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CERTAIN UNIFIED INTEGRALS INVOLVING A PRODUCT OF THE FOUR-PARAMETER BESSEL FUNCTION AND JACOBI POLYNOMIAL

S. C. Pandey and K. Chaudhary

Faculty of Mathematics and Computing,
Department of Mathematics and Statistics,
Banasthali Vidyapith, Niwai - 304022, Rajasthan, INDIA

E-mail : sharedpandey@yahoo.co.in, koshi1340@gmail.com

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Abstract: The present paper is devoted to derive a generalized Oberhettinger-type integral formula. The derived form of an integral involving the product of the four-parameter Bessel function and Jacobi polynomial. The outcomes are expressed in terms of the Kampé de Fériet and Srivastava and Daoust functions. Also, four Corollaries of the both Theorems are derived in terms of the Kampé de Fériet and Srivastava and Daoust functions. Some of the significant particular cases are also determined. Furthermore, we drive an interesting relationship between Kampé de Fériet and Srivastava and Daoust functions.

Keywords and Phrases: Bessel function, Hypergeometric function, Srivastava and Daoust function.

2020 Mathematics Subject Classification: 33C10, 33C20, 33C65.

1. Introduction

The Bessel function frequently appears in a wide variety of problems pertaining to applied Sciences. Daniel Bernoulli's analysis of the oscillations of a uniform heavy flexible chain is the first application of the Bessel function in the physical problems [9]. Bessel function and modified Bessel function play an important role in the analysis of optical transmission and microwave in waveguides [11, 19] including fiber and coaxial. Additionally, the Bessel function can be seen inverse problem in wave propagation, which has applications in acoustic imaging [7], astronomy and medicine. Solving the Helmholtz and Laplace equations in cylindrical and spherical coordinate systems to used half-integer and integer orders of the Bessel function [12] respectively in the coordinate system and furthermore, solution to the radial schrödinger equation for a free particle in cylindrical and spherical coordinate systems [10], electromagnetic waves in a cylindrical waveguides, heat conduction in a cylindrical objects, diffusion problems on a lattice [14], modes of vibrations of a thicker plates such as sheet metal (Mindlin-Reissner and Kirchhoff-Love plate theory) or thin circuler or annular acoustic membrane (membranophone and drumhead), frequency-dependent friction in circular pipelines [22], cooling of a heated cylinder [1], pressure amplitudes of inviscid rotational flow, angular resolution, geophysics and seismology analysis of the surface waves produced by microtremors and signal processing [13, 16] (FM audio synthesis, Bessel filter and Kaiser window). In some recent investigations [2, 5, 6, 8], several authors have proposed a number of interesting integral formulas associated with Bessel functions.

Also, $(\varrho)_l$ represents the Pochhammer's symbol or rising factorial [18] defined by

$$(\varrho)_l = \frac{\Gamma(\varrho + l)}{\Gamma(\varrho)} = \begin{cases} 1 & \text{if } l = 0, \\ \varrho(\varrho+1)...(\varrho+l-1) & \text{if } l \in \mathbb{N}. \end{cases}$$

Chaudhry and Zubair [3] defined the generalized Gamma function

$$\Gamma_\varrho(x) = \int_0^\infty z^{x-1} e^{-z-\frac{\varrho}{z}} dz, \ \mathfrak{Re}(\varrho) > 0. \tag{1.1}$$

Srivastava et al. [20] introduced the generalized Pochhammer symbol as

$$(\beta;\varrho)_l = \begin{cases} \frac{\Gamma_\varrho(\beta+l)}{\Gamma(\beta)} & \text{if } \mathfrak{Re}(\varrho) > 0,\ l,\ \beta \in \mathbb{C}, \\ (\beta)_l & \text{if } \varrho = 0,\ l,\ \beta \in \mathbb{C}. \end{cases}$$

Özarslan and Yaşar ([24], p. 5, eq. (1.8)) introduced the four-parameter Bessel function in the following series form

$$G_\nu^{(\alpha,\beta)}(x;\varrho) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k (\beta;\varrho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{x}{2})^{2k+\nu}}{k!}, \tag{1.2}$$

where $\nu,\ x,\ \beta \in \mathbb{C}$, $\mathfrak{Re}(\nu) > -1$, $\mathfrak{Re}(\varrho) > 0$.

For $\alpha = 1$ the four-parameter Bessel function reduced to the three-parameter Bessel function of the first kind ([24], p. 5)

$$J_\nu^{(\beta)}(x;\varrho) = \sum_{k=0}^{\infty} \frac{(-1)^k(\beta;\varrho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{x}{2})^{2k+\nu}}{k!}, \tag{1.3}$$

where x, ν, $\beta \in \mathbb{C}$, $\mathfrak{Re}(\varrho) > 0$, $\mathfrak{Re}(\nu) > -1$.

In case $\alpha = -1$ the four-parameter Bessel function reduced to the three-parameter Bessel function of the second kind ([24], p. 5)

$$I_\nu^{(\beta)}(x;\varrho) = \sum_{k=0}^{\infty} \frac{(\beta;\varrho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{x}{2})^{2k+\nu}}{k!}, \tag{1.4}$$

where ν, x, $\beta \in \mathbb{C}$, $\mathfrak{Re}(\nu) > -1$, $\mathfrak{Re}(\varrho) > 0$.

Assuming, $\beta = 1$ and $\varrho = 0$, $J_\nu^{(\beta)}(x;\varrho)$ is reduced to Bessel function [23] of the first kind $J_\nu(x)$. Furthermore, $\beta = -1$ and $\varrho = 0$, $I_\nu^{(\beta)}(x;\varrho)$ is reduced to modified Bessel function [23] of the first kind $I_\nu(x)$.

The classical Jacobi polynomial $P_n^{(\tau,\varsigma)}(x)$ can be presented in the following series form [18]

$$P_n^{(\tau,\varsigma)}(x) = \sum_{k=0}^{n} \frac{(1+\tau)_n(\varsigma+\tau+1)_{n+k}}{(n-k)!k!(1+\tau)_k(\varsigma+\tau+1)_n} \left(\frac{x-1}{2}\right)^k, \tag{1.5}$$

it also is equivalently a function of the Gauss hypergeometric function [19]

$$P_n^{(\tau,\varsigma)}(x) = \frac{(1+\tau)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, & (n+\varsigma+\tau+1); \\ (1+\tau); & \end{matrix} \; \frac{1-x}{2} \right]. \tag{1.6}$$

Kampé de Fériet ([21], p. 27) introduced the general hypergeometric series in two variables defined as follows

$$F_{l:m;n}^{i:j;k} \left[\begin{matrix} (u_i) : (v_j); (w_k); \\ (\xi_l) : (\eta_m); (\zeta_n); \end{matrix} x_1, x_2 \right] = \sum_{k_1,k_2=0}^{\infty} \frac{\Pi_{h=1}^{i}(u_h)_{k_1+k_2}\Pi_{h=1}^{j}(v_h)_{k_1}\Pi_{h=1}^{k}(w_h)_{k_2}}{\Pi_{h=1}^{l}(\xi_h)_{k_1+k_2}\Pi_{h=1}^{m}(\eta_h)_{k_1}\Pi_{h=1}^{n}(\zeta_h)_{k_2}} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!}, \tag{1.7}$$

for convergence
(i) $i+j < 1+l+m$, $i+k < 1+l+n$, $|x_1| < \infty$, $|x_2| < \infty$, or
(ii) $i+j = 1+l+m$, $i+k = 1+l+n$ and

$$\begin{cases} |x_1|^{-\frac{1}{(l-i)}} + |x_2|^{-\frac{1}{(l-i)}} < 1 & \text{if } l < i, \\ max\{|x_1|, |x_2|\} < 1 & \text{if } l \geq i. \end{cases}$$

Srivastava and Daoust [21] proposed multivariable generalized hypergeometric function, given as

$$F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = F^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}}\left[\begin{matrix}[(a):\phi',...,\phi^{(n)}]:[(b'):\theta'];...;\\ [(c):\delta',...,\delta^{(n)}]:[(d'):\psi'];...;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\theta^{(n)}];\\ [(d^{(n)}):\psi^{(n)}];\end{matrix} x_1,...,x_n\right] = \sum_{k_1,...,k_n=0}^{\infty} \Theta(k_1,...,k_n)\frac{x^{k_1}}{k_1!}...\frac{x^{k_n}}{k_n!}, \tag{1.8}$$

where, for convenience

$$\Theta(k_1,...,k_n) = \frac{\prod_{j=1}^{A}(a_j)_{\phi'_j k_1+...+\phi_j^{(n)}k_n}\prod_{j=1}^{B'}(b'_j)_{\theta'_j k_1}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{\theta_j^{(n)}k_n}}{\prod_{j=1}^{C}(c_j)_{\delta'_j k_1+...+\delta_j^{(n)}k_n}\prod_{j=1}^{D'}(d'_j)_{\psi'_j k_1}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{\psi_j^{(n)}k_n}},$$

the coefficients $\phi_j^{(l)}$, $j=1,...,A$; $\theta_j^{(l)}$, $j=1,...,B^{(l)}$; $\delta_j^{(l)}$, $j=1,...,C$; $\psi_j^{(l)}$, $j=1,...,D^{(l)}$ are real and positive, and (a) abbreviates the array of A parameters $a_1,...,a_A$, $(b^{(l)})$ abbreviates the array of $B^{(l)}$ parameters $b_j^{(l)}$, $j=1,...,B^{(l)}$; $\forall\ l \in \{1,...,n\}$, with similar interpretations for (c) and $(d^{(l)})$, $\forall\ l \in \{1,...,n\}$; etcetera. For applications of Srivastava and Daoust function of Pandey [17] and Chaurasia and Pandey [4].

In the present work, we recall the following integral mentioned in the classical monograph by Oberhettinger (see [15], p. 22)

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\eta}dx = 2\eta h^{-\eta}\left(\frac{h}{2}\right)^{\delta}\frac{\Gamma(2\delta)\Gamma(\eta-\delta)}{\Gamma(1+\delta+\eta)}, \tag{1.9}$$

provided $0<\mathfrak{Re}(\delta)<\mathfrak{Re}(\eta)$.

In this paper, we discuss Oberhettinger-type integral that contains the four-parameter Bessel function and Jacobi polynomial. The current investigations are given as two Theorems whose outcomes in terms of the Kampé de Fériet and Srivastava and Daoust functions. Further, four Corollaries of the both results in Theorems are derived in terms of the Kampé de Fériet and Srivastava and Daoust functions. Also, determines a some interesting well-known special cases. Moreover, we establish a remarkable relation between Kampé de Fériet and Srivastava and Daoust functions.

2. Main Theorems

In this section, we derive an integral formula involving four-parameter Bessel function and Jacobi polynomial. The outcomes are expressed in terms of the Kampé de Fériet and Srivastava and Daoust function, defined above in (1.7) and (1.8).

Theorem 2.1. *For $\mathfrak{Re}(\lambda+\nu) > \mathfrak{Re}(\delta) > 0, \mathfrak{Re}(\varrho) > 0$ and $\mathfrak{Re}(\nu) > -1$, the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} G_\nu^{(\alpha,\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx = \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^\nu(\beta)_\nu(1+\varsigma)_n\Gamma(2\delta)}{\Gamma^2(\nu+1)n!\Gamma(\lambda+\delta+\nu+1)}$$

$$\times\frac{\Gamma(\nu+\lambda+1)\Gamma(\lambda-\delta+\nu)}{\Gamma(\lambda+\nu)}\left[F^{4:2;4}_{4:3;3}\left[\begin{matrix}\triangle(2;\nu-\delta+\lambda),\triangle(2;\lambda+\nu+1):\\ \triangle(2;\nu+\lambda),\triangle(2;\nu+\delta+\lambda+1):\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\triangle(2;(\beta+\nu;\varrho));\triangle(2;-n),\triangle(2;1+\varsigma+\tau+n);\\ \triangle(2;\nu+1),(\nu+1);\triangle(2;1+\varsigma),\tfrac{1}{2};\end{matrix}\ -\frac{\alpha\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]$$

$$-\frac{n(\nu+\lambda+1)(\nu-\delta+\lambda)(1+\varsigma+\tau+n)}{(\nu+\lambda)(1+\varsigma)(\nu+\delta+\lambda+1)}\left(\frac{\zeta}{2h}\right)$$

$$\times F^{4:2;4}_{4:3;3}\left[\begin{matrix}\triangle(2;1+\nu+\lambda-\delta),\triangle(2;2+\nu+\lambda):\triangle(2;(\beta+\nu;\varrho));\triangle(2;1-n),\\ \triangle(2;1+\lambda+\nu),\triangle(2;\nu+\delta+\lambda+2):\triangle(2;1+\nu),(\nu+1);\end{matrix}\right.$$

$$\left.\left.\begin{matrix}\triangle[2;2+\varsigma+\tau+n];\\ \triangle(2;2+\varsigma),\tfrac{3}{2};\end{matrix}\ -\frac{\alpha\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]\right], \tag{2.1}$$

where $\triangle(m_1;l_1)$ abbreviates the array of m_1 parameters $\frac{l_1}{m_1},\frac{l_1+1}{m_1},\ldots,\frac{l_1+m_1-1}{m_1}, m_1 \geq 1$.

Proof. To prove Theorem 2.1, we first express the four-parameter Bessel function and Jacobi polynomial in series forms given by (1.2) and (1.6) respectively. Now, we interchange the order of summations and integration (permissible with the uniform convergence of the series), we get

$$=\frac{(\beta)_\nu(1+\varsigma)_n\left(\frac{\xi}{2}\right)^\nu}{n!\Gamma^2(\nu+1)}\sum_{l=0}^{\infty}\sum_{m=0}^{n}\frac{(\beta+\nu;\varrho)_{2l}(\varsigma+\tau+n+1)_m(-n)_m}{l!m!(1+\nu)_{2l}(\nu+1)_l(1+\varsigma)_m}\left(-\frac{\alpha\xi^2}{4}\right)^l\left(\frac{\zeta}{2}\right)^m$$

$$\times\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-[\lambda+\nu+2l+m]}dx. \tag{2.2}$$

Using the Oberhettinger integral (1.9) formula, we obtain

$$= \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^{\nu}(\beta)_{\nu}\Gamma(\lambda+\nu+1)(1+\varsigma)_n\Gamma(2\delta)\Gamma(\nu-\delta+\lambda)}{n!\Gamma^2(\nu+1)\Gamma(\delta+\nu+\lambda+1)\Gamma(\nu+\lambda)}$$

$$\times \sum_{l=0}^{\infty}\sum_{m=0}^{n} \frac{(1+\lambda+\nu)_{2l+m}(\lambda+\nu-\delta)_{2l+m}(\beta+\nu;\varrho)_{2l}(1+\varsigma+\tau+n)_m(-n)_m}{l!m!(\nu+\lambda)_{2l+m}(\nu+\delta+\lambda+1)_{2l+m}(1+\nu)_{2l}(\nu+1)_l(1+\varsigma)_m}$$

$$\times \left(-\frac{\alpha\xi}{4h^2}\right)^l \left(\frac{\zeta}{2h}\right)^m. \tag{2.3}$$

Now, separating m series into its even and odd terms (2.3) then we obtain

$$= \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^{\nu}(\beta)_{\nu}\Gamma(\nu-\delta+\lambda)(1+\varsigma)_n\Gamma(2\delta)\Gamma(\lambda+\nu+1)}{n!\Gamma^2(\nu+1)\Gamma(\nu+\lambda)\Gamma(\delta+\nu+\lambda+1)}$$

$$\times \sum_{l=0}^{\infty}\sum_{m=0}^{n} \left[\frac{(1+\lambda+\nu)_{2l+2m}(\lambda+\nu-\delta)_{2l+2m}(\beta+\nu;\varrho)_{2l}(1+\varsigma+\tau+n)_{2m}(-n)_{2m}}{(\nu+\lambda)_{2l+2m}(\nu+\delta+\lambda+1)_{2l+2m}(\nu+1)_{2l}(\nu+1)_l(1+\varsigma)_{2m}(\frac{1}{2})_m l!m!}\right.$$

$$\times \left(-\frac{\alpha\xi^2}{4h^2}\right)^l \left(\frac{\zeta^2}{16h^2}\right)^m - \frac{n(1+\lambda+\nu)(\lambda+\nu-\delta)(1+\varsigma+\tau+n)}{(\nu+\lambda)(1+\varsigma)(1+\lambda+\nu+\delta)}\left(\frac{\zeta}{2h}\right)$$

$$\times \frac{(2+\nu+\lambda)_{2l+2m}(1+\nu+\lambda-\delta)_{2l+2m}(\beta+\nu;\varrho)_{2l}(2+\varsigma+\tau+n)_{2m}(1-n)_{2m}}{(1+\nu+\lambda)_{2l+2m}(2+\lambda+\nu+\delta)_{2l+2m}(\nu+1)_{2l}(\nu+1)_l(2+\varsigma)_{2m}(\frac{3}{2})_m l!m!}$$

$$\left.\times \left(-\frac{\alpha\xi^2}{4h^2}\right)^l \left(\frac{\zeta^2}{16h^2}\right)^m\right]. \tag{2.4}$$

Now, using the Kampé de Fériet function (1.7), we arrive at the desire form given in *RHS* of (2.1).

Theorem 2.2. *For $\mathfrak{Re}(\lambda+\nu) > \mathfrak{Re}(\delta) > 0, \mathfrak{Re}(\varrho) > 0$ and $\mathfrak{Re}(\nu) > -1$, the following integral formula holds true:*

$$\int_0^{\infty} x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} G_{\nu}^{(\alpha,\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx$$

$$= \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^{\nu}(\beta)_{\nu}\Gamma(\nu-\delta+\lambda)\Gamma(2\delta)\Gamma(1+\nu+\lambda)}{\Gamma^2(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\nu+\delta+\lambda)}$$

$$\times F^{5:0;0}_{6:0;1}\left[\begin{matrix}(\nu+\lambda+1:2,3),(\nu-\delta+\lambda:2,3),((\beta+\nu;\varrho):2,2),(\varsigma+\tau+1:1,2)\\(\nu+\lambda:2,3),(\nu+\lambda+\delta+1:2,3),(\nu+1:2,2),(\nu+1:1,1),(1:1,1),\end{matrix}\right.$$

$$\left.\begin{matrix}(1+\tau:1,1):\_;\_;\\(1+\varsigma+\tau:1,1):\_;(1+\varsigma:1);\end{matrix}-\frac{\alpha\xi^2}{4h^2},\frac{\alpha\xi^2\zeta}{8h^3}\right]. \tag{2.5}$$

Proof. In order to prove Theorem 2.2, we first express the four-parameter Bessel function and Jacobi polynomial in series forms given by (1.2) and (1.5) respectively and using the Lemma ([18], *p.* 57)

$$\sum_{n_1=0}^{\infty}\sum_{r_1=0}^{n_1}B(r_1,n_1)=\sum_{n_1=0}^{\infty}\sum_{r_1=0}^{\infty}B(r_1,n_1+r_1), \tag{2.6}$$

we get

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\xi}{2}\right)^{\nu}(\beta)_{\nu}(1+\varsigma)_{n+m}(1+\varsigma+\tau)_{n+2m}(\beta+\nu;\varrho)_{2n+2m}}{n!\Gamma^2(\nu+1)m!(n+m)!(1+\nu)_{2n+2m}(\nu+1)_{n+m}(1+\varsigma+\tau)_{n+m}(1+\varsigma)_m}$$

$$\times\left(-\frac{\alpha\xi^2}{4}\right)^{n+m}\left(-\frac{\zeta}{2}\right)^{m}\int_0^{\infty}x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-[\lambda+\nu+2n+3m]}dx. \tag{2.7}$$

Now, using the Oberhettinger integral (1.9) formula, we get

$$=\frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^{\nu}(\beta)_{\nu}\Gamma(2\delta)\Gamma(\nu+\lambda+1)\Gamma(\nu-\delta+\lambda)}{\Gamma(\lambda+\nu)\Gamma^2(1+\nu)\Gamma(\lambda+\delta+\nu+1)}$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\nu+\lambda+1)_{2n+3m}(\nu-\delta+\lambda)_{2n+3m}(\beta+\nu;\varrho)_{2n+2m}(1+\varsigma+\tau)_{n+2m}}{(\nu+\lambda)_{2n+3m}(\nu+\lambda+\delta+1)_{2n+3m}(\nu+1)_{2n+2m}(1+\varsigma+\tau)_{n+m}}$$

$$\times\frac{(1+\varsigma)_{n+m}}{(1+\varsigma)_m(1+\nu)_{n+m}(1)_{n+m}}\frac{\left(\frac{\alpha\xi^2\zeta}{8h^3}\right)^m}{m!}\frac{\left(-\frac{\alpha\xi^2}{4h^2}\right)^n}{n!}. \tag{2.8}$$

Now, using the Srivastava and Daoust function (1.8), we arrive at the desire form given in *RHS* of (2.5).

On taking $\alpha=1$ in Theorem 2.1 the four-parameter Bessel function $G_{\nu}^{(\alpha,\beta)}(x;\varrho)$ reduced to the three-parameter Bessel function of the first kind $J_{\nu}^{(\beta)}(x;\varrho)$ and we can deduce the following Corollary 2.1 based on the integral presented in Theorem 2.1.

Corollary 2.1. *For* $\mathfrak{Re}(\lambda+\nu) > \mathfrak{Re}(\delta) > 0, \mathfrak{Re}(\varrho) > 0$ *and* $\mathfrak{Re}(\nu) > -1$, *the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} J_\nu^{(\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx = \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^\nu(\beta)_\nu(1+\varsigma)_n\Gamma(2\delta)}{\Gamma^2(\nu+1)n!\Gamma(\lambda+\delta+\nu+1)}$$

$$\times\frac{\Gamma(\nu+\lambda+1)\Gamma(\lambda-\delta+\nu)}{\Gamma(\lambda+\nu)}\left[F_{4:3;3}^{4:2;4}\left[\begin{array}{l}\triangle(2;\nu-\delta+\lambda),\triangle(2;\lambda+\nu+1):\\ \triangle(2;\nu+\lambda),\triangle(2;\nu+\delta+\lambda+1):\end{array}\right.\right.$$

$$\left.\begin{array}{l}\triangle(2;(\beta+\nu;\varrho));\triangle(2;-n),\triangle(2;1+\varsigma+\tau+n);\\ \triangle(2;\nu+1),(\nu+1);\triangle(2;1+\varsigma),\frac{1}{2};\end{array} -\frac{\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]$$

$$-\frac{n(\nu+\lambda+1)(\nu-\delta+\lambda)(1+\varsigma+\tau+n)}{(\nu+\lambda)(1+\varsigma)(\nu+\delta+\lambda+1)}\left(\frac{\zeta}{2h}\right)$$

$$\times F_{4:3;3}^{4:2;4}\left[\begin{array}{l}\triangle(2;2+\nu+\lambda),\triangle(2;1+\nu+\lambda-\delta):\triangle(2;(\beta+\nu;\varrho));\triangle(2;1-n),\\ \triangle(2;\lambda+\nu+1),\triangle(2;\nu+\delta+\lambda+2):\triangle(2;\nu+1),(1+\nu);\end{array}\right.$$

$$\left.\left.\begin{array}{l}\triangle[2;2+\varsigma+\tau+n];\\ \triangle(2;2+\varsigma),\frac{3}{2};\end{array} -\frac{\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]\right]. \tag{2.9}$$

On substituting $\alpha = 1$ in Theorem 2.2 the four-parameter Bessel function $G_\nu^{(\alpha,\beta)}(x;\varrho)$ reduced to the three-parameter Bessel function of the first kind $J_\nu^{(\beta)}(x;\varrho)$ and we can deduce the following Corollary 2.2 based on the integral presented in Theorem 2.2.

Corollary 2.2. *For* $\mathfrak{Re}(\lambda+\nu) > \mathfrak{Re}(\delta) > 0, \mathfrak{Re}(\varrho) > 0$ and $\mathfrak{Re}(\nu) > -1$, *the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} J_\nu^{(\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx$$

$$=\frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^\nu(\beta)_\nu\Gamma(2\delta)\Gamma(\nu+\lambda+1)\Gamma(\nu-\delta+\lambda)}{\Gamma^2(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\delta)}$$

$$\times F^{5:0;0}_{6:0;1}\left[\begin{matrix}(\nu+\lambda+1:2,3),(\nu-\delta+\lambda:2,3),((\beta+\nu;\varrho):2,2),(\varsigma+\tau+1:1,2)\\(\nu+\lambda:2,3),(\nu+\lambda+\delta+1:2,3),(\nu+1:2,2),(\nu+1:1,1),(1:1,1),\end{matrix}\right.$$

$$\left.\begin{matrix}(1+\tau:1,1):\_;\_;\\(1+\varsigma+\tau:1,1):\_;(1+\varsigma:1);\end{matrix}\quad -\frac{\xi^2}{4h^2},\frac{\xi^2\zeta}{8h^3}\right]. \tag{2.10}$$

On setting $\alpha=-1$ in Theorem 2.1 the the four-parameter Bessel function $G_\nu^{(\alpha,\beta)}(x;\varrho)$ reduced to the three-parameter Bessel function of the second kind $I_\nu^{(\beta)}(x;\varrho)$ and based on the integral shown in Theorem 2.1, we may derive the following Corollary 2.3.

Corollary 2.3. *For* $\mathfrak{Re}(\lambda+\nu)>\mathfrak{Re}(\delta)>0, \mathfrak{Re}(\varrho)>0$ and $\mathfrak{Re}(\nu)>-1$, *the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} I_\nu^{(\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx=\frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^\nu(\beta)_\nu(1+\varsigma)_n\Gamma(2\delta)}{\Gamma^2(\nu+1)n!\Gamma(\lambda+\delta+\nu+1)}$$

$$\times\frac{\Gamma(\nu+\lambda+1)\Gamma(\lambda-\delta+\nu)}{\Gamma(\lambda+\nu)}\left[F^{4:2;4}_{4:3;3}\left[\begin{matrix}\triangle(2;\nu-\delta+\lambda),\triangle(2;\lambda+\nu+1):\\\triangle(2;\nu+\lambda),\triangle(2;\nu+\delta+\lambda+1):\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\triangle(2;(\beta+\nu;\varrho));\triangle(2;-n),\triangle(2;1+\varsigma+\tau+n);\\\triangle(2;\nu+1),(\nu+1);\triangle(2;1+\varsigma),\frac{1}{2};\end{matrix}\quad\frac{\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]$$

$$-\frac{n(\nu+\lambda+1)(\nu-\delta+\lambda)(1+\varsigma+\tau+n)}{(\nu+\lambda)(1+\varsigma)(\nu+\delta+\lambda+1)}\left(\frac{\zeta}{2h}\right)$$

$$\times F^{4:2;4}_{4:3;3}\left[\begin{matrix}\triangle(2;2+\nu+\lambda),\triangle(2;1+\nu+\lambda-\delta):\triangle(2;(\beta+\nu;\varrho));\triangle(2;1-n),\\\triangle(2;\nu+\lambda+1),\triangle(2;\nu+\delta+\lambda+2):\triangle(2;\nu+1),(\nu+1);\end{matrix}\right.$$

$$\left.\left.\begin{matrix}\triangle[2;2+\varsigma+\tau+n];\\\triangle(2;2+\varsigma),\frac{3}{2};\end{matrix}\quad\frac{\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]\right]. \tag{2.11}$$

On assuming $\alpha=-1$ in Theorem 2.2 the four-parameter Bessel function $G_\nu^{(\alpha,\beta)}(x;\varrho)$ reduced to the three-parameter Bessel function of the second kind $I_\nu^{(\beta)}(x;\varrho)$ and by using integral presented in Theorem 2.2, we can deduce the following Corollary 2.4.

Corollary 2.4. *For* $\mathfrak{Re}(\lambda+\nu) > \mathfrak{Re}(\delta) > 0, \mathfrak{Re}(\varrho) > 0$ and $\mathfrak{Re}(\nu) > -1$, *the following integral formula holds true:*

$$\int_0^\infty x^{\delta-1}\left(x+h+\sqrt{x^2+2hx}\right)^{-\lambda} I_\nu^{(\beta)}\left(\frac{\xi}{x+h+\sqrt{x^2+2hx}};\varrho\right)$$

$$\times P_n^{(\varsigma,\tau)}\left(1-\frac{\zeta}{x+h+\sqrt{x^2+2hx}}\right)dx$$

$$= \frac{2^{1-\delta-\nu}h^{\delta-\lambda-\nu}\xi^\nu(\beta)_\nu\Gamma(2\delta)\Gamma(\nu+\lambda+1)\Gamma(\nu-\delta+\lambda)}{\Gamma^2(\nu+1)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\delta)}$$

$$\times F^{5:0;0}_{6:0;1}\left[\begin{array}{l}(\nu+\lambda+1:2,3),(\nu-\delta+\lambda:2,3),((\beta+\nu;\varrho):2,2),(\varsigma+\tau+1:1,2)\\(\nu+\lambda:2,3),(\nu+\lambda+\delta+1:2,3),(\nu+1:2,2),(\nu+1:1,1),(1:1,1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\tau:1,1):\_;\_;\\(1+\varsigma+\tau:1,1):\_;(1+\varsigma:1);\end{array}\ \frac{\xi^2}{4h^2},-\frac{\xi^2\zeta}{8h^3}\right]. \quad (2.12)$$

3. Relationship between Kampé de Fériet and Srivastava and Daoust function

In this section, we gives an interesting connection between Kampé de Fériet and Srivastava and Daoust function by the comparing (2.1) and (2.5).

$$F^{4:2;4}_{4:3;3}\left[\begin{array}{l}\triangle(2;\lambda-\delta+\nu),\triangle(2;\lambda+\nu+1):\triangle(2;(\beta+\nu;\varrho));\triangle(2;\varsigma+\tau+n+1),\\\triangle(2;\nu+\delta+\lambda+1),\triangle(2;\nu+\lambda):\triangle(2;\nu+1),(1+\nu);\triangle(2;1+\varsigma),\end{array}\right.$$

$$\left.\begin{array}{l}\triangle(2;-n);\\\frac{1}{2};\end{array}\ -\frac{\alpha\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]-\frac{n(\nu+\lambda+1)(\lambda+\nu-\delta)(1+\varsigma+\tau+n)}{(\nu+\lambda)(1+\varsigma)(1+\nu+\lambda+\delta)}\left(\frac{\zeta}{2h}\right)$$

$$\times F^{4:2;4}_{4:3;3}\left[\begin{array}{l}\triangle(2;\nu+\lambda+2),\triangle(2;1+\lambda-\delta+\nu):\triangle(2;(\beta+\nu;\varrho));\triangle(2;1-n),\\\triangle(2;\lambda+1+\nu),\triangle(2;2+\lambda+\nu+\delta):\triangle(2;\nu+1),(\nu+1);\end{array}\right.$$

$$\left.\begin{array}{l}\triangle[2;2+\varsigma+\tau+n];\\\triangle(2;2+\varsigma),\frac{3}{2};\end{array}\ -\frac{\alpha\xi^2}{4h^2},\frac{\zeta^2}{16h^2}\right]=\frac{n!}{(1+\varsigma)_n}$$

$$\times F^{5:0;0}_{6:0;1}\left[\begin{array}{l}(\nu-\delta+\lambda:2,3),(\nu+\lambda+1:2,3),((\beta+\nu;\varrho):2,2),(1+\varsigma+\tau:1,2)\\(\nu+\lambda:2,3),(\nu+\delta+\lambda+1:2,3),(\nu+1:2,2),(1+\nu:1,1),(1:1,1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\tau:1,1):\_;\_;\\(1+\varsigma+\tau:1,1):\_;(1+\varsigma:1);\end{array}\ -\frac{\alpha\xi^2}{4h^2},\frac{\alpha\xi^2\zeta}{8h^3}\right]. \quad (3.1)$$

4. Special Cases

In this section, we present some of the well-known and interesting special cases which can be determined by specializing the parameters of the Corollaries 2.1 and 2.2.

(i) Substituting $\beta = 1, \varrho = 0$ and $P_n^{(0,0)}(1) = 1$, Corollary 2.1 reduces in to [5, Theorem 1, p. 3, eq. (2.1)] investigated by Choi and Agarwal.

(ii) Taking $\beta = 1, \varrho = 0$ and $P_n^{(0,0)}(1) = 1$, Corollary 2.2 reduces to another result due to Choi and Agarwal [5, Theorem 2, p. 3, eq. (2.2)].

5. Special Cases

In the current investigation by the applications of the Oberhettinger integral formula we have established some of the results involving the four-parameter Bessel function and Jacobi polynomial whose outcomes are expressed in terms of the Kampé de Fériet and Srivastava and Daoust functions and we drive an interesting relationship between Kampé de Fériet and Srivastava and Daoust functions from our main results. Also, the Bessel function of the first kind is a special case of Fox H-function [5, p. 9, eq. (4.1)]. Consequently, all the result of this paper can easily converted in terms of the Fox H-function for the appropriate settings of parameters. We can find some other results in terms of the Kampé de Fériet and Srivastava and Daoust functions for the proper settings of parameters in the Jacobi polynomial.

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