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FRACTIONAL INTEGRATIONS FOR THE NEW GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract: In this article, authors discussed about image formulas for the some new extended hypergeometric function using elementary properties of the fractional calculus integral operators. Furthermore, using integral transforms some image formulas are also established.

Keywords and Phrases: Beta function, SUM transform, Fox-Wright function, Saigo fractional operator, Pathway transform.

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1. Introduction and Definitions

Fractional calculus is a branch of mathematics that deals with the study of fractional order integrals and derivatives. Many real-life applications of fractional calculus can be found in earthquake prediction, mathematical bio-science, electronic system design, image processing and optics, interested reader can refer to [12]. Saigo [11] studied the following fractional integral operators with Gauss hypergeoemtric kernel:

$$\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu}f\right)(x)=\frac{x^{-\kappa-\lambda}}{\Gamma(\kappa)}\int_0^x(x-t)^{\kappa-1}{}_2F_1\left(\kappa+\lambda,-\mu;\kappa;1-\frac{t}{x}\right)f(t)dt, \tag{1}$$

and

$$\left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu}f\right)(x)=\frac{1}{\Gamma(\kappa)}\int_x^\infty(t-x)^{\kappa-1}t^{-\kappa-\lambda}{}_2F_1\left(\kappa+\lambda,-\mu;\kappa;1-\frac{x}{t}\right)f(t)dt. \tag{2}$$

where $\kappa,\lambda,\mu\in\mathbb{C}$, $x\in\mathbb{R}^+$ and ${}_2F_1(;)$ is the Gauss hypergeometric function defined in Rainville [10], as:

$${}_2F_1(\eta,\rho;\tau;z)=\sum_{n=0}^{\infty}\frac{(\eta)_n(\rho)_n}{(\tau)_n}\frac{z^n}{n!}\qquad(\eta,\rho,\tau,z\in\mathbb{C})$$

If $\lambda=-\kappa$, then (1) and (2) reduces to the Riemann-Liouville and Weyl type fractional integral operators as follows (Agarwal and Choi [1]):

$$\left(\mathcal{R}_{0,x}^{\kappa}f\right)(x)=\left(\mathcal{I}_{0,x}^{\kappa,-\kappa,\mu}f\right)(x)=\frac{1}{\Gamma(\kappa)}\int_0^x(x-t)^{\kappa-1}f(t)dt,$$

and

$$\left(\mathcal{W}_{x,\infty}^{\kappa}f\right)(x)=\left(\mathcal{I}_{x,\infty}^{\kappa,-\kappa,\mu}f\right)(x)=\frac{1}{\Gamma(\kappa)}\int_x^\infty(t-x)^{\kappa-1}f(t)dt.$$

If $\lambda=0$, then (1) and (2) reduces to the following Erdelyi-Kober fractional integral operators as follows (Agarwal and Choi [1]):

$$\left(\mathcal{E}_{0,x}^{\kappa,\mu}f\right)(x)=\left(\mathcal{I}_{0,x}^{\kappa,0,\mu}f\right)(x)=\frac{x^{-\kappa-\mu}}{\Gamma(\kappa)}\int_0^x(x-t)^{\kappa-1}t^{\mu}f(t)dt,$$

and

$$\left(\mathcal{K}_{x,\infty}^{\kappa,\mu}f\right)(x)=\left(\mathcal{I}_{x,\infty}^{\kappa,0,\mu}f\right)(x)=\frac{x^{\mu}}{\Gamma(\kappa)}\int_x^\infty(t-x)^{\kappa-1}t^{-\kappa-\mu}f(t)dt$$

The following new extended beta function is investigated by Kaurangini *et al* [5].

$$\begin{aligned}{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\Omega,\mho)&={}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}\left[\begin{array}{c|c|c}(r_k,R_k)_{1,h}&(\tau_i,T_i)_{1,\varepsilon}&\\ &&\Omega,\mho\\ (d_\iota,D_\iota)_{1,g}&(\ell_j,L_j)_{1,\zeta}&\end{array}\right]\\&=\int_0^1(1-t)^{\mho-1}t^{\Omega-1}{}_{\varepsilon}\Psi_{\zeta}\left(\frac{-\Im}{(1-t)^{\Upsilon}}\right){}_{h}\Psi_{g}\left(\frac{-\wp}{t^{\Lambda}}\right)dt,\end{aligned} \tag{3}$$

where $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Omega), Re(\mho)\} > 0$, and ${}_h\Psi_g$ is the generalized Wright function defined for $z \in \mathbb{C}$, complex r_k, d_ι and $R_k, D_\iota \in \mathbb{R}$ $(k = 1, 2, 3 \ldots, h; \iota = 1, 2, 3, \ldots, g)$ by the series in Kilbas and Srivastava [6].

$$
{}_h\Psi_g(z) = {}_h\Psi_g \left[\begin{array}{c|c} (r_k, R_k)_{1,h} & \\ & z \\ (d_\iota, D_\iota)_{1,g} & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{h} \Gamma(r_k + nR_k)}{\prod_{\iota}^{g} \Gamma(d_\iota + nD_\iota)} \frac{z^n}{n!} \tag{4}
$$

They also (Kaurangini *et al.* [5]) studied the following hypergeometric function:

$$
{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi) = {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon} \left[\begin{array}{c|c|c} (r_k, R_k)_{1,h} & (\tau_i, T_i)_{1,\varepsilon} & \\ & & \upsilon, \phi; \varphi; z \\ (d_\iota, D_\iota)_{1,g} & (\ell_j, L_j)_{1,\zeta} & \end{array} \right]
$$

$$
= \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi + n, \varphi - \phi)}{B(\phi, \varphi - \phi)} \frac{z^n}{n!}, \tag{5}
$$

where $Re(\varphi) > Re(\phi) > 0$ and $|z| < 1$; and

$$
{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi; \varphi) = {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon} \left[\begin{array}{c|c|c} (r_k, R_k)_{1,h} & (\tau_i, T_i)_{1,\varepsilon} & \\ & & \phi; \varphi; z \\ (d_\iota, D_\iota)_{1,g} & (\ell_j, L_j)_{1,\zeta} & \end{array} \right]
$$

$$
= \sum_{n=0}^{\infty} \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi + n, \varphi - \phi)}{B(\phi, \varphi - \phi)} \frac{z^n}{n!}, \tag{6}
$$

where $Re(\varphi) > Re(\phi) > 0$.
The main purpose of this article is to apply the Saigo's fractional integral operators in (1) and (2) to the newly introduced Gauss and confluent hypergeoetric functions in (5) and (6). Furthermore, some new images formulas are also obtain by applying integral transforms to the obtained fractional integral operators.

2. Fractional Integration of the New Extended Hypergeometric Function

2.1. The Left-sided Saigo Fractional Integral Operator

In this section the following power function formulas are required:

Lemma 1. [1]:

$$
\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1} \right)(x) = \frac{\Gamma(\tau)\Gamma(\tau + \mu - \lambda)}{\Gamma(\tau - \lambda)\Gamma(\tau + \mu + \kappa)} x^{\tau-\lambda-1}, \tag{7}
$$

where $\kappa, \lambda, \mu, \tau \in \mathbb{C}$ with $Re(\kappa) > 0$ and $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$.

Lemma 2. [1]:

$$\left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\right)(x) = \frac{\Gamma(\tau)}{\Gamma(\tau+\kappa)} x^{\tau+\kappa-1}, \tag{8}$$

where $\kappa, \tau \in \mathbb{C}$ with $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$.

Lemma 3. [1]:

$$\left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1}\right)(x) = \frac{\Gamma(\tau+\mu)}{\Gamma(\tau+\kappa+\mu)} x^{\tau-1}, \tag{9}$$

where $\kappa, \mu, \tau \in \mathbb{C}$ with $Re(\tau + \mu) > 0$.
The following definition is also needed in this section.

Definition 4: *Pohlen in [8] defined the following Hadamard convolution (product) for the two power series $h(z) = a_n z^n$ $(|z| < R_h)$ and $k(z) = b_n z^n$ $(|z| < R_k)$, where R_h and R_k are the radii of convergence defined by*

$$(h * k)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (k * h)(z) \quad (R_h R_k \leq R) \tag{10}$$

Theorem 5. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following holds:*

$$\begin{aligned}&\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi; zt^{\ell})\right)(x) \\ &= x^{\tau-\lambda-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi; zx^{\ell}) * {}_3\Psi_2\left[\begin{array}{c|c}(\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1) & \\ & zx^{\ell} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell) & \end{array}\right]\end{aligned} \tag{11}$$

Proof. Let F_1 be the left-hand side of (11), using (5) and changing the order of summation and integral operator leads us to

$$F_1 = \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n, \varphi-\phi)}{B(\phi, \varphi-\phi)} \frac{z^n}{n!} \left(I_{0,x}^{\kappa,\lambda,\mu} t^{\tau+n\ell-1}\right)(x) \tag{12}$$

Applying (7) to (17) and simplifying, yields

$$F_1 = \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n, \varphi-\phi)}{B(\phi, \varphi-\phi) n!} \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell) n!} (x^{\ell} z)^n \tag{13}$$

Using the Hadamard convolution (product) in (10) and Fox-Wright function in (4) to (13) the required result in (11) is obtained.

Corollary 6. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following result is valid:*

$$\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^{\ell})\right)(x) = x^{\tau-\lambda-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zx^{\ell}) * {}_3\Psi_2 \left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1) & \\ & zx^{\ell} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell) & \end{array} \right]$$

Corollary 7. *Let $x > 0$, $\kappa, \lambda, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ and $Re(\kappa) > 0$ be such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$\left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x) = x^{\tau+\kappa-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\ell}) * {}_2\Psi_1 \left[\begin{array}{c|c} (\tau,\ell),(1,1) & \\ & zx^{\ell} \\ (\tau+\kappa,\ell) & \end{array} \right]$$

Corollary 8. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ and $Re(\kappa) > 0$ be such that $Re(\tau + \mu) > 0$. Then, the following holds:*

$$\left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x) = x^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\ell}) * {}_2\Psi_1 \left[\begin{array}{c|c} (\tau+\mu,\ell),(1,1) & \\ & zx^{\ell} \\ (\tau+\kappa+\mu,\ell) & \end{array} \right]$$

2.2. The Right-sided Saigo Fractional Integral Operator

Lemma 9. [1]:

$$\left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}\right)(x) = \frac{\Gamma(1-\tau+\lambda)\Gamma(1-\tau+\mu)}{\Gamma(1-\tau)\Gamma(1-\tau+\kappa+\lambda+\mu)} x^{\tau-\lambda-1}, \tag{14}$$

where $\kappa, \lambda, \mu, \tau \in \mathbb{C}$ with $Re(\kappa) > 0$ and $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$.

Lemma 10. [1]:

$$\left(\mathcal{W}_{x,\infty}^{\kappa} t^{\tau-1}\right)(x) = \frac{\Gamma(1-\tau-\kappa)}{\Gamma(1-\tau)} x^{\tau+\kappa-1}, \tag{15}$$

where $\kappa, \tau \in \mathbb{C}$ *with* $Re(\tau) < 1 + Re(\kappa)$.

Lemma 11. [1]:

$$\left(\mathcal{K}^{\kappa,\mu}_{x,\infty} t^{\tau-1}\right)(x) = \frac{\Gamma(1-\tau+\mu)}{\Gamma(1-\tau+\kappa+\mu)} x^{\tau-1}, \tag{16}$$

where $\kappa, \mu, \tau \in \mathbb{C}$ *with* $Re(\mu) > Re(\tau) > -1$.

Theorem 12. *Let* $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ *and* $Re(\kappa) > 0$ *be such that* $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. *Then, the following holds:*

$$\begin{aligned}&\left(\mathcal{J}^{\kappa,\lambda,\mu}_{x,\infty} t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\\ &= x^{\tau-\lambda-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zx^{-\ell}) * {}_3\Psi_2\left[\begin{array}{c|c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1) & \\ & zx^{-\ell}\\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell) & \end{array}\right]\end{aligned} \tag{17}$$

Proof. The proof of (17) follows directly from Theorem 5 and equation (14).

Corollary 13. *Let* $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ *and* $Re(\kappa) > 0$ *be such that* $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. *Then, the following holds:*

$$\begin{aligned}&\left(\mathcal{J}^{\kappa,\lambda,\mu}_{x,\infty} t^{\tau-1}\ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;zt^{-\ell})\right)(x)\\ &= x^{\tau-\lambda-1}\ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;zx^{-\ell}) * {}_3\Psi_2\left[\begin{array}{c|c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1) & \\ & zx^{-\ell}\\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell) & \end{array}\right]\end{aligned}$$

Corollary 14. *Let* $x > 0$, $\kappa, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$ *and* $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *be such that*$Re(\tau) < 1 + Re(\kappa) >$. *Then, the following holds:*

$$\begin{aligned}&\left(\mathcal{W}^{\kappa}_{x,\infty} t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\\ &= x^{\tau+\kappa-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zx^{-\ell}) * {}_2\Psi_1\left[\begin{array}{c|c}(1-\kappa-\tau,\ell),(1,1) & \\ & zx^{-\ell}\\ (1-\tau,\ell) & \end{array}\right]\end{aligned}$$

Corollary 15. *Let $x > 0$, $\kappa, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ and $Re(\kappa) > 0$ be such that $Re(\mu) > Re(\tau) > -1$. Then, the following holds:*

$$\left(\mathcal{K}^{\kappa,\mu}_{x,\infty} t^{\tau-1\,\Psi} F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon, \phi; \varphi; zt^{-\ell})\right)(x)$$
$$= x^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon, \phi; \varphi; zx^{-\ell}) * {}_2\Psi_1 \left[\begin{array}{c|c} (\mu-\tau+1,\ell), (1,1) & \\ & zx^{-\ell} \\ (\kappa+\mu-\tau+1,\ell) & \end{array} \right].$$

3. Integral Transforms of Fractional Integral Operator with the New Extended Hypergeometric Function

3.1. Euler-beta Transform for the Fractional Integral and Derivative Operators

Definition 16. *The Beta transform of $f(z)$ is defined in [8], as:*

$$\mathcal{B}\{f(z); h, m\} = \int_0^1 f(z) z^{h-1} (1-z)^{m-1} dz \tag{18}$$

Theorem 17. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, be such that $Re(\tau) > \max\{0, Re(\lambda-\mu)\}$. Then, the following holds:*

$$\mathcal{B}\left\{\left(\mathcal{I}^{\kappa,\lambda,\mu}_{0,x} t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon, \phi; \varphi; zt^{\ell})\right)(x); h, m\right\} = x^{\tau-\lambda-1}\Gamma(m)\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon, \phi; \varphi; zx^{\ell})$$
$$* {}_4\Psi_3 \left[\begin{array}{c|c} (\tau,\ell), (\tau+\mu-\lambda,\ell), (h,1), (1,1) & \\ & x^{\ell} \\ (\tau-\lambda,\ell), (\tau+\kappa+\mu,\ell), (h+m,1) & \end{array} \right] \tag{19}$$

Proof. Let E be the left-hand side of (19), using (11), (18) and changing the order of summation and integral leads us to

$$E = \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\phi+n, \varphi-\phi)}{B(\phi, \varphi-\phi) n!} \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell) n!} x^{n\ell}$$
$$\times \int_0^1 z^{h+n-1}(1-z)^{m-1} dz \tag{20}$$

Simplifying (20), gives

$$E = \Gamma(m) \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B^{\Lambda,\Upsilon}_{\wp,\Im}(\phi+n, \varphi-\phi)}{B(\phi, \varphi-\phi) n!} \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(h+n)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)\Gamma(h+m+n) n!} x^{\ell} \tag{21}$$

By applying the Hadamard convolution (product) in (10) and Fox-Wright function in (4) to (21) the desired result in (19) is obtained.

Corollary 18. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following result is valid:*

$$\mathcal{B}\left\{\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^{\ell})\right)(x);h,m\right\}$$
$$= x^{\tau-\lambda-1}\Gamma(m)\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zx^{\ell}) * {}_4\Psi_3\left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(h,1),(1,1) & \\ & x^{\ell} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(h+m,1) & \end{array}\right]$$

Corollary 19. *Let $x > 0$, $\kappa, \lambda, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$\mathcal{B}\left\{\left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x);h,m\right\}$$
$$= x^{\tau+\mu-1}\Gamma(m)\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\ell}) * {}_3\Psi_2\left[\begin{array}{c|c} (\tau,\ell),(h,1),(1,1) & \\ & x^{\ell} \\ (\tau+\mu,\ell),(h+m,1) & \end{array}\right]$$

Corollary 20. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau+\mu) > 0$. Then, the following holds:*

$$\mathcal{B}\left\{\left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x);h,m\right\}$$
$$= x^{\tau-1}\Gamma(m)\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{\ell}) * {}_3\Psi_2\left[\begin{array}{c|c} (\tau+\mu,\ell),(h,1),(1,1) & \\ & x^{\ell} \\ (\tau+\kappa+\mu,\ell),(h+m,1) & \end{array}\right]$$

Theorem 21. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. Then, the following holds:*

$$\mathcal{B}\left\{\left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x);h,m\right\} = x^{\tau-\lambda-1}\Gamma(m)\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zx^{-\ell})$$
$$* {}_4\Psi_3\left[\begin{array}{c|c} (\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(h,1),(1,1) & \\ & x^{-\ell} \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),(h+m,1) & \end{array}\right] \tag{22}$$

Proof. Equation (22) follows directly from (19) and (17).

Corollary 22. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. Then, the following holds:*

$$\begin{aligned}&\mathcal{B}\left\{\left(\mathcal{J}^{\kappa,\lambda,\mu}_{x,\infty}t^{\tau-1}\ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;zt^{-\ell})\right)(x);h,m\right\}\\ &= x^{\tau-\lambda-1}\Gamma(m)\ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;zx^{-\ell})\\ &* {}_4\Psi_3\left[\begin{array}{c|c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(h,1),(1,1) & \\ & x^{-\ell}\\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),(h+m,1) & \end{array}\right]\end{aligned}$$

Corollary 23. *Let $x > 0$, $\kappa, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, be such that $Re(\tau) < 1 + Re(\kappa) >$. Then, the following holds:*

$$\begin{aligned}&\mathcal{B}\left\{\left(\mathcal{W}^{\kappa}_{x,\infty}t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x);h,m\right\}\\ &= x^{\tau+\kappa-1}\Gamma(m)\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zx^{-\ell}) * {}_3\Psi_2\left[\begin{array}{c|c}(\kappa-\tau+1,\ell),(h,1),(1,1) & \\ & x^{-\ell}\\ (1-\tau,\ell),(h+m,1) & \end{array}\right]\end{aligned}$$

Corollary 24. *Let $x > 0$, $\kappa, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\mu) > Re(\tau) > -1$. Then, the following holds:*

$$\begin{aligned}&\mathcal{B}\left\{\left(\mathcal{K}^{\kappa,\mu}_{x,\infty}t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x);h,m\right\}\\ &= x^{\tau-1}\Gamma(m)\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zx^{-\ell}) * {}_3\Psi_2\left[\begin{array}{c|c}(\mu-\tau+1,\ell),(h,1),(1,1) & \\ & x^{-\ell}\\ (\kappa+\mu-\tau+1,\ell),(h+m,1) & \end{array}\right]\end{aligned}$$

3.2. SUM Transform for the Fractional Integral and Derivative Operators

Definition 25. *The SUM transform is defined by Hasan et al. [3] by*

$$S_a\{f(z)\}_{(s)} = \frac{1}{s^r}\int_0^\infty f(z)a^{-st}dt, \tag{23}$$

where $t \geq 0$, $r \in \mathbb{Z}$, $a \in (0,\infty)\setminus\{1\}$, $m_1 \leq s \leq m_2$ and $m_1, m_2 > 0$.

Theorem 26. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following holds:*

$$\begin{aligned} &S_a\left\{z^{k-1}\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi; zt^{\ell})\right)(x)\right\} \\ &= \frac{x^{\tau-\lambda-1}}{s^r[slog(a)]^k}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon, \phi; \varphi; \frac{zx^{\ell}}{[slog(a)]}\right) \\ &* {}_4\Psi_2\left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(k,1),(1,1) & \\ & \frac{x^{\ell}}{[slog(a)]} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(k+m,1) & \end{array}\right] \end{aligned} \tag{24}$$

Proof. Let S be the left-hand side of (24), using (11), (23) and changing the order of summation and integral leads us to

$$S = \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi + n, \varphi - \phi)}{B(\phi, \varphi - \phi) n!} \frac{\Gamma(\tau + n\ell)\Gamma(\tau + \mu - \lambda + n\ell)\Gamma(1 + n)}{\Gamma(\tau - \lambda + n\ell)\Gamma(\tau + \kappa + \mu + n\ell) n!} x^{\ell} \times \left\{\frac{1}{s^r}\int_0^{\infty} a^{-sz} z^{k+n-1} dz\right\} \tag{25}$$

Simplifying (25), gives

$$S = \frac{x^{\tau-\lambda-1}}{s^r[slog(a)]^k} \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi + n, \varphi - \phi)}{B(\phi, \varphi - \phi) n!} \times \frac{\Gamma(\tau + n\ell)\Gamma(\tau + \mu - \lambda + n\ell)\Gamma(k + n)\Gamma(1 + n)}{\Gamma(\tau - \lambda + n\ell)\Gamma(\tau + \kappa + \mu + n\ell) n!} \left(\frac{x^{\ell}}{[slog(a)]}\right)^n \tag{26}$$

By applying the Hadamard convolution (product) in (10) and Fox-Wright function in (4) to (26) the desired result in (24) is obtained.

Corollary 27. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following result is valid:*

$$\begin{aligned} &S_a\left\{z^{k-1}\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi; \varphi; zt^{\ell})\right)(x)\right\} = \frac{x^{\tau+\kappa-1}}{s^r[slog(a)]^k} \\ &\times {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi; \varphi; \frac{zx^{\ell}}{[slog(a)]}\right) * {}_4\Psi_2\left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(k,1),(1,1) & \\ & \frac{x^{\ell}}{[slog(a)]} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(k+m,1) & \end{array}\right] \end{aligned}$$

Corollary 28. *Let $x > 0$, $\kappa, \lambda, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$S_a\left\{z^{k-1}\left(\mathcal{R}^{\kappa}_{0,x}t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\}$$
$$= \frac{x^{\tau+\kappa-1}}{s^r[slog(a)]^k}\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{zx^{\ell}}{[slog(a)]}\right) * {}_3\Psi_1\left[\begin{array}{c|c} (\tau,\ell),(k,1),(1,1) & \\ & \frac{x^{\ell}}{[slog(a)]} \\ (\tau+\kappa,\ell) & \end{array}\right]$$

Corollary 29. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau + \mu) > 0$. Then, the following holds:*

$$S_a\left\{z^{k-1}\left(\mathcal{E}^{\kappa,\mu}_{0,x}t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\}$$
$$= \frac{x^{\tau-1}}{s^r[slog(a)]^k}\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{zx^{\ell}}{[slog(a)]}\right) * {}_3\Psi_1\left[\begin{array}{c|c} (\tau+\mu,\ell),(k,1),(1,1) & \\ & \frac{x^{\ell}}{[slog(a)]} \\ (\tau+\kappa+\mu,\ell) & \end{array}\right]$$

Theorem 30. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. Then, the following holds:*

$$\begin{aligned} &S_a\left\{z^{k-1}\left(\mathcal{J}^{\kappa,\lambda,\mu}_{x,\infty}t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\} \\ &= \frac{x^{\tau-\lambda-1}}{s^r[slog(a)]^k}\,{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{zx^{-\ell}}{[slog(a)]}\right) \\ &* {}_4\Psi_2\left[\begin{array}{c|c} (\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(k,1),(1,1) & \\ & \frac{x^{-\ell}}{[slog(a)]} \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell) & \end{array}\right] \end{aligned} \tag{27}$$

Proof. Theorem 30 follows from Theorem 26.

Corollary 31. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be* *such that* $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. *Then, the following holds:*

$$S_a\left\{z^{k-1}\left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$
$$= \frac{x^{\tau-\lambda-1}}{s^r[slog(a)]^k}\,{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi;\varphi;\frac{zx^{-\ell}}{[slog(a)]}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(k,1),(1,1)\\ \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell)\end{array}\middle|\ \frac{x^{-\ell}}{[slog(a)]}\right]$$

Corollary 32. *Let* $x > 0$, $\kappa,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *be such that* $Re(\tau) < 1 + Re(\kappa) >$. *Then, the following holds:*

$$S_a\left\{z^{k-1}\left(\mathcal{W}_{x,\infty}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$
$$= \frac{x^{\tau+\kappa-1}}{s^r[slog(a)]^k}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{zx^{-\ell}}{[slog(a)]}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c}(1-\tau-\kappa,\ell),(k,1),(1,1)\\ \\ (1-\tau,\ell)\end{array}\middle|\ \frac{x^{-\ell}}{[slog(a)]}\right]$$

Corollary 33. *Let* $x > 0$, $\kappa,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *and* $Re(\kappa) > 0$ *be such that* $Re(\mu) > Re(\tau) > -1$. *Then, the following holds:*

$$S_a\left\{z^{k-1}\left(\mathcal{K}_{x,\infty}^{\kappa,\mu} t^{\tau-1\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$
$$= \frac{x^{\tau-1}}{s^r[slog(a)]^k}\,{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{zx^{-\ell}}{[slog(a)]}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c}(\mu-\tau+1,\ell),(k,1),(1,1)\\ \\ (\kappa+\mu-\tau+1,\ell)\end{array}\middle|\ \frac{x^{-\ell}}{[slog(a)]}\right]$$

If $a = e$ and $r = 0$, then (24) and (27) reduce to the classical Laplace transform as follows:

Corollary 34. *Let* $x > 0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *and* $Re(\kappa) > 0$ *be* *such that $Re(\tau) > \max\{0, Re(\lambda-\mu)\}$. Then, the following holds:*

$$\mathcal{L}\left\{z^{k-1}\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu}t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\}$$

$$= \frac{x^{\tau-\lambda-1}}{s^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{zx^{\ell}}{s}\right) * {}_4\Psi_2\left[\begin{array}{c}(\tau,\ell),(\tau+\mu-\lambda,\ell),(k,1),(1,1)\\ \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(k+m,1)\end{array}\middle|\frac{x^{\ell}}{s}\right]$$

Corollary 35. *Let $x>0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon\in\mathbb{C}$, $Re(\upsilon)>0$, $Re(\varphi)>Re(\phi)>0$, $Re(\ell)>0$, $\min\{Re(\wp),Re(\Im)\}>0$, $\min\{Re(\Lambda),Re(\Upsilon)\}>0$, and $Re(\kappa)>0$ be such that $Re(\tau)<1+\min\{Re(\lambda),Re(\mu)\}$. Then, the following holds:*

$$\mathcal{L}\left\{z^{k-1}\left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu}t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$

$$= \frac{x^{\tau-\lambda-1}}{s^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{zx^{-\ell}}{s}\right)$$

$$* {}_4\Psi_2\left[\begin{array}{c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(k,1),(1,1)\\ \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell)\end{array}\middle|\frac{x^{-\ell}}{s}\right]$$

3.3. Whittaker Transform for the Fractional Integral and Derivative Operators

Theorem 36. *Let $x>0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon\in\mathbb{C}$, $Re(\upsilon)>0$, $Re(\varphi)>Re(\phi)>0$, $Re(\ell)>0$, $\min\{Re(\wp),Re(\Im)\}>0$, $\min\{Re(\Lambda),Re(\Upsilon)\}>0$, and $Re(\kappa)>0$ be such that $Re(\tau)>\max\{0,Re(\lambda-\mu)\}$. Then, the following holds:*

$$\int_0^{\infty} z^k\exp\left(\frac{-mz}{2}\right)W_{\varrho,\rho}(mz)\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu}t^{\tau-1\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)dz$$

$$= \frac{x^{\tau-\lambda-1}}{m^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right)$$

$$* {}_5\Psi_3\left[\begin{array}{c}(\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1)\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k+n,1\right)\\ \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(1-\varrho+k,1)\end{array}\middle|\frac{x^{\ell}}{m}\right] \quad (28)$$

Proof. For simplicity, let W be the left-hand side of (28), using (11) and changing the order of summation and integral leads us to

$$W = \sum_{n=0}^{\infty}(\upsilon)_n\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!}\frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)n!}x^{\ell}$$

$$\times\int_0^{\infty} z^{k+n-1}\exp\left(\frac{-mz}{2}\right)W_{\varrho,\rho}(mz)dz \quad (29)$$

Setting $mz = \vartheta$ in (29) leads us to

$$W = \frac{x^{\tau-\lambda-1}}{m^k}\sum_{n=0}^{\infty}(\upsilon)_n\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!}\frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)n!}\left(\frac{x^\ell}{m}\right)^n \times \int_0^\infty \vartheta^{k+n-1}\exp\left(\frac{-\vartheta}{2}\right)W_{\varrho,\rho}(\vartheta)d\vartheta \tag{30}$$

Using the result from Bhatnagar and Pandey [2], we obtained:

$$\int_0^\infty \vartheta^{k-1}\exp\left(\frac{-\vartheta}{2}\right)W_{\varrho,\rho}(\vartheta)d\vartheta = \frac{\Gamma\left(\frac{1}{2}+\rho+k\right)\Gamma\left(\frac{1}{2}-\rho+k\right)}{\Gamma\left(1-\varrho+k\right)}, \tag{31}$$

where $Re(\rho \pm k) > -\frac{1}{2}$, $W_{\varrho,\rho}(;)$ the Whittaker confluent hypergeometric function and so, Simplifying (30) using (31) gives

$$W = \frac{x^{\tau-\lambda-1}}{m^k}\sum_{n=0}^{\infty}(\upsilon)_n\frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!} \times \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma\left(\frac{1}{2}+\rho+k+n\right)\Gamma\left(\frac{1}{2}-\rho+k+n\right)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)\Gamma\left(1-\varrho+k+n\right)n!}\left(\frac{x^\ell}{m}\right)^n \tag{32}$$

By using the Hadamard convolution (product) in (10) and Fox-Wright function in (4) to (32) the desired result in (28) is obtained.

Corollary 37. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda-\mu)\}$. Then, the following result is valid:*

$$\int_0^\infty z^k\exp\left(\frac{-mz}{2}\right)W_{\varrho,\rho}(mz)\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu}t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^\ell)\right)(x)dz = \frac{x^{\tau-\lambda-1}}{m^k}{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi;\varphi;\frac{x^\ell}{m}\right) * {}_5\Psi_3\left[\begin{array}{c}(\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1)\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right)\\ \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),(1-\varrho+k,1)\end{array}\middle|\ \frac{x^\ell}{m}\right]$$

Corollary 38. *Let $x > 0$, $\kappa, \lambda, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be* *such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz) \left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)dz$$
$$= \frac{x^{\tau-\lambda-1}}{m^k} {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right)$$
$$* {}_4\Psi_2 \left[\begin{array}{c} (\tau,\ell),(1,1)\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right) \\ \\ (\tau+\kappa,\ell),(1-\varrho+k,1) \end{array} \middle| \frac{x^{\ell}}{m} \right]$$

Corollary 39. *Let $x > 0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau+\mu) > 0$. Then, the following holds:*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz) \left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)dz$$
$$= \frac{x^{\tau-1}}{m^k} {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right)$$
$$* {}_4\Psi_2 \left[\begin{array}{c} (\tau+\mu,\ell),(1,1)\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right) \\ \\ (\tau+\kappa+\mu,\ell),(1-\varrho+k,1) \end{array} \middle| \frac{x^{\ell}}{m} \right]$$

Theorem 40. *If $x > 0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$,such $Re(\kappa) > 0$ and $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz) \left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)dz$$
$$= \frac{x^{\tau-\lambda-1}}{m^k} {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$* {}_5\Psi_3 \left[\begin{array}{c} (\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1),\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right) \\ \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),(1-\varrho+k,1) \end{array} \middle| \frac{x^{-\ell}}{m} \right] \tag{33}$$

Proof. Theorem 40 follows from Theorem 36.
Using equation (28), (31) and (33) the following formulas are obtained:

Corollary 41. *Let $x > 0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) >$*

0, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ *and* $Re(\kappa) > 0$ *be such that* $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. *Then, the following holds:*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz)\left(\mathcal{J}^{\kappa,\lambda,\mu}_{x,\infty} t^{\tau-1}\ {}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}(\phi;\varphi;zt^{-\ell})\right)(x)dz$$
$$= \frac{x^{\tau-\lambda-1}}{m^k}{}^{\Psi}\Phi^{\Lambda,\Upsilon}_{\wp,\Im}\left(\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$*\ {}_5\Psi_3\left[\begin{array}{c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1),\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right)\\ \\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),(1-\varrho+k,1)\end{array}\middle|\ \frac{x^{-\ell}}{m}\right]$$

Corollary 42. *Let* $x > 0$, $\kappa,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *be such that* $Re(\tau) < 1 + Re(\kappa) >$. *Then, the following holds:*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz)\left(\mathcal{W}^{\kappa}_{x,\infty} t^{\tau-1\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)dz$$
$$= \frac{x^{\tau+\kappa-1}}{m^k}{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c}(1-\kappa-\tau+1,\ell),(1,1),\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right)\\ \\ (1-\tau,\ell),(1-\varrho+k,1)\end{array}\middle|\ \frac{x^{-\ell}}{m}\right]$$

Corollary 43. *Let* $x > 0$, $\kappa,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$ *and* $Re(\kappa) > 0$ *be such that* $Re(\mu) > Re(\tau) > -1$. *Then, the following holds:*

$$\int_0^\infty z^k \exp\left(\frac{-mz}{2}\right) W_{\varrho,\rho}(mz)\left(\mathcal{K}^{\kappa,\mu}_{x,\infty} t^{\tau-1\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)dz$$
$$= \frac{x^{\tau-1}}{m^k}{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c}(\mu-\tau+1,\ell),(1,1),\left(\frac{1}{2}+\rho+k,1\right),\left(\frac{1}{2}-\rho+k,1\right)\\ \\ (\kappa+\mu-\tau+1,\ell),(1-\rho+k,1)\end{array}\middle|\ \frac{x^{-\ell}}{m}\right]$$

3.4. Verma Transform for the Fractional Integral and Derivative Operators

Theorem 44. *Let* $x > 0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$,

$Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following holds:

$$\mathcal{V}\left\{\left(I_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1\Psi} F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\} = \frac{x^{\tau-\lambda-1}}{m^k} {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right) * {}_5\Psi_3\left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1)\,(2\rho+k,1),(k,1) & \\ & \frac{x^{\ell}}{m} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right] \tag{34}$$

Proof. The proof of this theorem follows from Theorem 36.

Corollary 45. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda - \mu)\}$. Then, the following result is valid:*

$$\mathcal{V}\left\{\left(\mathcal{I}_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^{\ell})\right)(x)\right\} = \frac{x^{\tau-\lambda-1}}{m^k} {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi;\varphi;\frac{x^{\ell}}{m}\right) * {}_5\Psi_3\left[\begin{array}{c|c} (\tau,\ell),(\tau+\mu-\lambda,\ell),(1,1)\,(2\rho+k,1),(k,1) & \\ & \frac{x^{\ell}}{m} \\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right]$$

Corollary 46. *Let $x > 0$, $\kappa, \lambda, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$\mathcal{V}\left\{\left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\} = \frac{x^{\tau+\kappa-1}}{m^k} {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right) * {}_4\Psi_2\left[\begin{array}{c|c} (\tau+\kappa,\ell),(1,1)\,(2\rho+k,1),(k,1) & \\ & \frac{x^{\ell}}{m} \\ (\tau+\kappa,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right]$$

Corollary 47. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be* *such that $Re(\tau+\mu)>0$. Then, the following holds:*

$$\mathcal{V}\left\{\left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\}$$
$$=\frac{x^{\tau-1}}{m^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}}{m}\right)*{}_4\Psi_2\left[\begin{array}{c|c}(\tau+\mu,\ell),(1,1)\,(2\rho+k,1)\,,(k,1) & \\ & \frac{x^{\ell}}{m}\\ (\tau+\kappa+\mu,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right]$$

Theory 48. *Let $x>0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon\in\mathbb{C}$, $Re(\upsilon)>0$, $Re(\varphi)>Re(\phi)>0$, $Re(\ell)>0$, $\min\{Re(\wp),Re(\Im)\}>0$, $\min\{Re(\Lambda),Re(\Upsilon)\}>0$, and $Re(\kappa)>0$ be such that $Re(\tau)<1+\min\{Re(\lambda),Re(\mu)\}$. Then, the following holds:*

$$\mathcal{V}\left\{\left(J_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}=\frac{x^{\tau-\lambda-1}}{m^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$\times *{}_5\Psi_3\left[\begin{array}{c|c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1),(2\rho+k,1)\,,(k,1) & \\ & \frac{x^{-\ell}}{m}\\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right] \quad (35)$$

Proof. Proof of this Theorem follow from Theorem 40.

Corollary 49. *Let $x>0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon\in\mathbb{C}$, $Re(\varphi)>Re(\phi)>0$, $Re(\ell)>0$, $\min\{Re(\wp),Re(\Im)\}>0$, $\min\{Re(\Lambda),Re(\Upsilon)\}>0$, and $Re(\kappa)>0$ be such that $Re(\tau)<1+\min\{Re(\lambda),Re(\mu)\}$. Then, the following holds:*

$$\mathcal{V}\left\{\left(J_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1}{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$
$$=\frac{x^{\tau-\lambda-1}}{m^k}{}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi;\varphi;\frac{x^{-\ell}}{m}\right)$$
$$*{}_5\Psi_3\left[\begin{array}{c|c}(\lambda-\tau+1,\ell),(\mu-\tau+1,\ell),(1,1),(2\rho+k,1)\,,(k,1) & \\ & \frac{x^{-\ell}}{m}\\ (1-\tau,\ell),(\kappa+\lambda+\mu-\tau+1,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right]$$

Corollary 50. *Let $x>0$, $\kappa,\tau,\wp,\Im,\Lambda,\Upsilon\in\mathbb{C}$, $Re(\upsilon)>0$, $Re(\varphi)>Re(\phi)>0$, $Re(\ell)>0$, $\min\{Re(\wp),Re(\Im)\}>0$, $\min\{Re(\Lambda),Re(\Upsilon)\}>0$, be such that $Re(\tau)<1+Re(\kappa)>$. Then, the following holds:*

$$\mathcal{V}\left\{\left(\mathcal{W}_{x,\infty}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}$$
$$=\frac{x^{\tau+\kappa-1}}{m^k}{}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)*{}_4\Psi_2\left[\begin{array}{c|c}(1-\kappa-\tau,\ell),(1,1),(2\rho+k,1)\,,(k,1) & \\ & \frac{x^{-\ell}}{m}\\ (1-\tau,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right) & \end{array}\right]$$

Corollary 51. *Let $x > 0$, $\kappa, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\mu) > Re(\tau) > -1$. Then, the following holds:*

$$\begin{aligned}&\mathcal{V}\left\{\left(\mathcal{K}^{\kappa,\mu}_{x,\infty} t^{\tau-1\,\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x)\right\}\\&=\frac{x^{\tau-1}}{m^k}{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^{-\ell}}{m}\right)*{}_4\Psi_2\left[\begin{array}{c}(\mu-\tau+1,\ell),(1,1),(2\rho+k,1),(k,1)\\ \\(\kappa+\mu-\tau+1,\ell),\left(\frac{1}{2}+\rho-\varrho+k,1\right)\end{array}\middle|\frac{x^{-\ell}}{m}\right]\end{aligned}$$

3.5. Pathway Transform for the Fractional Integral and Derivative Operators

Definition 52. *Pathway integral transform is also called $\mathcal{P}_\vartheta$-transform an is defined by Kumar [7] and Kachhia et al., [4] by*

$$\mathcal{P}_\vartheta\{f(z);s\} = \int_0^\infty f(z)[1+(\vartheta-1)s]^{-\frac{z}{(\vartheta-1)}}dz, \tag{36}$$

where $\vartheta > 1$ and the $\mathcal{P}_\vartheta$-transform of power function is given by

$$\mathcal{P}_\vartheta\{z^{\tau-1};s\} = \Gamma(\tau)\left\{\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right\}^\tau, \tag{37}$$

where $\tau \in \mathbb{C}$, $Re(\tau) > 0$ and $\vartheta > 1$.

Theorem 53. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda-\mu)\}$. Then, the following holds:*

$$\begin{aligned}&\mathcal{P}_\vartheta\left\{z^{l-1}\left(I^{\kappa,\lambda,\mu}_{0,x}t^{\tau-1}\;{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{\ell})\right)(x)\right\}\\&=x^{\tau-\lambda-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right)\\&*{}_4\Psi_2\left[\begin{array}{c}(\tau,\ell),(\tau+\mu-\lambda,\ell),(l,1),(1,1)\\ \\(\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell)\end{array}\middle|\frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]\end{aligned} \tag{38}$$

Proof. For convenience, let Q be the left-hand side of (38), using (11), (36) and changing the order of summation and pathway transform give us to

$$Q = \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!} \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)n!} x^{n\ell} \times \int_0^\infty z^{l+n-1}[1+(\vartheta-1)s]^{-\frac{z}{\vartheta-1}} dz \quad (39)$$

Simplifying (39) using P_ϑ of power function in (37) gives

$$Q = x^{\tau-\lambda-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]^l \sum_{n=0}^{\infty} (\upsilon)_n \frac{{}^{\Psi}B_{\wp,\Im}^{\Lambda,\Upsilon}(\phi+n,\varphi-\phi)}{B(\phi,\varphi-\phi)n!} \times \frac{\Gamma(\tau+n\ell)\Gamma(\tau+\mu-\lambda+n\ell)\Gamma(l+n)\Gamma(1+n)}{\Gamma(\tau-\lambda+n\ell)\Gamma(\tau+\kappa+\mu+n\ell)n!}\left[\frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right] \quad (40)$$

Using the Hadamard convolution (product) in (10) and Fox-Wright function in (4) to (40) the desired result in (38) is obtained.

Corollary 54. *Let $x>0$, $\kappa,\lambda,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) > \max\{0, Re(\lambda-\mu)\}$. Then, the following result is valid:*

$$\mathcal{P}_\vartheta\left\{z^{l-1}\left(I_{0,x}^{\kappa,\lambda,\mu} t^{\tau-1}\ {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi;\varphi;zt^\ell)\right)(x)\right\} = x^{\tau-\lambda-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right] {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}\left(\phi;\varphi;\frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right) * {}_4\Psi_2\left[\begin{matrix}(\tau,\ell),(\tau+\mu-\lambda,\ell),(l,1),(1,1)\\ (\tau-\lambda,\ell),(\tau+\kappa+\mu,\ell)\end{matrix}\middle| \frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]$$

Corollary 55. *Let $x>0$, $\kappa,\lambda,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\kappa) > 0$ and $Re(\kappa) > Re(\tau)$. Then, the following formula holds:*

$$\mathcal{P}_\vartheta\left\{z^{l-1}\left(\mathcal{R}_{0,x}^{\kappa} t^{\tau-1}\ {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon,\phi;\varphi;zt^\ell)\right)(x)\right\} = x^{\tau+\kappa-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right] {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}\left(\upsilon,\phi;\varphi;\frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right) * {}_3\Psi_1\left[\begin{matrix}(\tau,\ell),(l,1),(1,1)\\ (\tau+\kappa,\ell)\end{matrix}\middle| \frac{x^\ell(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]$$

Corollary 56. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau + \mu) > 0$. Then, the following holds:*

$$\mathcal{P}_\vartheta \left\{ z^{l-1} \left(\mathcal{E}_{0,x}^{\kappa,\mu} t^{\tau-1} \, {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi; zt^{\ell}) \right)(x) \right\}$$
$$= x^{\tau-\lambda-1} \left[\frac{(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right] {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon} \left(\upsilon, \phi; \varphi; \frac{x^{\ell}(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right)$$
$$* \, {}_4\Psi_2 \left[\begin{array}{c|c} (\tau + \mu, \ell), (l, 1), (1, 1) & \\ & \frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]} \\ (\tau + \kappa + \mu, \ell) & \end{array} \right]$$

Theorem 57. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. Then, the following holds:*

$$\mathcal{P}_\vartheta \left\{ \left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1} \, {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon}(\upsilon, \phi; \varphi; zt^{-\ell}) \right)(x); k, m \right\}$$
$$= x^{\tau-\lambda-1} \left[\frac{(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right] {}^{\Psi}F_{\wp,\Im}^{\Lambda,\Upsilon} \left(\upsilon, \phi; \varphi; \frac{x^{\ell}(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right)$$
$$* \, {}_4\Psi_2 \left[\begin{array}{c|c} (\lambda - \tau + 1, \ell), (\mu - \tau + 1, \ell), (1, 1), (l, 1) & \\ & \frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]} \\ (1 - \tau, \ell), (\kappa + \lambda + \mu - \tau + 1, \ell) & \end{array} \right] \quad (41)$$

Proof. Theorem follows directly from Theorem 53.

Corollary 58. *Let $x > 0$, $\kappa, \lambda, \mu, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, and $Re(\kappa) > 0$ be such that $Re(\tau) < 1 + \min\{Re(\lambda), Re(\mu)\}$. Then, the following holds:*

$$\mathcal{P}_\vartheta \left\{ \left(\mathcal{J}_{x,\infty}^{\kappa,\lambda,\mu} t^{\tau-1} \, {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon}(\phi; \varphi; zt^{-\ell}) \right)(x); k, m \right\}$$
$$= x^{\tau-\lambda-1} \left[\frac{(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right] {}^{\Psi}\Phi_{\wp,\Im}^{\Lambda,\Upsilon} \left(\phi; \varphi; \frac{x^{\ell}(\vartheta - 1)}{\ln[1 + (\vartheta - 1)s]} \right)$$
$$* \, {}_4\Psi_2 \left[\begin{array}{c|c} (\lambda - \tau + 1, \ell), (\mu - \tau + 1, \ell), (1, 1), (l, 1) & \\ & \frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]} \\ (1 - \tau, \ell), (\kappa + \lambda + \mu - \tau + 1, \ell) & \end{array} \right]$$

Corollary 59. *Let $x > 0$, $\kappa, \tau, \wp, \Im, \Lambda, \Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, be such that $Re(\tau) <$*

$1 + Re(\kappa) >$. *Then, the following holds:*

$$\mathcal{P}_{\vartheta}\left\{\left(\mathcal{W}^{\kappa}_{x,\infty} t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x);k,m\right\}$$
$$= x^{\tau+\mu-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c|c} (1-\tau-\kappa,\ell),(1,1),(l,1) & \\ & \frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]} \\ (1-\tau,\ell) & \end{array}\right]$$

Corollary 60. *Let* $x > 0$, $\kappa,\mu,\tau,\wp,\Im,\Lambda,\Upsilon \in \mathbb{C}$, $Re(\upsilon) > 0$, $Re(\varphi) > Re(\phi) > 0$, $Re(\ell) > 0$, $\min\{Re(\wp), Re(\Im)\} > 0$, $\min\{Re(\Lambda), Re(\Upsilon)\} > 0$, *and* $Re(\kappa) > 0$ *be such that* $Re(\mu) > Re(\tau) > -1$. *Then, the following holds:*

$$\mathcal{P}_{\vartheta}\left\{\left(\mathcal{K}^{\kappa,\mu}_{x,\infty} t^{\tau-1}\ {}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}(\upsilon,\phi;\varphi;zt^{-\ell})\right)(x);k,m\right\}$$
$$= x^{\tau-1}\left[\frac{(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right]{}^{\Psi}F^{\Lambda,\Upsilon}_{\wp,\Im}\left(\upsilon,\phi;\varphi;\frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]}\right)$$
$$*\ {}_4\Psi_2\left[\begin{array}{c|c} (\mu-\tau+1,\ell),(1,1),(l,1) & \\ & \frac{x^{\ell}(\vartheta-1)}{\ln[1+(\vartheta-1)s]} \\ (\kappa+\mu-\tau+1,\ell) & \end{array}\right].$$

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