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CERTAIN RESULTS ON CONTINUED FRACTIONS

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Abstract: In this paper certain results on Ramanujan's continued fractions have been discussed.

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1. Introduction, Notations and Definitions

The subject of continued fractions (CF) is an old subject although many people are not aware of it. Actually, continued fractions have so many applications in algebra and in various fields such as mathematics, physics, and chemistry. The easiest way of forming a continued fraction is by writing a certain amount in the form of a numerator and a denominator, and each denominator is composed of a numerator and a denominator and so on. Usually, the successive numerators are equal to one.

Continued fractions have a long history; they were known since the appearance of Euclidean algorithm for finding the greatest common divisor (GCD) of two numbers. That was around the year 300 B.C.. Research works and papers continue then to be performed and a huge accumulation of applications arise; this is due to their simplicity to deal with and the smooth way of the calculations involved. We should add that the subject of continued fractions is still very fruitful and interesting for researchers [4, 5, 6, 9, 10] all over the globe.

One of the most celebrated continued fractions of Ramanujan is,

$$\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}=\frac{(a-b)}{(1-q)+}\frac{(a-bq)(aq-b)}{(1-q^3)+}\frac{q(a-bq^2)(aq^2-b)}{(1-q^5)+}\frac{q^2(a-bq^3)(aq^3-b)}{(1-q^7)+...}, \quad (1.1)$$

[Ramanujan S. 8; Entry 11, p. 195]

where $q-$rising factorial is defined as,

$$(a;q)_n=(1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \qquad n\in 1,2,3,...,$$

for $k-$ complex number,

$$(a;q)_k=\frac{(a;q)_\infty}{(aq^k;q)_\infty}, \quad (a;q)_0=1$$

and

$$(a;q)_\infty=\prod_{r=0}^{\infty}(1-aq^r)=\lim_{n\to\infty}(a;q)_n. \quad (1.2)$$

The $q-$binomial theorem is defined as,

$$\frac{(a;q)_\infty}{(b;q)_\infty}=\sum_{r=0}^{\infty}\frac{(a/b;q)_r}{(q;q)_r}b^r. \quad (1.3)$$

In this paper, we give proof of (1.1) and also deduce certain more results on continued fractions from (1.1).

Proof of (1.1).

In order to prove (1.1) we start by taking left hand side of (1.1). Dividing numerator and denominator of the left hand side of (1.1) by $(a;q)_\infty(-a;q)_\infty$ we have,

$$\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}=\frac{\dfrac{(b;q)_\infty}{(a;q)_\infty}-\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}{\dfrac{(b;q)_\infty}{(a;q)_\infty}+\dfrac{(-b;q)_\infty}{(-a;q)_\infty}}. \quad (1.4)$$

Now, applying (1.3) we get

$$=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n a^n}{(q;q)_n}-\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n(-a)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n a^n}{(q;q)_n}+\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n(-a)^n}{(q;q)_n}}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n a^n}{(q;q)_n}\{1-(-1)^n\}}{\displaystyle\sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n a^n}{(q;q)_n}\{1+(-1)^n\}},$$

taking n odd in numerator and even in denominator we have

$$= \frac{\sum_{n=0}^{\infty} \frac{\left(\frac{b}{a};q\right)_{2n+1} a^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty} \frac{\left(\frac{b}{a};q\right)_{2n} a^{2n}}{(q;q)_{2n}}} = \frac{\frac{(a-b)}{(1-q)} \sum_{n=0}^{\infty} \frac{\left(\frac{bq}{a};q\right)_{2n} a^{2n}}{(q^2;q)_{2n}}}{\sum_{n=0}^{\infty} \frac{\left(\frac{b}{a};q\right)_{2n} a^{2n}}{(q;q)_{2n}}}. \tag{1.5}$$

$$= \frac{\frac{(a-b)}{(1-q)} \sum_{n=0}^{\infty} \frac{\left(\frac{bq}{a};q^2\right)_n \left(\frac{bq^2}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n (q^3;q^2)_n}}{\sum_{n=0}^{\infty} \frac{\left(\frac{b}{a};q^2\right)_n \left(\frac{bq}{a};q^2\right)_n a^{2n}}{(q;q^2)_n (q^2;q^2)_n}}.$$

$$= \frac{\frac{(a-b)}{(1-q)}}{1 + \frac{\sum_{n=0}^{\infty} \frac{\left(\frac{bq}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n} \left\{ \frac{\left(\frac{b}{a};q^2\right)_n}{(q;q^2)_n} - \frac{(bq^2;q^2)_n}{(q^3;q^2)_n} \right\}}{\sum_{n=0}^{\infty} \frac{\left(\frac{bq}{a};q^2\right)_n \left(\frac{bq^2}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n (q^3;q^2)_n}}},$$

which on simplification gives,

$$= \frac{\frac{(a-b)}{(1-q)}}{1 + \frac{\frac{(aq-b)(a-bq)}{(1-q)(1-q^3)}}{\left\{ \frac{\sum_{n=0}^{\infty} \frac{\left(\frac{bq}{a};q^2\right)_n \left(\frac{bq^2}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n (q^3;q^2)_n}}{\sum_{n=0}^{\infty} \frac{\left(\frac{bq^2}{a};q^2\right)_n \left(\frac{bq^3}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n (q^5;q^2)_n}} \right\}}}. \tag{1.6}$$

$$= \cfrac{\frac{(a-b)}{(1-q)}}{1+\cfrac{\frac{(aq-b)(a-bq)}{(1-q)(1-q^3)}}{1+\cfrac{\sum_{n=0}^{\infty}\frac{\left(\frac{bq^2}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n}\left\{\frac{\left(\frac{bq}{a};q^2\right)_n}{(q^3;q^2)_n}-\frac{\left(\frac{bq^3}{a};q^2\right)_n}{(q^5;q^2)_n}\right\}}{\sum_{n=0}^{\infty}\frac{\left(\frac{bq^2}{a};q^2\right)_n\left(\frac{bq^3}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n(q^5;q^2)_n}}}}, \tag{1.7}$$

which on simplification gives,

$$= \cfrac{\frac{(a-b)}{(1-q)}}{1+\cfrac{\frac{(aq-b)(a-bq)}{(1-q)(1-q^3)}}{1+\cfrac{\frac{q(aq^2-b)(a-bq^2)}{(1-q^3)(1-q^5)}}{\left\{\cfrac{\sum_{n=0}^{\infty}\frac{\left(\frac{bq^2}{a};q^2\right)_n\left(\frac{bq^3}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n(q^5;q^2)_n}}{\sum_{n=0}^{\infty}\frac{\left(\frac{bq^3}{a};q^2\right)_n\left(\frac{bq^4}{a};q^2\right)_n a^{2n}}{(q^2;q^2)_n(q^7;q^2)_n}}\right\}}}}. \tag{1.8}$$

Iterating the process and applying [Jones W.B. and Thron W. J. 3; (2.3.14), p. 33] we get (1.1). Taking $b = 0$ in (1.1) we have

$$\frac{(-a;q)_\infty-(a;q)_\infty}{(-a;q)_n+(a;q)_n} = \frac{a}{(1-q)+}\frac{a^2q}{(1-q^3)+}\frac{a^2q^3}{(1-q^5)+}\frac{a^2q^5}{(1-q^7)+...}. \tag{1.9}$$

2. Another Results on Continued Fraction

Now, making use of componendo and dividendo rule, viz.,
If $\frac{A}{B} = \frac{C}{D}$ then $\frac{A+B}{A-B} = \frac{C+D}{C-D}$ on (1.1) we obtain,

$$\frac{(-a;q)_\infty(b;q)_\infty}{(a;q)_\infty(-b;q)_\infty} = 1 + \frac{2(a-b)}{(1-q-a+b)+}\frac{(a-bq)(aq-b)}{(1-q^3)+}\frac{q(a-bq^2)(aq^2-b)}{(1-q^5)+}\frac{q^2(a-bq^3)(aq^3-b)}{(1-q^7)+...}. \tag{2.1}$$

Taking $b = -a$ in (2.1) we have,

$$\frac{(-a;q)_\infty^2}{(a;q)_\infty^2} = 1 + \frac{2a^2}{(1-q-2a)+}\frac{a^2(1+q)^2}{(1-q^3)+}\frac{a^2q(1+q^2)^2}{(1-q^5)+}\frac{a^2q^2(1+q^3)^2}{(1-q^7)+...}. \tag{2.2}$$

For $a = -q$, (2.2) yields

$$(q;q)_\infty^2(q;q^2)_\infty^2 = 1 + \frac{2q^2}{(1+q)+}\frac{q^2(1+q)^2}{(1-q^3)+}\frac{q^3(1+q^2)^2}{(1-q^5)+}\frac{q^4(1+q^3)^2}{(1-q^7)+...}. \tag{2.3}$$

Taking $b = 0$ in (2.1) we have,

$$\frac{(-a;q)_\infty}{(a;q)_\infty} = 1 + \frac{2a}{(1-q-a)+}\frac{a^2q}{(1-q^3)+}\frac{a^2q^3}{(1-q^5)+}\frac{a^2q^5}{(1-q^7)+...}. \tag{2.4}$$

Comparing (2.2) and (2.4) we find,

$$\left(1 + \frac{2a}{(1-q-a)+}\frac{a^2q}{(1-q^3)+}\frac{a^2q^3}{(1-q^5)+}\frac{a^2q^5}{(1-q^7)+...}\right)^2 = 1 + \frac{2a^2}{(1-q-2a)+}\frac{a^2(1+q)^2}{(1-q^3)+}\frac{a^2q(1+q^2)^2}{(1-q^5)+}\frac{a^2q^2(1+q^3)^2}{(1-q^7)+...} \tag{2.5}$$

Putting q^2 for q in (2.1) we get,

$$\frac{(-a;q^2)_\infty(b;q^2)_\infty}{(a;q^2)_\infty(-b;q^2)_\infty} = 1 + \frac{2(a-b)}{(1-q^2-a+b)+}\frac{(a-bq^2)(aq^2-b)}{(1-q^6)+}\frac{q^2(a-bq^4)(aq^4-b)}{(1-q^{10})+}\frac{q^4(a-bq^6)(aq^6-b)}{(1-q^{14})+...}. \tag{2.6}$$

Taking $a = q$ and $b = q^2$ in (2.6) and comparing with [Andrews G.E. and Berndt B.C. 1; (15.2.2), p. 328] we get,

$$\Phi(q) = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} = 1 + \frac{2q(1-q)}{(1-q)+}\frac{q^3(1-q^3)(q-1)}{(1-q^6)+}\frac{q^5(1-q^5)(q^3-1)}{(1-q^{10})+}\frac{q^7(1-q^7)(q^5-1)}{(1-q^{14})+...}, \tag{2.7}$$

where $\Phi(q)$ is Ramanujan's theta function defined as,

$$\Phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Putting $a = -q$ in (2.4) we get

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} = 1 - \frac{2q}{1+} \frac{q^3}{(1-q^3)+} \frac{q^5}{(1-q^5)+} \frac{q^7}{(1-q^7)+...}, \tag{2.8}$$

where $\theta_4(q)$ is Jacobi's fourth theta function [Rainville E.D. 7; chapetr 20].

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