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ON SOLUTIONS TO THE ARMS RACE MODEL USING SOME TECHNIQUES OF FRACTIONAL CALCULUS

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Abstract: In this paper, we investigate the fractional-order arms race model. The model has emerged as an important tool for the investigation of international conflict and arms races. The variational iteration method, the homotopy perturbation method, and the adomian decomposition method are used to solve the mathematical model with Caputo's fractional derivative. Several numerical computations have been provided to establish the validity and accuracy of the acquired results. It is shown that the fractional-order model can be solved easily using semi-analytical methods. The results obtained by all methods are compared.

Keywords and Phrases: Richard's Arms Race Model, Reimann-Liouville Fractional Integral, Caputo Fractional Derivative, Variational Iteration Method, Adomian Decomposition Method, Homotopy Perturbation Method.

2020 Mathematics Subject Classification: 26A33, 35A99, 91B74.

1. Introduction

Fractional calculus is the study of derivatives and integrals of arbitrary real or complex orders. It has attracted a lot of attention in recent decades and has evolved into a potent tool for better modelling of real-world phenomena, such as in mathematical biology, electric circuits, astronomy, and others [1, 3, 4, 5, 6, 12, 14]. Several systems with physical phenomena in diverse disciplines are mathematically modelled, resulting in many different differential equations. An efficient approach is required to analyze these mathematical models and provide solutions that are consistent with physical reality. Several powerful mathematical methods, including the adomian decomposition method, the homotopy perturbation method, the variational iterative method, the Laplace decomposition method, and the modified Laplace decomposition method, are employed to obtain both exact and approximate analytical solutions [1, 2, 6, 7, 9, 11, 13, 15]. In this study, we employ the variational iteration method [2], the homotopy perturbation method [9], and the adomian decomposition method [12] to solve the arms race model [8] of fractional order. The solutions obtained by all these methods are compared.

2. Preliminaries

In this section, we see the definitions of fractional operators and the arms race model of fractional order.

Definition 2.1. *The generalization of factorial functions known as the Gamma function [10] is defined as follows:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \;\; \Re(z) > 0. \tag{1}$$

Definition 2.2. *The Riemann-Liouville (R-L) fractional integral of order* $\alpha(\alpha > 0)$ *[10] is defined as follows:*

$${}_a^{RL}I_t^\alpha\, f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \; n-1 < \;\alpha\; \leq n. \tag{2}$$

Definition 2.3. *The Caputo fractional derivative of a function* $f(t)$ *of order* α *[10] is defined as follows:*

$${}_a^{C}\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}} d\tau, \; n-1 < \alpha < n. \tag{3}$$

Next, we will discuss the arms race model of fractional order.

Arms Race Model of Fractional Order

Let x(t) and y(t) be the armaments of nations X and Y at time t, respectively. The rate of change of the armaments on one side depends on the number of armaments on the opposing side, as if one nation increases its armaments, the other will follow suit. Considering the derivatives in the Caputo sense, the arms race model [8] of fractional order can be developed by the following system of differential equations:

$$\begin{cases} D^m x(t) & = ky(t) \\ D^n y(t) & = lx(t) \end{cases} \tag{4}$$

where k and l are proportionality constants, $0 < m \leq 1, 0 < n \leq 1$. The initial conditions are considered as $x(0) = x_0$ and $y(0) = y_0$. From the view point of the model, x_0 and y_0 are assumed to be positive.

3. Methodologies

3.1. Variational Iteration Method (VIM)

VIM iterations rapidly converge on the exact solution. This approach does not need linearization, differentiation, or the computation of Adomian polynomials etcetera. After selecting an initial guess, the solution of the system (4) can be obtained through iterations of VIM [13].

$$x_{n+1} = x_n + J^m[\lambda(D^m x(t) - ky(t))] \tag{5}$$

$$y_{n+1} = y_n + J^n[\lambda(D^n y(t) - lx(t))] \tag{6}$$

where J^m, J^n and x_0, y_0 are fractional integration and initial guesses, respectively. Furthermore, Lagrange's multiplier can be chosen as $\lambda = -1$ for simplicity. We finally get the approximate solution in iterative form by using initial approximations, x_0, y_0.

3.2. Homotopy Perturbation Method(HPM)

This method eliminates the need for transformation, linearization, and discretization [7]. Following the HPM, we construct the homotopy structure of system (4) as follows:

$$D^m x(t) = p[ky] \tag{7}$$

$$D^n y(t) = p[lx] \tag{8}$$

where $0 < m, n \leq 1$ and $p \in [0, 1]$ is the homotopy parameter.
Thus, the solution of the system can be written as a power series of p :

$$x(t) = \sum_{n=0}^{\infty} p^n x_n(t) = x_0(t) + px_1(t) + p^2 x_2(t) + p^3 x_3(t) \cdots \tag{9}$$

$$y(t) = \sum_{n=0}^{\infty} p^n y_n(t) = y_0(t) + py_1(t) + p^2 y_2(t) + p^3 y_3(t) \cdots \tag{10}$$

Substituting the values of Eq.(9) and Eq.(10) in Eq.(7) and Eq.(8) and equating the powers of p from both sides, we get

$$\begin{aligned} p^0 : D^m x_0(t) &= x_0, \\ D^n y_0(t) &= y_0 \end{aligned}$$

$$\begin{aligned} p^1 : D^m x_1(t) &= k y_0, \\ D^n y_1(t) &= l x_0 \end{aligned}$$

$$\begin{aligned} p^2 : D^m x_2(t) &= k y_1, \\ D^n y_2(t) &= l x_1 \end{aligned}$$

and so on.
Taking $p \to 1$, we obtain approximate solution of the system (4) as

$$x(t) = x_0 + x_1(t) + x_2(t) + \cdots .$$

$$y(t) = y_0 + y_1(t) + y_2(t) + \cdots .$$

3.3. Adomian Decomposition Method(ADM)

This method does not need transformation, linearization, or discretization [9]. One of its key characteristics is its quick convergence towards the solution. This approach considers the solutions for $x(t)$ and $y(t)$ of the system (4) as the following series:

$$x(t) = \sum_{n=0}^{\infty} x_n, \quad y(t) = \sum_{n=0}^{\infty} y_n$$

Let $L = D^m$ and L^{-1} be the inverse operator of L, then the system (4) in operator form is given by

$$\begin{aligned} L(x(t)) &= k y(t) \\ L(y(t)) &= l(x(t)) \end{aligned}$$

Applying the inverse operator L^{-1} on both the sides of the above equations, we get

$$\begin{aligned} x(t) &= x_0 + L^{-1}(k y(t)) \\ y(t) &= y_0 + L^{-1}(l x(t)) \end{aligned}$$

Furthermore, we get

$$
\begin{aligned}
x_1 &= L^{-1}(ky_0(t)), \\
y_1 &= L^{-1}(lx_0(t)) \\
x_2 &= L^{-1}(ky_1(t)), \\
y_2 &= L^{-1}(lx_2(t))
\end{aligned}
$$

and so on.

Solution for Arms Race model

The exact solution for the system (4) is given by

$$
\begin{aligned}
x(t) &= \sqrt{\frac{k}{l}}\left(Ae^{t\sqrt{kl}} - Be^{-t\sqrt{kl}}\right) \\
y(t) &= \sqrt{\frac{k}{l}}(Ae^{t\sqrt{kl}} - Be^{-t\sqrt{kl}})
\end{aligned}
$$

On applying the initial conditions $x(0) = x_0$ and $y(0) = y_0$, we get

$$A = \frac{1}{2}y_0 + \sqrt{\frac{l}{k}}x_0 \tag{11}$$

$$B = \frac{1}{2}y_0 - \sqrt{\frac{l}{k}}x_0 \tag{12}$$

We now use VIM, HPM, and ADM to solve the arms race model.
On applying the VIM with $\alpha = -1$ to solve system (4) for the initial conditions, we obtain the approximate solutions as

$$
\begin{aligned}
x_{n+1} &= x_n + J^m[(-D^m x(t) + ky(t))], \\
y_{n+1} &= y_n + J^n[(-D^n y(t) + lx(t))]
\end{aligned}
$$

Likewise, we have the following desired number of iterations for the solution

$$
\begin{aligned}
x_0(t) &= x_0, \\
y_0(t) &= y_0
\end{aligned}
$$

$$\begin{aligned}
x_1(t) &= J^m(ky_0) \\
&= ky_0 \frac{1}{\Gamma(m+1)} t^m, \\
y_1(t) &= J^n(lx_0) \\
&= lx_0 \frac{1}{\Gamma(n+1)} t^n \\
x_2(t) &= J^m(ky_1) \\
&= klx_0 \frac{\Gamma(mn+1)}{\Gamma(n+1)\Gamma(nm+m+1)} t^{nm+m}, \\
y_2(t) &= J^n(lx_1) \\
&= lky_0 \frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(nm+m+1)} t^{mn+n} \\
x_3(t) &= J^m(ky_2) \\
&= k^2 ly_0 \frac{\Gamma(mn+1)\Gamma(nm^2+nm+1}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(nm^2+nm+m+1)} t^{nm^2+nm+m+1}, \\
y_3(t) &= J^n(lx_2) \\
&= l^2 kx_0 \frac{\Gamma(mn+1)\Gamma(mn^2+nm+1)}{\Gamma(m+1)\Gamma(nm+n+1)\Gamma(nm^2+nm+n+1)} t^{mn^2+nm+n+1}
\end{aligned}$$

Thus, the solution obtained by VIM is

$$\begin{aligned}
x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + \cdots \\
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots
\end{aligned}$$

Next, we obtain the solution of the system (4) by employing HPM

$$\begin{aligned}
D^m[x_0 + px_1 + p^2x_2 + p^3x_3..] &= p[k(y_0 + py_1 + p^2y_2 + p^3y_3...)] \\
D^n[y_0 + py_1 + p^2y_2 + p^3y_3...] &= p[l(y_0 + py_1 + p^2y_2 + p^3y_3...)]
\end{aligned}$$

Equating the powers of p from both the sides, we obtain the following

$$\begin{aligned}
p^0 : D^m x_0(t) &= 0 \\
D^n y_0(t) &= 0
\end{aligned}$$

$$\begin{aligned}
p^1 : D^m x_1(t) &= ky_0 \\
D^n y_1(t) &= lx_1
\end{aligned}$$

$$
\begin{aligned}
p^2 : & D^m x_2(t) = k y_1 \\
& D^n y_2(t) = l x_1
\end{aligned}
$$

$$
\begin{aligned}
p^3 : & D^m x_3(t) = k y_2 \\
& D^n y_3(t) = l x_2
\end{aligned}
$$

and so on.

Applying the inverse operators J^m and J^n of the Caputo derivative D^m and D^n respectively we have,

$$
\begin{aligned}
x_0(t) &= x_0, \\
y_0(t) &= y_0 \\
x_1(t) &= J^m(k y_0) \\
&= k y_0 \frac{1}{\Gamma(m+1)} t^m, \\
y_1(t) &= J^n(l x_0) \\
&= l x_0 \frac{1}{\Gamma(n+1)} t^n \\
x_2(t) &= J^m(k y_1) \\
&= k l x_0 \frac{\Gamma(mn+1)}{\Gamma(n+1)\Gamma(nm+m+1)} t^{nm+m}, \\
y_2(t) &= J^n(l x_1) \\
&= l k y_0 \frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(nm+m+1)} t^{mn+n} \\
x_3(t) &= J^m(k y_2) \\
&= k^2 l y_0 \frac{\Gamma(mn+1)\Gamma(nm^2+nm+1}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(nm^2+nm+m+1)} t^{nm^2+nm+m+1}, \\
y_3(t) &= J^n(l x_2) \\
&= l^2 k x_0 \frac{\Gamma(mn+1)\Gamma(mn^2+nm+1}{\Gamma(m+1)\Gamma(nm+n+1)\Gamma(nm^2+nm+n+1)} t^{mn^2+nm+n+1}
\end{aligned}
$$

and so on.

The solution is obtained as

$$
\begin{aligned}
x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + x_4(t) \\
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t)
\end{aligned}
$$

Next, the solution for the system (4) by ADM, is given by

$$\begin{aligned} x(t) &= x_0 + L^{-1}(ky(t)), \\ y(t) &= y_0 + L^{-1}(lx(t)) \end{aligned}$$

Consider the series form of x and y as,

$$\begin{aligned} x &= x_0 + x_1 + x_2 + x_3 \cdots \\ y &= y_0 + y_1 + y_2 + y_3 \cdots \end{aligned}$$

Substituting the value of x and y in above equations, we get

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 ... &= x_0 + kL^{-1}(y_0 + y_1 + y_2 + y_3 \cdots), \\ y_0 + y_1 + y_2 + y_3 ... &= y_0 + lL^{-1}(x_0 + x_1 + x_2 + x_3 \cdots) \end{aligned}$$

On comparing the like terms, we have

$$\begin{aligned} x_1 &= L^{-1}(ky_0), \\ y_1 &= L^{-1}(lx_0) \\ x_2 &= L^{-1}(ky_1), \\ y_2 &= L^{-1}(lx_1) \\ x_3 &= L^{-1}(ky_2), \\ y_3 &= L^{-1}(lx_2) \end{aligned}$$

and so on.
Applying the inverse operators J^m and J^n of the Caputo derivatives D^m and D^n respectively, we have

$$\begin{aligned} x_0(t) &= x_0, \\ y_0(t) &= y_0 \\ x_1(t) &= J^m(ky_0) \\ &= ky_0 \frac{1}{\Gamma(m+1)} t^m, \\ y_1(t) &= J^n(lx_0) \\ &= lx_0 \frac{1}{\Gamma(n+1)} t^n \\ x_2(t) &= J^m(ky_1) \\ &= klx_0 \frac{\Gamma(mn+1)}{\Gamma(n+1)\Gamma(nm+m+1)} t^{nm+m}, \end{aligned}$$

$$
\begin{aligned}
y_2(t) &= J^n(lx_1)\\
&= lky_0\frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(nm+m+1)}t^{mn+n}\\
x_3(t) &= J^m(ky_2)\\
&= k^2ly_0\frac{\Gamma(mn+1)\Gamma(nm^2+nm+1}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(nm^2+nm+m+1)}t^{nm^2+nm+m+1},\\
y_3(t) &= J^n(lx_2)\\
&= l^2kx_0\frac{\Gamma(mn+1)\Gamma(mn^2+nm+1}{\Gamma(m+1)\Gamma(nm+n+1)\Gamma(nm^2+nm+n+1)}t^{mn^2+nm+n+1}
\end{aligned}
$$

Thus, the solution obtained is

$$
\begin{aligned}
x(t) &= x_0(t)+x_1(t)+x_2(t)+x_3(t)+x_4(t)\\
y(t) &= y_0(t)+y_1(t)+y_2(t)+y_3(t)+y_4(t)
\end{aligned}
$$

Hence, the solution by ADM and HPM is exactly same.
We compare the solution of system (4) by different methods with the exact solution in Table 1 for $k=l=0.9, x_0=20, y_0=0, A=10, B=-10, n=m=1$

Table 1: Solutions of Arms Race Model

t	$x(t)$	$y(t)$	$x(t)_{VIM}$	$y(t)_{VIM}$	$x(t)_{HPM}$	$y(t)_{HPM}$	$x(t)_{ADM}$	$y(t)_{ADM}$
0	20	0	20	0	20	0	20	0
1	28.66	20.43	28.1	20.43	28.127	20.43	28.127	20.43
2	62.1493	58.8435	52.4	74.88	52.836	55.44	52.836	55.44
3	148.672	147.328	92.9	250.833	129.11	119.61	129.11	119.61

One may take different values of the parameter to compare the differences between the solutions obtained by these methods.

4. More Realistic Model

$$
\begin{cases}
D^m x(t) &= ky(t)-\alpha x+g\\
D^n y(t) &= lx(t)-\beta y+h
\end{cases}
\tag{13}
$$

where $x(t)$ and $y(t)$ denote the armaments of nation X and Y respectively. k and l is the efficiency of increasing the armaments of X and Y respectively. g and h

are the ambitions of the grievances. Here, we solve the system (13) by above three discussed methods.

Variational Iterative Method

Consider the given system (13)

$$\begin{aligned} D^m x(t) &= ky(t) - \alpha x(t) + g \\ D^n y(t) &= lx(t) - \beta y(t) + h \end{aligned}$$

The iterations by VIM of system (13) are given by

$$\begin{aligned} x_{n+1} &= x_n + J^m[\alpha(D^m x_n(t) - ky_n(t) + \alpha x_n(t) - g)], \\ y_{n+1} &= y_n + J^n[\alpha(D^n y_n(t) - kx_n(t) + \beta y_n(t) - h)] \end{aligned}$$

Taking $\alpha = -1$, we get

$$\begin{aligned} x_{n+1} &= x_n + J^m[(-D^m x_n(t) + ky_n(t) - \alpha x_n(t) + g)], \\ y_{n+1} &= y_n + J^n[(-D^n y_n(t) + kx_n(t) - \beta y_n(t) + h))] \end{aligned}$$

$$\begin{aligned} x_1(t) &= J^m[(ky_0(t) - \alpha x_0(t) + g)] \\ &= (ky_0 - \alpha x_0(t) + g)\frac{1}{\Gamma(m+1)}t^m, \\ y_1(t) &= J^n[(lx_0 - \beta y_0(t) + h)] \\ &= (lx_0 - \beta y_0(t) + h)\frac{1}{\Gamma(n+1)}t^n \\ x_2(t) &= J^m[(ky_1 - \alpha x_1(t) + g)] \\ &= k(lx_0 - \beta y_0(t) + h)\frac{\Gamma(mn+1)}{\Gamma(n+1)\Gamma(nm+m+1)}t^{nm+m} \\ &- \alpha((ky_0(t) - \alpha x_0(t) + g)\frac{\Gamma(m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)}t^{m^2+m} + g\frac{1}{\Gamma(m+1)}t^m \\ y_2(t) &= J^n(lx_1 - \beta y_1(t) + h) \\ &= l(ky_0 - \alpha x_0(t) + g)\frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(nm+n+1)}t^{mn+n} \\ &- \beta(lx_0 - \beta y_0(t) + h)\frac{\Gamma(n^2+1)}{\Gamma(n+1)\Gamma(n^2+n+1)}t^{n^2+n} + h\frac{1}{\Gamma(n+1)}t^n \\ x_3(t) &= J^m[(ky_2 - \alpha x_2(t) + g)] \end{aligned}$$

$$
\begin{aligned}
&= kl(ky_0 - \alpha x_0(t) + g))\frac{\Gamma(mn+1)\Gamma(nm^2+nm+1}{\Gamma(nm+n+1)\Gamma(nm^2+nm+m+1)}t^{nm^2+nm+m+1} \\
&- k\beta(lx_0 - \beta y_0 + h)\frac{\Gamma(n^2+1)\Gamma(n^2m+1)}{\Gamma(n^2+n+1)\Gamma(n^2m+m+1)}t^{nm^2+m^2}+ \\
hk&\frac{\Gamma(nm+1)}{\Gamma(n+1)\Gamma(nm+m+1)}t^{nm+m} \\
&- \alpha k(lx_0 - \beta y_0 + h)\frac{\Gamma(mn+1)\Gamma(nm^2+m^2+1)}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(nm^2+m^2+m+1)}t^{nm^2+m^2+m} \\
&+ \alpha^2(ky_0 - \alpha x_0 + g)\frac{\Gamma(m^2+1)\Gamma(m^3+m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)\Gamma(m^3+m^2+m+1)}t^{m^3+m^2+m} \\
&- \alpha g\frac{\Gamma(m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)}t^{m^2+m}
\end{aligned}
$$

$$
\begin{aligned}
y_3(t) &= J^n[(lx_2 - \beta y_2 + h)]) \\
&= lk(x_0 - \beta y_0 + h) + \frac{\Gamma(mn+1)\Gamma(mn^2+nm+1}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(mn^2+nm+n+1)}t^{mn^2+nm+n} \\
&- \alpha l(ky_0 - \alpha x_0 + g)\frac{\Gamma(m^2+1)\Gamma(nm^2+mn+1}{\Gamma(m+1)\Gamma(m^2+m+1)\Gamma(m^2n+mn+n+1)}t^{m^2n+n^2+n} \\
&+ gl\frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(mn+n+1)}t^{mn+n} \\
&- \beta l(ky_0 - \alpha x_0 + g)\frac{\Gamma(mn+1)\Gamma(mn^2+n^2+1)}{\Gamma(mn+n+1)\Gamma(mn^2+n^2+n+1)}t^{mn^2+n^2+n} \\
&+ \beta^2(lx_0 - \beta y_0 + h)\frac{\Gamma(n^2+1)\Gamma(n^3+n^2+1)}{\Gamma(n^2+n+1)\Gamma(n^3+n^2+n+1)}t^{n^3+n^2+n} \\
&- \beta h\frac{\Gamma(n^2+1)}{\Gamma(n+1)\Gamma(n^2+n+1)}
\end{aligned}
$$

Thus, the solution obtained is

$$
\begin{aligned}
x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + ... \\
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + ...
\end{aligned}
$$

Homotopy Perturbation Method

Consider the homotopy structure of the above system (13) as follows:

$$D^m x(t) = p[ky(t) - \alpha x + g] \quad (14)$$

$$D^n y(t) = p[lx(t) - \beta y + h] \quad (15)$$

By writing the series form of x and y, we have

$$x = x_0 + px_1 + p^2x_2 + p^3x_3...$$

$$y = y_0 + py_1 + p^2y_2 + p^3y_3...$$

Substituting the value of x and y in Eq.(14) and Eq.(15), get

$$\begin{aligned} D^m[x_0 + px_1 + p^2x_2 + p^3x_3..] &= p[k(y_0 + py_1 + p^2y_2 + p^3y_3...) \\ &\quad - \alpha(x_0 + px_1 + p^2x_2 + p^3x_3..) + g] \\ D^n[y_0 + py_1 + p^2y_2 + p^3y_3...] &= p[l(x_0 + px_1 + p^2x_2 + p^3x_3..) \\ &\quad - \beta(y_0 + py_1 + p^2y_2 + p^3y_3...) + h] \end{aligned}$$

Equating the powers of p from both the sides, we obtain

$$\begin{aligned} p^0 :& D^m x_0(t) = 0 \\ & D^n y_0(t) = 0 \\ p^1 :& D^m x_1(t) = ky_0 - \alpha x_0 + g \\ & D^n y_1(t) = lx_0 - \beta y_0 + h \\ p^2 :& D^m x_2(t) = ky_1 - \alpha x_1 \\ & D^n y_2(t) = lx_1 - \beta y_1 \\ p^3 :& D^m x_3(t) = ky_2 - \alpha x_2 \\ & D^n y_3(t) = lx_2 - \beta y_2 \end{aligned}$$

and so on.
By applying the inverse operators J^m and J^n of the Caputo derivative D^m and D^n respectively, we get

$$\begin{aligned} x_1(t) &= J^m(ky_0 - \alpha x_0 + g) \\ &= (ky_0 - \alpha x_0 + g)\frac{1}{\Gamma(m+1)}t^m \\ y_1(t) &= J^n(lx_0 - \beta y_0 + h) \\ &= (lx_0 - \beta y_0 + h)\frac{1}{\Gamma(n+1)}t^n \\ x_2(t) &= J^m(ky_1 - \alpha x_1) \\ &= (lx_0 - \beta y_0 + h)\frac{\Gamma(nm+1)}{\Gamma(n+1)\Gamma(nm+m+1)}t^{nm+m} \\ &- (ky_0 - \alpha x_0 + g)\frac{\Gamma(m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)}t^{m^2+m} \end{aligned}$$

$$
\begin{aligned}
y_2(t) &= J^n(lx_1 - \beta y_1)\\
&= l(ky_0 - \alpha x_0 + g)\frac{\Gamma(nm+1)}{\Gamma(m+1)\Gamma(nm+n+1)}\\
&- \beta(lx_0 - \beta y_0 + h)\frac{\Gamma(n^2+1)}{\Gamma(n+1)(n^2+n+1)}t^{n^2+n}\\
x_3(t) &= J^m(ky_2 - \alpha x_2)\\
&= kl(ky_0 - \alpha x_0 + g)\frac{\Gamma(nm+1)\Gamma(m^2n+mn+1)}{\Gamma(m+1)\Gamma(nm+n+1)\Gamma(nm^2+nm+m+1)}t^{nm^2+nm+m}\\
&- k\beta((lx_0 - \beta y_0 + h)\frac{(\Gamma(n^2+n)\Gamma(mn^2+mn+1)}{\Gamma(n^2+n+1)\Gamma(n+1)\Gamma(mn^2+mn+m+1)}t^{m^2n+nm+m}\\
&- \alpha k(lx_0 - \beta y_0 + h)\frac{\Gamma(nm+1)(m^2n+m^2+1)}{\Gamma(n+1)\Gamma(nm+m+1)\Gamma(m^2n+m^2+m+1)}t^{nm^2+m^2+m}\\
&+ \alpha^2(ky_0 - \alpha x_0 + g)\frac{\Gamma(m^2+1)\Gamma(m^3+m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)\Gamma(m^3+m^2+m+1)}t^{m^3+m^2+m},\\
y_3(t) &= J^n(lx_2 - \beta y_2)\\
&= lk(lx_0 - \beta y_0 + h)\frac{\Gamma(nm+1)\Gamma(n^2m+mn+1)}{\Gamma(n+1)\Gamma(nm+n+1)\Gamma(mn^2+nm+m+1)}t^{mn^2+nm+n}\\
&- l\alpha(ky_0 - \alpha x_0 + g)\frac{(\Gamma(m^2+1)\Gamma(nm^2+mn+1)}{\Gamma(m^2+m+1)\Gamma(m+1)\Gamma(nm^2+mn+n+1)}t^{n^2m+nm+n}\\
&- \beta l(ky_0 - \alpha x_0 + g)\frac{\Gamma(nm+1)(n^2m+n^2+1)}{\Gamma(m+1)\Gamma(nm+n+1)\Gamma(n^2m+n^2+n+1)}t^{mn^2+n^2+n}\\
&+ \beta^2(lx_0 - \beta y_0 + h)\frac{\Gamma(n^2+1)\Gamma(n^3+n^2+1)}{\Gamma(n+1)\Gamma(n^2+n+1)\Gamma(n^3+n^2+n+1)}t^{n^3+n^2+n}
\end{aligned}
$$

Thus, the solution of the system is

$$x = x_0 + x_1 + x_2 + x_3 + ...$$

$$y = y_0 + y_1 + y_2 + y_3 + ...$$

Adomian Decomposition Method

Consider the given system

$$
\begin{aligned}
D^m x(t) &= ky(t) - \alpha x(t) + g\\
D^n y(t) &= lx(t) - \beta y(t) + h
\end{aligned}
$$

Let $L = D^m$ and L^{-1} be the inverse operator of L. Then the above system in operator form can be written as

$$L(x(t)) = ky(t) - \alpha x(t) + g,$$
$$L(y(t)) = lx(t) - \beta y(t) + h$$

On applying the inverse operator both the sides, we get

$$x(t) = x_0 + L^{-1}(ky(t)),$$
$$y(t) = y_0 + L^{-1}(lx(t))$$

Consider the series form of x and y as,

$$x = x_0 + x_1 + x_2 + x_3 \cdots$$

$$y = y_0 + y_1 + y_2 + y_3 \cdots$$

By substituting the value of x and y in above equations, we get

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 \cdots &= x_0 + L^{-1}k(y_0 + y_1 + y_2 + y_3 \cdots) \\ &\quad - \alpha(x_0 + x_1 + x_2 + x_3 + \cdots) + g, \\ y_0 + y_1 + y_2 + y_3 \cdots &= y_0 + L^{-1}l(x_0 + x_1 + x_2 + x_3 \cdots) \\ &\quad - \beta(y_0 + y_1 + y_2 + y_3 + \cdots) + h \end{aligned}$$

On comparing the like terms, we get

$$\begin{aligned} x_1 &= L^{-1}(ky_0 - \alpha x_0 + g), \\ y_1 &= L^{-1}(lx_0 - \beta + h) \\ x_2 &= L^{-1}(ky_1 - \alpha x_1), \\ y_2 &= L^{-1}(lx_1 - \beta y_1) \\ x_3 &= L^{-1}(ky_2 - \alpha x_2), \\ y_3 &= L^{-1}(lx_2 - \beta y_2) \end{aligned}$$

and so on. Applying the inverse operators J^m and J^n of the Caputo derivative D^m and D^n respectively, we have

$$\begin{aligned} x_0(t) &= x_0, \\ y_0(t) &= y_0 \\ x_1(t) &= J^m(ky_0 - \alpha x_0 + g) \\ &= (ky_0 - \alpha x_0 + g)\frac{1}{\Gamma(m+1)}, \\ y_1(t) &= J^n(lx_0 - \beta y_0 + h) \\ &= (lx_0 - \beta y_0 + h)\frac{1}{\Gamma(n+1)}t^n \end{aligned}$$

$$
\begin{aligned}
x_2(t) &= J^m(ky_1 - \alpha x_1) \\
&= k(lx_0 - \beta y_0 + h)\frac{\Gamma(mn+1)}{\Gamma(n+1)\Gamma(nm+m+1)}t^{nm+m} \\
&- \alpha(ky_0 - \beta x_0 + g)\frac{\Gamma(m^2+1)}{\Gamma(m+1)\Gamma(m^2+m+1)}t^{m^2+m}, \\
y_2(t) &= J^n(lx_1 - \beta y_1) \\
&= l(ky_0 - \alpha x_0 + g)\frac{\Gamma(mn+1)}{\Gamma(m+1)\Gamma(nm+m+1)}t^{mn+n} \\
&- \beta(lx_0 - \beta y_0 + h)\frac{n^2+1}{\Gamma(n+1)\Gamma(n^2+n+1)}t^{n^2+n}
\end{aligned}
$$

Thus, the solution obtained is

$$
\begin{aligned}
x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + \cdots \\
y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots
\end{aligned}
$$

One can calculate more terms in each method to get better approximation of the solution for both the models of fractional order.

We compare the solution of system (13) obtained by different methods with the exact solution in Table 2 for $k = 0.6, l = 0.9, x_0 = 100, y_0 = 80, \alpha = \beta = 0.2, n = m = 1, g = 0, h = 0$

Table 2: Solutions of Arms Race Model

t	$x(t)$	$y(t)$	$x(t)_{VIM}$	$y(t)_{VIM}$	$x(t)_{HPM}$	$y(t)_{HPM}$	$x(t)_{ADM}$	$y(t)_{ADM}$
0	100	80	100	80	100	80	100	80
1	145	152.717	143.6266	152.8534	144.906	152.8534	144.906	152.8534
2	229.373	259	223.04	234.027	225.648	259.627	225.648	259.627
3	372.304	427.432	345.2782	309.4418	345.262	427.432	345.262	427.432

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