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EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION FOR A NONLINEAR REACTION-DIFFUSION SYSTEM IN SPACES BY SOBOLEV

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Abstract: In this article, we present a weak solution existence result for a system of equations involved in the mathematical modeling of the flow of an inhomogeneous viscous and incompressible fluid. For this, two results have been established. In the first result, the differentiability is according to Frechet. In the second result, the differentiability is understood in a weaker sense than that of Frechet.

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1. Introduction

We consider a reproductive flow of a viscous, incompressible and inhomogeneous fluid (variable density) in a domain $\Omega \subset \mathbf{R}^d$ (d = 2 or 3) during an observation interval $[t_0, t_f]$. Let ϑ be the speed of the fluid, η the coefficient of viscosity, ρ the density and $\pi = \pi(x,t)$ the pressure. The model is then described, (see for example [10 , 9]) by the following equations

$$\partial_t(\rho\vartheta) + div(\rho\vartheta \otimes \vartheta) - \mathscr{B}\vartheta + \nabla\pi = \rho\, f_e \tag{1}$$

$$\partial_t \rho + div(\rho\vartheta) = 0 \quad (2)$$

$$div\,\vartheta = 0 \quad (3)$$

with $(x, t) \in \Omega \times (t_0, t_f)$. Here f_e denotes the density of the external forces and the operator $\mathscr{B}\vartheta$ is defined as $\mathscr{B}\vartheta = 2\mu div(\nabla\vartheta) - div(3\lambda\nabla \cdot \vartheta)$ where λ and μ respectively represent the bulk viscosity and the dynamic coefficients supposed to be constant. In this system, the pressure is given by the state law $\pi = k\rho^{C_a}$, $0 < k \leq 1$ and C_a adiabatic constant as $C_a \geq (d-1)/2$. In the following, we set $C_a = 1$.
The system is completed by initial conditions on the volume density and field velocity :

$$\rho\mid_{t_0=0} = \rho_0(x) \qquad \vartheta\mid_{t_0=0} = \vartheta_0(x) \quad \text{and} \quad \rho\vartheta\mid_{t_0=0} = q_0(x) \quad (4)$$

It is assumed that on the boundary $\partial\Omega$ the speed satisfies :

$$\vartheta\mid_{\partial\Omega} = 0 \quad \forall (x, t) \in \partial\Omega \times (t_0, t_f) \quad (5)$$

It is worth mentioning that $\rho\vartheta \otimes \vartheta \in \mathbb{R}^3$ in (1) is a tensor product of $\rho\vartheta$ and ϑ then,

$$\nabla \cdot (\rho\vartheta \otimes \vartheta) = \nabla \cdot (\rho\vartheta)\vartheta + \rho(\vartheta \cdot \nabla)\vartheta \quad (6)$$

The system (1)-(3) is represented by the conservative flow equations, whose equation (1) describes the motion of a viscous fluid (conservation of the quantity motion), equation (2) models continuity (conservation of mass) and (3) reflects the incompressibility of the fluid. The interest in the study of the model (1)-(5) is that it adapts to several situations real, in particular the evolution of several incompressible and immiscible fluids for example water and oil, and the flow in a river containing suspended solids, etc ... The classical initial value problem corresponding to the model (1)-(3) has been studied by several authors, for example the work of Antontzev and Kazhikhov [1], Antontzev et al. [2], Kim [8], Ladyzhenskaya, J. L. Lions [11], Padula [16], J. P. Lions [12], Fernandez-Cara and Guillén [6]. Antontzev and Kazhikhov [1] have obtained a locally weak solution in time with an additional initial hypothesis imposed on the moment $\rho\vartheta$ using mainly semi-type approximations Galerkin; then J. P. Lions [12] extended this result for only positive initial density and an initial condition in the weak sense for the moment $\rho\vartheta$. With the same techniques, Antontzev et al. [2] obtained a locally strong solution in time under assumptions extra for data. Padula [16] and Kim [8], obtained a similar result than [2]. J. L. Lions [11] and J. P. Lions [12] presented new versions of results from Antontzev and Kazhikhov [1] using arguments similar to those of

Antontzev et al. [2]. Fernandez and Guillén [6] have studied the existence of the weak solution of the problem (1)-(3) in a not necessarily bounded open set.

The main idea of this work is to obtain the existence, the uniqueness of the solution of a nonlinear dynamical system with $(\vartheta, \rho) \equiv \mathscr{R}\varepsilon(V)$, where $V = (f_e\, , q_0)$, in which q_0, and f_e are respectively the initial moment and the function that models the external forces, and $\mathscr{R}_\varepsilon$ the operator satisfying :

$$\left\|\mathscr{R}_\varepsilon(V) - \mathscr{R}_\varepsilon(\bar{V})\right\| \leq \frac{\eta}{\|\psi^{-1}\|}\left\|V - \bar{V}\right\| \tag{7}$$

where ψ is a continuous invertible operator. Our approach is therefore to perturb our system involving measurable functions and operators, twice continuously differentiable in Banach spaces in order to obtain the proof of the differentiability of the solution (ϑ, ρ). We end the introduction with a brief description of the content of the document. In section 2, we formulate the problem and give some preliminary notations that will be used in the sequel. Section 3 introduces the definition of a weak solution by rewriting the equations of the system in a particular framework, then we present a differentiability result of the general solution with constant viscosity coefficients. Then, we give the proof of the existence and uniqueness theorem of the solution when the system is perturbed according to a certain number of parameters.

2. Functional Spaces and Approximation of the Solution

2.1. Functional Spaces

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded with smooth boundary $\partial\Omega$ (class C^3 is enough). We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \Big\{ f \in L^q(D), ||\partial^\alpha f||_{L^q(D)} < +\infty, |\alpha| \leq m \Big\},$$

$m = 0, 1, 2, \ldots \quad 1 \leq q \leq +\infty, \quad D = \Omega \quad$ or $\quad \Omega \times (0, T)$,
$0 < t_f < +\infty$, with the usual norm. When $q = 2$, we denote by $H^m(D) = W^{2,q}(D)$ and H_0^m = closure of $C_0^{+\infty}$ in $H^m(D)$. If B is a Banach space, we denote by $L^q([t_0, t_f]); B)$ the Banach space of the B-valued functions defined in the interval $[t_0, t_f]$ that are L^q-intégrable in the sense of Bochner. We shall consider the following spaces of divergences free functions

$$C_{0,\sigma}^{+\infty}(\Omega) = \Big\{ \vartheta \in (C_0^{+\infty}(\Omega))^3 : div\vartheta = 0 \quad \text{in} \quad \Omega \Big\},$$

To relieve the notations, we pose for

$$
\begin{aligned}
X(t,\mathbb{R}^3) &= L^\infty((t_0,t_f);(H_0^1(\Omega))^3)\\
Y(t,\mathbb{R}^3) &= L^2((t_0,t_f);(L^2(\Omega))^3)\\
W(t,\mathbb{R}^3) &= L^2((t_0,t_f);[H^1(\Omega)]^3)
\end{aligned}
$$

2.2. Parametric Sensitivity of Solutions

We briefly demonstrate some a priori estimates relative to the solutions of the evolution equation (1)-(3). Multiplying the evolution equation (1) by ϑ, and integrating over Ω, we get :

$$
\int_\Omega \rho\Big(\frac{\partial\vartheta}{\partial t}\Big)\vartheta dx + \int_\Omega \rho(\vartheta\nabla\vartheta)\vartheta dx - \int_\Omega (2\mu\Delta\vartheta)\vartheta dx - \int_\Omega 3\lambda\nabla div(\vartheta)\vartheta dx + \int_\Omega \nabla\pi\vartheta dx = \int_\Omega \rho f_e\vartheta dx
$$

By applying the differentiation theorem, the first member of the left gives the following estimate :

$$
\int_\Omega \rho\Big(\frac{\partial\vartheta}{\partial t}\Big)\vartheta dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho\|\vartheta\|^2_{H_0^1}dx, \quad \forall t\in[t_0,t_f] \tag{8}
$$

The Navier-Stokes equations in slow report that the integral over the volume Ω of the term $(\vartheta\nabla\vartheta)$ is null due to the assumption of low speed.

$$
\int_\Omega \rho(\vartheta\nabla\vartheta)\vartheta dx = 0, \qquad \forall t\in[t_0,t_f] \tag{9}
$$

In order to solve the (1)-(3) problem, several estimates are required.

(i) Estimate of $\int_\Omega (\mu\Delta\vartheta)\vartheta dx$

$$
\int_\Omega (\mu\Delta\vartheta)\vartheta dx = \mu\int_{\partial\Omega} y_0\vartheta(\nabla\vartheta\cdot\vec{n})ds - \mu\int_\Omega tr(\nabla\vartheta\cdot\nabla^t\vartheta)dx
$$

(where y_0 is an unique continuous linear application defined from $W_2^1(\Omega)\to L^2(\Omega)$ such as $y_0\vartheta = 0$, $\vec{n}$ is the normal to the edge of Ω, denoted by $\partial\Omega$ and ds its elementary surface element). It therefore follows that :

$$
\int_\Omega (\mu\Delta\vartheta)\vartheta dx = -\mu\sum_{i,j}^{3}\int_\Omega \frac{\partial\vartheta_i\partial\vartheta_j}{\partial x_i\partial x_j}dx \le \mu\int_\Omega \Big\|\frac{D\vartheta}{Dt}\Big\|^2 dx \tag{10}
$$

(ii) Estimate of $\displaystyle\int_\Omega \lambda \nabla div(\vartheta) dx$

$$\int_\Omega \lambda \nabla div(\vartheta)\vartheta dx = \lambda\Big(\int_\Omega \nabla(\vartheta div(\vartheta) dx - \int_\Omega \Delta\vartheta^2 dx)\Big) \leq \lambda \int_\Omega \Big\|\frac{D\vartheta}{Dt}\Big\|^2 dx \qquad (11)$$

(iii) Estimate of $\displaystyle\int_\Omega \nabla\pi\vartheta dx$

$\displaystyle\int_\Omega \nabla\pi\vartheta dx = \int_\Omega \nabla k\rho\vartheta dx$ after integration by parts we have :

$$\int_\Omega \nabla k\rho\vartheta dx = \int_{\partial\Omega} k\rho y_0 \vartheta.\vec{n} ds - \int_\Omega \frac{\partial}{\partial t} k\rho dx = -\frac{d}{dt}\int_\Omega k\rho dx \qquad (12)$$

Finally the force provided by the membrane : $\forall t \in [t_0, t_f]$

$$\int_\Omega \rho f_e \vartheta dx \leq \|\rho\vartheta\|_{(L^4(\Omega))^3} \|f_e\|_{(L^2(\Omega))^3} \qquad (13)$$

It is of the greatest interest to an estimate of the solution (ϑ, ρ) under the assumption of low speeds. Putting these different estimates together, we have the following result:

Theorem 2.1. (Estimated solution with low speed hypothesis)
Let $\vartheta_0 \in (H_0^1(\Omega))^3$, $\rho_0 \in (L^2(\Omega))^3$ and $f_e \in L^2((t_0, t_f); (L^2(\Omega))^3)$. We suppose there exists $\beta > 0$ such as $\forall (x,t) \in \Omega \times (t_0, t_f)$,

$$|\rho|^{-1} \leq \beta \quad and \quad \beta < \rho_0$$

Then there exists a solution (ϑ, ρ) of the system (1)-(3) satisfying the initial conditions (4) and the following inequality :

$$\|\vartheta\|_X \leq \beta\left[\Big(\|q_0\|^2_{(L^4(\Omega))^3} + \|f_e\|^2_Y\Big) e^{\delta t}\right]^{1/2}. \qquad (14)$$

Proof. If we assume that the viscosity coefficients λ, μ are regular then, it is readily seen that [17] implies that $\lambda = \frac{-2}{3}\mu$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho(\|\vartheta\|_{(H_0^1(\Omega))^3} - k) dx \leq \|\rho\vartheta\|_{(L^4(\Omega))^3}\|f_e\|_{(L^2(\Omega))^3}$$

On the other hand, by an integration by parts of (2) we get $\displaystyle\frac{d}{dt}\int_\Omega \rho(x,t) dx = 0$,we thus obtain :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho\|\vartheta\|_{(H_0^1(\Omega))^3}) dx \leq \|q\|_{(L^4(\Omega))^3}\|f_e\|_{(L^2(\Omega))^3}$$

By applying the Young inequality, the estimation becomes :

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(\rho\|\vartheta\|_{H_0^1(\Omega)^3})dx \leq \frac{1}{2}\|q\|^2_{(L^4(\Omega))^3} + \frac{1}{2}\|f_e\|^2_{(L^2(\Omega))^3}$$

Integrate the inequality on (t_0, t_f) we get:

$$|\rho|^2\|\vartheta\|^2_X \leq \int_{t_0}^{t_f}(\|q\|^2_{(L^4(\Omega))^3})ds + \|q_0\|^2_{(L^4(\Omega))^3} + \int_{t_0}^{t_f}(\|f_e\|^2_{(L^2(\Omega))^2})dt$$

$$|\rho|^2\|\vartheta\|^2_X \leq \int_{t_0}^{t_f}(\|q\|^2_{(L^4(\Omega))^3})ds + \|q_0\|^2_{(L^4(\Omega))^3} + \|f_e\|^2_Y$$

Applying the Gronwall Lemma (see [18]) we obtain for any $t \geq 0$

$$|\rho|^2\|\vartheta\|^2_X \leq (\|q_0\|^2_{(L^4(\Omega))^3} + \|f_e\|^2_Y)exp(\int_{t_0}^{t_f} dt)$$

$$\|\vartheta\|^2_X \leq |\rho|^{-2}(\|q_0\|^2_{(L^4(\Omega))^3} + \|f_e\|^2_Y)e^{\delta t}$$

Under the increase imposed on $|\rho^{-1}|$ in the statement of the theorem, we can establish that:

$$\|\vartheta\|_X \leq \beta\left[(\|q_0\|^2_{(L^4(\Omega))^3} + \|f_e\|^2_Y)e^{\delta t}\right]^{1/2}$$

3. Linerization System

The characteristics are defined as above, with the same initial conditions and a domain Ω which is still bounded. We are still interested in studying the system under the assumption of compressibility of cancer cells.
However, let's look at the character $\vartheta\nabla\vartheta$ that appears in the (1), it is at the origin of difficulties when solving this problem. We will linearize this term by substituting the following disturbance:

$$\mathfrak{F}(\mathcal{H}, \varphi) = \mathcal{H}(x, t) + \varphi(x, t, \varrho, \chi) \tag{15}$$

Where $\mathcal{H}$ is an integrated linear operator [19] that will be later and φ a function given by: $\Omega \times [t_0, t_f] \times \mathbb{R}^3 \times \mathbb{R}^9 \to [t_0, t_f] \times \mathbb{R}^9, \quad (x, t, \varrho, \chi) \mapsto \varphi(x, t, \varrho, \chi)$
Then for all $(x, t) \in \Omega \times (t_0, t_f)$, equation (1) becomes :

$$\partial_t(\rho\nu) + div(\rho\vartheta)\vartheta + \mathfrak{F}(\mathcal{H}, \varphi) + \nabla\pi = \rho f_e + \mathcal{B}\vartheta \tag{16}$$

This approach has introduced new variables ϱ, χ which are considered as a field argument $\vartheta(x, t)$ and its divergence(describes the increase in the volume) respectively.

Proposition 3.1. *For our study, let consider the functions $\varphi(x,t,\varrho,\chi)$ and $\mathcal{U}$ (x,t,ϱ,χ) defined on $\Omega \times [t_0,t_f] \in \mathbb{R}^3 \times \mathbb{R}^9$ and satisfying the following assumptions:*
*Assumptions(**H**):*
H-1 *: For all $(\varrho,\chi) \in \mathbb{R}^3 \times \mathbb{R}^9$, $\exists\ \beta,\ \beta' > 0$ such as functions $(x,t,\varrho,\chi) \mapsto \varphi(x,t,\varrho,\chi)$ and $(x,t) \mapsto \mathcal{U}(x,t,\varrho,\chi)$ are measurable and satisfy the following conditions :*

$$\left|\varphi(x,t,\varrho,\chi)\right| \leq \beta(\varrho^2+\chi^2)e^{\delta t} \tag{17}$$

$$\left|\mathcal{U}(x,t,\varrho,\chi)\right| \leq \beta(\varrho^2+\chi^2)e^{\delta t} \tag{18}$$

H-2 *: For almost all $(x,t) \in \Omega \times [t_0,t_f]$, there exists $\omega, \bar{\omega} > 0$ such that the functions $(x,t,\varrho,\chi) \mapsto \varphi(x,t,\varrho,\chi)$ and $(x,t) \mapsto \mathcal{U}(x,t,\varrho,\chi)$ are twice continuous and differentiable on $\mathbb{R}^3 \times \mathbb{R}^9$ in addition :*

$$|\Delta_\varrho \varphi| + |\Delta_\chi \varphi| \leq 4\omega e^{\delta t} \quad \textit{and} \quad |\Delta_\varrho \mathcal{U}| + |\Delta_\chi \mathcal{U}| \leq 4\bar{\omega} e^{\delta t} \tag{19}$$

H-3 *: let $\mathcal{U} = \mathcal{P}u$ be a continuous linear integral operator, which any function u matches $\mathcal{H}$ such that:*

$$\mathcal{H}_p u(.,t) := \int_{t_0}^{t_f} \int_\Omega P(x,t,y)u(y,t)dydt \quad \textit{defined} \quad \textit{by}: \tag{20}$$

$$\mathcal{H} : L^2(\Omega) \times [t_0,t_f] \mapsto L^2(\Omega) \times [t_0,t_f]$$

H-4*: let $\mathcal{A}_\epsilon'$ and $\mathcal{T}_\epsilon'$ be two non-linear differentiable operators in $L^2((t_0,t_f) \times W_2^1(\Omega))$. Note by $d\left[\mathcal{A}_\epsilon'(\vartheta)g,h\right]$ $\left(respd\left[\mathcal{T}_\epsilon'(\vartheta)g,h\right]\right)$ the second differential of $\mathcal{A}_\epsilon$ (resp $\mathcal{T}_\epsilon$) in ϑ where $\mathcal{A}_\epsilon'(\vartheta)g = d\mathcal{A}_\epsilon(\vartheta,g)$. For an increase h independent of g we have the increase :*

$$\mathcal{A}_\varepsilon'(\vartheta+g)h - \mathcal{A}_\varepsilon'(\vartheta)h = \sum_{i=1}^{3} \partial_\chi^2 \varphi \frac{\partial^2 gh}{\partial x \partial t} + \partial_\varrho^2 \varphi gh + o(g)h$$

For $h = g$ from this we deduce the following formulas :

$$d\left[\mathcal{A}_\epsilon'(\vartheta)g,h\right]_{h=g} = \sum_{i=1}^{3} \partial_\chi^2 \varphi \frac{\partial^2 g^2}{\partial x \partial t} + \partial_\varrho^2 \varphi g^2 \quad \text{and} \tag{21}$$

$$d\left[\mathcal{T}_\epsilon'(\vartheta)g,h\right]_{h=g} = \sum_{i=3}^{3} \partial_\chi^2 \mathcal{U} \frac{\partial^2 g^2}{\partial x \partial t} + \partial_\varrho^2 \mathcal{U} g^2 \tag{22}$$

4. Study of Strict ε-differentiability

In this section, let Ω_p be the disruption of domain Ω and define a displacement field of Ω defined from $\mathbb{R}^3 \to \mathbb{R}^3$; $\Omega_p = \{(x,t) \in \Omega_t, \vartheta \in \Omega + \tau\Omega\}$

Definition 4.1. *Let E_1 and E_2, two normed spaces Ω an open set in E_1. Let M_ε all compact systems E_1. if $J^{'}(\vartheta+g)h - J^{'}(\vartheta)h = J^{'}(\vartheta)h^2 + \mathcal{J}(\|h^2\|)$ or $J^{'}(\vartheta)h^2 \in \mathcal{L}(E_1, \mathcal{L}(E_1, E_2))$ with $J^{'}(\vartheta)h^2$ is a bilinear operator. The function $J : \Omega \to E_2$ is called strictly ε-differentiable on Ω if the condition (D_ε) is satisfied :*

$$(D_\varepsilon) : \left\{ \begin{pmatrix} \forall\eta > 0, \forall h \in M_\varepsilon, \forall\vartheta \in \Omega, \exists\lambda > 0 \\ \|\vartheta - \vartheta_f\| < \lambda, |d| < \lambda, \vartheta + th^2 \in \Omega \end{pmatrix} \Rightarrow \|\mathcal{J}(h^2)\| \le \eta|d| \right\}$$

Proposition 4.1. *Let Ω_p a disturbed area of Ω defined as follows :*

$$\Omega_p = \{(x,t) \in \Omega_p, \vartheta \in \Omega, \vartheta + \tau\Omega\}.$$

The operator defined is φ ε-continuous and ε-differentiable on X.

Proof. Suppose that $\mathcal{A}_\varepsilon$ is Frechet-differentiable and ϑ it a first variation, that is

$$\frac{\mathcal{A}_\varepsilon(\vartheta+\tau g) - \mathcal{A}_\varepsilon(\vartheta)}{\tau} \to_{\tau\to 0} \delta\mathcal{A}_\varepsilon(\vartheta, g)$$

It is therefore clear that for all $\forall g \in X$, the quantity $\mathcal{A}_\varepsilon(\vartheta, \tau g)$ is defined for τ small enough. After we suppose that $\delta\mathcal{A}_\varepsilon(\vartheta, g) = \mathcal{A}^{'}_\varepsilon(\vartheta)g$. Let show that $\mathcal{A}_\varepsilon$ is twice differentiable according to Gateau X.

Assume that $\mathcal{A}_\varepsilon$ is Fréchet differentiable. We have for all $|\tau|$ small enough and for all $\forall g \in X$

$$\mathcal{A}_\varepsilon(\vartheta+g) - \mathcal{A}_\varepsilon(\vartheta) = d\mathcal{A}_\varepsilon(\vartheta, g) + o(g)$$

For $\tau \in]-1,1[$, $\tau \neq 0$, $\mathcal{A}_\varepsilon(\vartheta+\tau g) - \mathcal{A}_\varepsilon(\vartheta) = \delta\mathcal{A}_\varepsilon(\vartheta+g) + o(\tau g)$

we have $\mathcal{A}^{'}_\varepsilon(\vartheta+g)h - \mathcal{A}^{'}_\varepsilon(\vartheta)h - d[\mathcal{A}^{'}_\varepsilon(\vartheta)g, h]_{h=g} = \mathcal{A}^{'}_\varepsilon(\vartheta+g)h - \mathcal{A}^{'}_\varepsilon(\vartheta)h - \mathcal{A}^{''}_\varepsilon(\vartheta)h^2$

Taking the L^2-norm in X, we have:

$$\left\| \frac{\mathcal{A}^{'}_\varepsilon(\vartheta+\tau g)h - \mathcal{A}^{'}_\varepsilon(\vartheta)h}{\tau} - \mathcal{A}^{'}_\varepsilon(\vartheta)h^2 \right\|^2_X = \left\| \frac{\mathcal{A}^{'}_\varepsilon(\vartheta+\tau g)h}{\tau} - \mathcal{A}^{''}_\varepsilon(\vartheta)h^2 - \frac{\mathcal{A}^{'}_\varepsilon(\vartheta)h}{\tau} \right\|^2_X$$

$$= \left\| \frac{\mathcal{A}^{'}_\varepsilon(x,t,\vartheta+\tau g, \nabla\vartheta + \tau\nabla g)h}{\tau} - \partial^2_\varrho\varphi h^2 - \sum_i^3 \partial^2_\chi\varphi\frac{\partial^2 h^2}{\partial x\partial t} - \frac{\mathcal{A}^{'}_\varepsilon(x,t,\vartheta,\nabla\vartheta)h}{\tau} \right\|^2_X$$

$$= \left\| \frac{\mathcal{A}^{'}_\varepsilon(x,t,\vartheta+\tau g, \nabla\vartheta + \tau g)h}{\tau} - \frac{\mathcal{A}^{'}_\varepsilon(x,t,\vartheta,\nabla\vartheta+\tau\nabla g)h}{\tau} - \partial^2_\chi\varphi h^2 + \right.$$

$$\left.\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta,\nabla\vartheta+\tau\nabla g)h}{\tau}-\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta,\nabla\vartheta)h}{\tau}-\sum_i^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right\|^2_X$$

$$\leq\int_{t_0}^{t_f}\int_\Omega\left(\left|\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta+\tau g,\nabla\vartheta+\tau\nabla g)h}{\tau}-\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta,\nabla\vartheta+\tau\nabla g)h}{\tau}-\partial^2_\varrho\varphi h^2\right|^2\right)dxdt+$$

$$\int_{t_0}^{t_f}\int_\Omega\left(\left|\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta,\nabla\vartheta+\tau\nabla g)h}{\tau}-\frac{\mathcal{A}'_\varepsilon(x,t,\vartheta,\nabla\vartheta)h}{\tau}-\sum_i^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2\right)dxdt$$

Using Lagrange's formula [18] for some $\theta\in[0;1]$

$$\leq\int_{t_0}^{t_f}\int_\Omega\left(\left|\int_0^1\mathcal{A}''_\varepsilon(x,t,\vartheta+\theta\tau g,\nabla\vartheta+\tau\nabla g)h^2-\partial^2_\varrho\varphi h^2\right|^2d\theta\right)dxdt+$$

$$\int_{t_0}^{t_f}\int_\Omega\left(\left|\int_0^1\mathcal{A}''_\varepsilon(x,t,\vartheta,\nabla\vartheta+\theta\tau\nabla g)h^2-\sum_i^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2d\theta\right)dxdt$$

$$\leq\int_{t_0}^{t_f}\int_\Omega\left(\left|\int_0^1\left((\partial^2_\varrho\varphi(x,t,\vartheta+\theta\tau g,\nabla\vartheta+\tau\nabla g)h^2-\partial^2_\varrho\varphi h^2)\times\sum_{i=1}^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2d\theta\right)\right)dxdt+$$

$$\int_{t_0}^{t_f}\int_\Omega\left(\left|\int_0^1\sum_i(\partial^2_\chi\varphi(x,t,\nabla\vartheta+\tau\nabla g)h^2-\partial^2_\chi\varphi h^2)\times\sum_i\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2d\theta\right)dxdt$$

$$\leq\int_0^1\left(\int_{t_0}^{t_f}\int_\Omega\left(\left|\partial^2_\varrho\varphi(x,t,\vartheta+\theta\tau g,\nabla\vartheta+\tau\nabla g)h^2-\partial^2_\chi\varphi h^2\right|^2\right)\times\left|\sum_{i=1}^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2dxdt\right)d\theta+$$

$$\int_0^1\left(\int_{t_0}^{t_f}\int_\Omega\left(\sum_{i=1}^3\left|\partial^2_\varrho\varphi(x,t,\vartheta,\nabla\vartheta+\tau\nabla g)h^2-\partial^2_\chi\varphi h^2\right|^2\times\left|\sum_{i=1}^3\partial^2_\chi\varphi\frac{\partial^2h^2}{\partial x\partial t}\right|^2\right)dxdt\right)d\theta$$

From Newton-Leibniz formula, Cauchy inequality and using (18), when we go to the limit for $\tau\to0$ we obtain :

$$\left\langle d\left[\mathcal{A}'_\varepsilon(\vartheta)h,g\right]_{h=g},h\right\rangle\to o(\|g^2\|)$$

On the other hand, let $m\in[t_0;t_f]$ suppose that there exists a sequence ϑ_m of X such that for all integer m.
We have : $\vartheta_m\to\vartheta_f$ in X and $\nabla\vartheta_m\to\nabla\vartheta_f$ in Y.

Then there exists $h \in X$ such as $d\left[\mathcal{A}_{\varepsilon}^{'}h, g\right]_{h=g} \notin L^2$ space so that

$\left\|d\left[\mathcal{A}_{\varepsilon}^{'}(\vartheta_m)h, g\right]_{h=g} - d\left[\mathcal{A}_{\varepsilon}^{'}(\vartheta_f)h, g\right]_{h=g}\right\|_X^2 \neq 0$ for $m \to 0$, there exists, $\exists \alpha \geq 1$ such as $\left\|d\left[\mathcal{A}_{\varepsilon}^{'}(\vartheta_m)h, g\right]_{h=g} - d\left[\mathcal{A}_{\varepsilon}^{'}(\vartheta_f)h, g\right]_{h=g}\right\|_X^2 \geq \dfrac{\alpha}{2}$

Indeed

$$\begin{aligned}
&\left\|d\Big[\mathcal{A}_{\varepsilon}^{'}(\vartheta_m)h, g\Big]_{h=g} - d\Big[\mathcal{A}_{\varepsilon}^{'}(\vartheta_f)h, g\Big]_{h=g}\right\|_X^2 \\
&= \Bigg\|\partial_{\chi}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m)h^2 + \sum_{i=1}^{3}\partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_m)\frac{\partial^2 h^2}{\partial x\partial t} - \partial_{\chi}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)h^2 \\
&+ \sum_{i=1}^{3}\partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\frac{\partial^2 h^2}{\partial x\partial t}\Bigg\|_X^2
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t_f}\int_{\Omega}\left|\partial_{\chi}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m) - \partial_{\chi}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)h^2\right|^2 dxdt + \\
&\int_{t_0}^{t_f}\int_{\Omega}\Big|\sum_{i=1}^{3}\partial_{\varrho}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m) - \partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\frac{\partial^2 h^2}{\partial x\partial t}\Big|^2 dxdt
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t_f}\int_{\Omega}\alpha\Big|\partial_{\varrho}^2\varphi(x,t,\vartheta_m,\nabla v_m) - \partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\Big|^2\Big|h^2\Big|dxdt \\
&+ \int_{t_0}^{t_f}\int_{\Omega}\alpha\left|\sum_{i}\partial_{\varrho}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m) - \partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\right|^2\left|\frac{\partial^2 h^2}{\partial x\partial t}\right|^2 dxdt \\
&\leq 16\alpha\omega^2 e^{\delta t}\int_{t_0}^{t_f}\left|h^2\right|dt + 24\alpha\omega^2 e^{\delta t}\int_{t_0}^{t_f}\left|\frac{\partial^2 h^2}{\partial x\partial t}\right|dt \\
&\leq 40\alpha\omega^2 e^{\delta t}\left\|h\right\|_X^2
\end{aligned} \tag{25}$$

According to the **H-2** hypothesis for all $m \in [1; f]$, $\vartheta_m \to \vartheta_1$ et $\nabla\vartheta_m \to \nabla\vartheta_f$ pp. in Y.

$$\alpha\left|\partial_{\chi}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m) - \partial_{\chi}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\right|^2|h|^2 \to 0$$

in the same way,

$$\alpha\left|\sum_{i=1}^{3}\partial_{\varrho}^2\varphi(x,t,\vartheta_m,\nabla\vartheta_m) - \partial_{\varrho}^2\varphi(x,t,\vartheta_f,\nabla\vartheta_f)\right|^2\left|\frac{\partial^2 h^2}{\partial x\partial t}\right|^2 \to 0$$

Using double integration, we obtain:

$$\left\|d[\mathcal{A}_{\varepsilon}^{'}(\vartheta_m)h,g]_{h=g}-d[\mathcal{A}_{\varepsilon}^{'}(\vartheta_f)h,g]_{h=g}\right\|_X^2\to 0,$$

for $m\to f$ which contradicts our hypothesis.
However, it was therefore $d[\mathcal{A}_{\varepsilon}^{'}(\cdot)h,g]_{h=g}$ belongs to the space (X,Y). We can therefore conclude that the second variation of the operator $\mathcal{A}_{\varepsilon}^{'}$ equals $d[\mathcal{A}_{\varepsilon}^{'}(\cdot)h,g]_{g=h}$, $\forall\vartheta,h\in X$ and for a given speed $\vartheta(x,t)$,$d[\mathcal{A}_{\varepsilon}^{'}(\cdot)h,g]_{h=g}$ in general it will be a linear operator space $E_{ep}(X;Y)$. However, according to the above we can say that $\mathcal{A}_{\varepsilon}^{'}$ is ε-continuous and ε-differentiable on X.

Proposition 4.2 *Let Ω to a bounded open set in $\mathbb{R}^3$. Let $l(x,t)\in X$ and $h(x,t)\in X$, $\forall n$ there exists $d_n>0$ such that for $\tau^n\in]0;1[$.*
If $|\tau^n|<d_n$, and h_n small enough such that $\|h_n\|_X\le 1$, then

$$\Big|<\frac{1}{\tau^n}\mathcal{J}_{\varepsilon}(\|\tau^n h_n^2),\psi(x,t)>\Big|\le|\tau^n|$$

for $\|\vartheta_n-\vartheta_f\|\to 0$, (we say that $\vartheta(x,\cdot)\to\vartheta_f(x,t_f)$ almost over Ω_t).
Proof. Let ϑ_f,$\vartheta_n\in(H_0^1(\Omega))^3$ such that for $n\in[1;f]$, $\vartheta_n\to\vartheta_f$ pp.in $\Omega\times[t_0;t_f]$.
Let h_n be small enough as $\|h_n\|_X\le 1$.
Let $\mathcal{J}_{\varepsilon}(\|h_n^2\|)=\mathcal{A}_{\varepsilon}^{'}(\vartheta+h_n)h_n-\mathcal{A}_{\varepsilon}^{'}(\vartheta)h_n-\mathcal{A}_{\varepsilon}^{''}(\vartheta_f)h_n^2$ such that for $\tau^n\in]0;1[$
$\mathcal{J}(\|\tau^n h_n^2\|)=\mathcal{A}_{\varepsilon}^{'}(\vartheta+\tau^n h_n)h_n-\mathcal{A}_{\varepsilon}^{'}(\vartheta)h_n-\mathcal{A}_{\varepsilon}^{''}(\vartheta_f)\tau^n h_n^2$, we get
$\Big|\left\langle\frac{1}{\tau^n}\mathcal{J}_{\varepsilon}\left(\|\tau^n h_n^2\|\right),\psi(x,t)\right\rangle\Big|=\left|\int_{t_0}^{t_f}\int_{\Omega}\frac{1}{\tau^n}\mathcal{J}_{\varepsilon}\left(\left\|\tau^n h_n^2\right\|\right)\times\psi(x,t)dxdt\right|=$
$\Big|\int_{t_0}^{t_f}\int_{\Omega}\frac{1}{\tau^n}\Big[\partial_{\varrho}\varphi(x,t,\vartheta_n+\tau^n h_n,\nabla\vartheta_n+\tau^n\nabla\vartheta_n)h_n-\partial_{\varrho}\varphi(x,t,\vartheta_n,\nabla\vartheta_n)h_n$
$-\partial_{\varrho}^2\varphi(x,t,\vartheta_f)\tau^n h_n^2\Big]\times l(x,t)dxdt\Big|$
So from Lagrange's formula [8] for a some $\theta\in[0;1]$ the equality becomes :
$=\left|\int_{t_0}^{t_f}\int_{\Omega}\frac{1}{\tau^n}\Big[\int_0^1\left(\partial_{\varrho}^2\varphi(x,t,\vartheta_n+\theta\tau^n h_n)\tau^n h_n^2-\partial_{\varrho}^2\varphi(x,t,\vartheta_f)\tau^n h_n^2 d\theta\right)\Big]\times\psi(x,t)dxdt\right|$
$=\left|\int_{t_0}^{t_f}\int_{\Omega}\frac{1}{\tau^n}\left[\int_0^1\partial_{\varrho}^2\psi(x,t,\vartheta_n+\theta\tau^n h_n)-\partial_{\varrho}^2\psi(x,t,\vartheta_f)d\theta\right]\tau^n h_n^2\times\psi(x,t)dxdt\right|$
From Cauchy Schwarz inequality, we deduce that

$$\le\int_{t_0}^{t_f}\int_{\Omega}\left(\left|\int_0^1\left(\partial_{\varrho}^2\varphi(x,t,\vartheta_n+\theta\tau^n h_n)-\partial_{\varrho}^2\varphi(x,t,\vartheta_f)\right)d\theta\right|^2 h_n^2dtdx\right)^{1/2}\times\left(\int_{t_0}^{t_f}\int_{\Omega}\psi^2dxdt\right)^{1/2}$$

$$\le\int_{t_0}^{t_f}\int_{\Omega}\left(\left|\int_0^1\partial_{\varrho}^2\varphi(x,t,\vartheta_n+\theta\tau^n h_n)-\partial_{\varrho}^2\varphi(x,t,\vartheta_f)d\theta\right|^2dxdt\right)^{1/2}\times\left(\int_{t_0}^{t_f}\int_{\Omega}\psi^2dxdt\right)^{1/2}\|h_n\|_X$$

On the other hand, $\|\vartheta_n - \vartheta_f\| \to 0$ and $\tau^n \to 0$ pp in $\Omega \times [t_0; t_f]$ then

$$(\Delta_\varrho \varphi(x,t,\vartheta_n + \theta\tau^n h_n) - \Delta_\varrho \varphi(x,t,\vartheta_f)) \to 0$$

This ends the proof.

Proposition 4.3. *Let Ω be a bounded Lipchitz open interval in $\mathbb{R}^3$; and let $\hat{\vartheta} \in X$ such that $\nabla\vartheta$ and $\nabla\hat{\vartheta} \in \mathrm{Y}$. Let g be small enough such that $\|g\|_X \leq 1$.*
Suppose that the operator ∇ at any point of $\Omega \times [t_0; t_f]$ satisfies the following inequality:

$$\|\nabla\vartheta(x,t) - \nabla\vartheta^*(x,t)\|_Y \leq k\, \|\vartheta(x,t) - \vartheta^*(x,t)\|_X \tag{26}$$

Then for $k, \omega > 0$, the operator $\mathcal{A}_\varepsilon$ satisfies :

$$\left\|\mathcal{A}_\varepsilon'(\vartheta)(x,t) - \mathcal{A}_\varepsilon'(\vartheta^*)(x,t)\right\|_W \leq 4\omega(k+1)e^{\delta t}\, \|\vartheta - \vartheta^*\|_X \tag{27}$$

Proof. $\left\|\mathcal{A}_\varepsilon'(\vartheta) - \mathcal{A}_\varepsilon'(\hat{\vartheta})\right\|_W =$

$$\left\|\sum_{i=1}^{3} \partial_\chi \varphi(\vartheta, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t} + \partial_\varrho \varphi(\vartheta, \nabla\vartheta) g - \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\hat{\vartheta})\frac{\partial^2 g}{\partial x \partial t} - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\hat{\vartheta}) g\right\|_W$$

$$\leq \left\|\partial_\varrho(\vartheta, \nabla\vartheta) g + \sum_{i=1}^{3} \partial_\chi \varphi(\vartheta, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t} - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta) g - \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t}\right\|$$

$$+ \left\|\partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta) g + \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\hat{\vartheta})\frac{\partial^2 g}{\partial x \partial t} - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\hat{\vartheta}) g\right\|$$

$$\leq \left\|\partial_\varrho \varphi(\vartheta, \nabla\vartheta) g - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta) g\right\| + \left\|\sum_{i=1}^{3} \partial_\chi \varphi(\vartheta, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t}\right\|$$

$$+ \left\|\partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta) g - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\hat{\vartheta}) g\right\| + \left\|\sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\vartheta)\frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_\chi \varphi(\hat{\vartheta}, \nabla\hat{\vartheta})\frac{\partial^2 g}{\partial x \partial t}\right\|$$

$$\leq \left[\int_{t_0}^{t_f}\int_\Omega \left|\partial_\varrho \varphi(\vartheta, \nabla\vartheta) - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta)\right|^2 dx dt\right]^{1/2} + \left[\int_{t_0}^{t_f}\int_\Omega |\partial_\varrho \varphi(\hat{\vartheta}, \nabla\vartheta) - \partial_\varrho \varphi(\hat{\vartheta}, \nabla\hat{\vartheta})|^2 dx dt\right]^{1/2}$$

$$\leq \left[\int_{t_0}^{t_f}\int_\Omega 16\omega^2 e^{2\delta t}|\vartheta(x,t) - \hat{\vartheta}(x,t)|^2 dx dt\right]^{1/2} + \left[\int_{t_0}^{t_f}\int_\Omega 16\omega^2 e^{2\delta t}|\nabla\vartheta(x,t) - \nabla\hat{\vartheta}(x,t)|^2 dx dt\right]^{1/2}$$

$$\leq 4\omega e^{\delta t}\left[\left\|\vartheta(x,t)-\hat{\vartheta}(x,t)\right\|_X+\left\|\nabla\vartheta(x,t)-\nabla\hat{\vartheta}(x,t)\right\|_X\right]$$

$$\left\|\mathcal{A}'_\varepsilon(\vartheta)-\mathcal{A}'_\varepsilon(\hat{\vartheta})\right\|_W \leq 4\omega e^{\delta t}(k+1)\left\|\vartheta(x,t)-\hat{\vartheta}(x,t)\right\|_X$$

Remark 4.1. *In the same way, we can also show that, the operator $\mathcal{T}_\varepsilon$ satisfies the inequality (5.2) on the other hand, $\mathcal{P}$ and $\mathcal{T}_\varepsilon$ are two continuous linear application and from the proposition 5.3, the operator $\mathcal{P}[\mathcal{T}_\varepsilon(\vartheta)]$ is also lipschitz. Indeed if we take $\mathcal{A}_\vartheta=\mathcal{A}_\varepsilon(\vartheta)+\mathcal{P}[\mathcal{T}_\varepsilon(\vartheta)]$, we simply show that:*

$$\left\|\mathcal{A}_\varepsilon(\vartheta)+\mathcal{P}[\mathcal{T}_\varepsilon(\vartheta)]-\mathcal{A}_\varepsilon(\hat{\vartheta})-\mathcal{P}[\mathcal{T}_\varepsilon(\hat{\vartheta})]\right\|_W \leq c\max(\omega,\bar{\omega})\left[4\omega e^{\delta t}(k+1)\left\|\vartheta-\hat{\vartheta}\right\|_X\right] \tag{28}$$

Theorem 4.1. *Assume that the initial terms (3), (4) and assumption **H-1**,**H-2** on φ and $\mathcal{U}$ are satisfied. Suppose that there exists a real $\gamma>-1$ such that for all b, with $0<b<\gamma$ or $b=\max_{j=1;2}|4\omega_j e^{\delta t}|$, then there exists a time $t_f\in[t_0;t_f]$and an unique solution $\vartheta=\mathscr{R}_\varepsilon(\vartheta_0,\rho_0,f_e)$ of the problem (17) for all $\vartheta_0\in(H_0^1(Q))^3$,$\rho_0\in L^2(Q)^3$, $f_e\in Y$*
More: $(H_0^1(Q))^3\times(L^2(Q))^3\times Y\to X$, $(\vartheta_0,\rho_0,f_e)\mapsto\mathscr{R}_\varepsilon(\vartheta_0,\rho_0,f_e)$ is ε-continuous and ε-differentiable. On the other hand the operator $\mathscr{R}_\varepsilon$ is strongly differentiable on $(H_0^1(Q))^3\times(L^2(Q))^3\times Y$ as an application on the space $(X;\sigma)$ and a σ weak topology in X.

Proof. Consider Q a sub space of X :

$$Q:=\left\{\vartheta\in X,\exists f_e\in Y,\exists\,\vartheta_0\in(H_0^1(\Omega))^3 \text{ and } \rho_0\in L^2(\Omega)^3\text{such that } L\vartheta:=(\vartheta_0,\rho_0,f_e)\right\}$$

Let $\mathcal{Z}\vartheta$ an operator defined from the condition (3),

$$\mathcal{Z}\vartheta: Q\to Y\times(H_0^1(\Omega))^3\times(L^2(\Omega))^3,\quad \vartheta\mapsto(L\vartheta,\vartheta_0,\rho_0)$$

Using the norm on Q, we how that $L\vartheta$ is linear, continuous and has inverse which is also continuous, more $\|\mathcal{Z}\vartheta\|^{-1}\leq\dfrac{1}{4\omega e^{\delta t}}$
Furthermore, if the inequality (27) and (28) are satisfied, $\mathcal{Z}\vartheta$ is continuous and reversible, more as $\mathcal{A}_\varepsilon$, is lipschitz then using Hadamard theorem, we can write that for all $\vartheta_0\in Q$. The operator

$$\mathscr{R}(\vartheta)\equiv\left(L\vartheta+d[\mathcal{A}'_\varepsilon(\vartheta_0)h,g]_{h=g}+\int_{t_0}^{t_f}\int_\Omega\mathcal{P}d[\mathcal{T}'_\varepsilon(\vartheta)g,h]_{h=g}dxdt,\vartheta_0,\rho_0\right)$$

defined Q in $Y\times(H_0^1(\Omega))^3\times(L^2(\Omega))^3$ has a continuous inverse function in the following form :

$$\Re(\vartheta)^{-1}\equiv\left(L\vartheta+[\mathcal{A}'_\varepsilon(\vartheta)h,g]_{h=g}+\int_{t_0}^{t_f}\int_\Omega\mathcal{P}[\mathcal{T}'_\varepsilon(\vartheta)g,h]_{h=g}dxdt,\vartheta_0,\rho_0\right)$$

from $Y \times (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$ in Q. $\mathscr{R}_\varepsilon(\vartheta)^{-1}$ has an inverse Lipchitz function , then there is an unique solution $\vartheta \equiv \mathscr{R}_\varepsilon(V)$. However according to the Proposition 5.3, $\mathscr{R}^{-1}$ is strongly ε-differentiable function, then for all $\vartheta_0 \in Q$ obtained by the strong theorems of differentiable function that $\mathscr{R}_\epsilon$ is ε-continuous and ε-differentiable function and (X, σ) is strongly differentiable on space $Y \times (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$.

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