty &= \prod_{n=0}^{\infty}(1-q^{2n+2}) \\
&= \prod_{n=0}^{\infty}(1-q^{2(12n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+3)+2}) \\
&\times \prod_{n=0}^{\infty}(1-q^{2(12n+4)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+5)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+6)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+7)+2}) \\
&\times \prod_{n=0}^{\infty}(1-q^{2(12n+8)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+9)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+10)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(12n+11)+2}) \\
&= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16},q^{18},q^{20},q^{22},q^{24};q^{24})_\infty,
\end{aligned}$$

$$(q^6;q^{12})_\infty = (q^6,q^{12},q^{18},q^{24};q^{24})_\infty,$$

and

$$(-q^{12};q^{24})_\infty(q^{12};q^{24})_\infty = (q^{24};q^{48})_\infty.$$

To prove identity (31), using identity (23) and further applying q-product identities, we have:

$$\begin{aligned}
&\rho_3(q)+\rho_3(-q)+\frac{1}{2}\left(\omega_3(q)+\omega_3(-q)\right) \\
&= \frac{3(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty} \\
&= \frac{3(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty(q^6,q^{12},q^{18},q^{24};q^{24})_\infty} \\
&= \frac{3(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16},q^{18},q^{20},q^{22},q^{24};q^{24})_\infty(q^6,q^{12},q^{18},q^{24};q^{24})_\infty} \\
&= \frac{3(q^{24};q^{48})_\infty(-q^{12};q^{24})_\infty(q^{24};q^{24})^2_\infty}{(q^2,q^4,q^6,q^8,q^{10},q^{14},q^{16},q^{18},q^{20},q^{22},q^{24};q^{24})_\infty(q^6,q^{18},q^{24};q^{24})_\infty}\frac{(q^{24};q^{24})^2_\infty}{(q^{12};q^{24})^2_\infty} \\
&= \frac{3(q^{24};q^{48})_\infty(-q^{12};q^{24})_\infty(q^{24};q^{24})^2_\infty}{(q^2,q^4,q^6,q^8,q^{10},q^{14},q^{16},q^{18},q^{20},q^{22},q^{24};q^{24})_\infty(q^6,q^{18},q^{24};q^{24})_\infty}F(q^3,q^3;q^6) \\
&= 3\frac{(q^{24};q^{48})_\infty(q^{24};q^{24})_\infty(-q^{12};q^{24})_\infty}{(q^2,q^4,q^6,q^6,q^8,q^{10},q^{14},q^{18},q^{18},q^{20},q^{22},q^{24},q^{24};q^{24})_\infty} \\
&\times \cfrac{1}{1-q^6+\cfrac{(q^3-q^9)^2}{(1-q^6)(1+q^{12})+\cfrac{(q^3-q^{21})^2}{(1-q^6)(1+q^{84})+\vdots}}}.
\end{aligned}$$

Which completes our demonstration of the second assertion (31).
Next, we prove our second identity (32), using identity (24) and further applying q-product identities, we have:

$$\begin{aligned}\rho_3(q)+\frac{1}{2}\omega_3(q) &= \frac{3}{2}(-q;q)_\infty\frac{A(q^3)}{A(q)}\\ &= \frac{3}{2}(-q;q)_\infty\times\end{aligned}$$

$$\left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}\right]^{-1}\times\left[\cfrac{1}{1-\cfrac{q^3}{1+\cfrac{q^3(1-q^3)}{1-\cfrac{q^9}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{15}}{1+\cfrac{q^9(1-q^9)}{1-\vdots}}}}}}}\right].$$

Which completes our demonstration of the second assertion (32).
Then we show the third identity (33), we have

$$\begin{aligned}V_1(q)-V_1(-q) &= 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\frac{(q^8;q^8)_\infty^2}{(q^2;q^4)_\infty}\\ &= 2q(-q^4;q^4)_\infty(q^8;q^8)_\infty\frac{(q^8;q^8)_\infty}{(q^2;q^2)_\infty}\\ &= 2q(-q^4;q^4)_\infty(q^8;q^8)_\infty\frac{(q^8;q^8)_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty}\\ &= \frac{2q}{(q^2,q^6,q^8;q^8)_\infty}F(q,q,q^2)\\ &= \frac{2q}{(q^2,q^6,q^8;q^8)_\infty}\cfrac{1}{1-q^2+\cfrac{(q-q^3)^2}{(1-q^2)(1+q^4)+\cfrac{(q-q^7)^2}{(1-q^2)(1+q^{28})+\vdots}}}.\end{aligned}$$

Which completes our demonstration of the third assertion (33).
Further, we prove our fourth identity (34), we have

$$U_0(q) + 2U_1(q) = (-q;q^2)^3_\infty (q^2;q^2)_\infty (q^2;q^4)_\infty$$
$$= \frac{(-q;q^2)^3_\infty (-q^2;q^2)^3_\infty}{(-q^2;q^2)^3_\infty} (q^2;q^2)_\infty (q^2;q^4)_\infty$$
$$= \frac{(-q;q)^3_\infty (q^2;q^2)^2\infty (q^2;q^2)_\infty (q^2;q^4)_\infty}{(-q^2;q^2)^3_\infty (q^2;q^2)^2_\infty}$$
$$= \frac{(-q;q)^3_\infty (q^2;q^2)^3}{(-q^2;q^2)^2\infty (q^4;q^4)^2} = \frac{A^3(q)}{A^2(q^2)}$$
$$= (q^2;q^4)_\infty \left[\cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}\right]^3 \left[\cfrac{1}{1-\cfrac{q^2}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^6}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{10}}{1+\cfrac{q^6(1-q^6)}{1-\vdots}}}}}}}\right]^{-2}.$$

Which completes our demonstration of the fourth assertion (34).
Similarly, we can prove our fifth identity (35), and left for the readers as an exercise. Hence, finally we complete the proof of our Theorem 7.1.

8. Conclusion

The result of Theorem 7.1 gives us an idea to write relationships between each mock theta function and q-Continued Fractions. So we can always see mock theta functions as q-products and continuous q-fractions.

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