J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 10, No. 2 (2023), pp. 37-44

DOI: 10.56827/JRSMMS.2023.1002.3 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

ON CERTAIN RESULTS ASSOCIATED WITH THE TRANSFORMATIONS OF GENERALIZED HYPERGEOMETRIC SERIES

Satya Prakash Singh and Akash Rawat

Department of Mathematics,
T. D. P. G. College, Jaunpur,
Jaunpur - 222002, Uttar Pradesh, INDIA

E-mail : snsp39@gmail.com, arofficial26@gmail.com

(**Received:** *Mar. 05, 2023* **Accepted:** *May 30, 2023* **Published:** *Jun. 30, 2023)*

Abstract: In this paper, certain transformation formulas for ordinary hypergeometric series and also for q-hypergeometric series have been established.

Keywords and Phrases: Ordinary hypergeometric series, basic (q-) hypergeometric series, transformation formula, summation formula, Bailey's transform, identity.

2020 Mathematics Subject Classification: 33C15, 33C20, 33D15.

1. Introduction, Notations and Definitions

The theory of generalized hypergeometric series is the most important topic in the entire special function theory. In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper in which he considered the infinite series

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)1.2}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)1.2.3}z^3+...,\tag{1.1}$$

as a function of a, b, c, z where it is assumed that $c \neq 0, -1, -2, ...$, so no zero factors appear in the denominators of the terms of the series (1.1). It is now customary to use ${}_2F_1(a, b; c; z)$ or ${}_2F_1\left[\begin{matrix} a, b; z \\ c \end{matrix}\right]$ for this series. Now, ${}_2F_1(a, b; c; z)$ stands thus,

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\tag{1.2}$$

where $(a)_n$ denotes the shifted factorial defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1)....(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, 3, ... \quad .$$

The series (1.2) converges for $|z| < 1$.
The generalization of Gauss series is the generalized hypergeometric series with r numerator parameters $a_1, a_2, ..., a_r$ and s denominator parameters $b_1, b_2, ..., b_s$ defined by

$${}_rF_s[a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; z] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r)_n}{(b_1, b_2, ..., b_s)_n} \frac{z^n}{n!}, \tag{1.3}$$

where $(a_1, a_2, ..., a_r)_n = (a_1)_n (a_2)_n ... (a_r)_n$.
The series in (1.3) converges for $|z| < \infty$ if $r \leq s$ and for $|z| < 1$ if $r = s+1$. If $r > s+1$, it converges nowhere except $z = 0$. Transformation theory plays a very important role in the development of this subject. Some of the identities, transformations and summations formulas have been established and any can refers Singh Satya Prakash and Yadav Vijay [2], Singh S. P., Singh S. N. and Yadav Vijay [3], Singh Satya Prakash, Yadav Vijay and Singh Priyanka [4], Singh S. N. and Singh Satya Prakash [5], Srivastava H. M., Singh S. N., Singh S. P. and Yadav Vijay [7]. In 1879 Thomae [8] established following transformations for ${}_3F_2$ series.

$${}_3F_2\left[\begin{array}{l} -n, a, b; 1 \\ c, d \end{array}\right] = \frac{(d-b)_n}{(d)_n} \; {}_3F_2\left[\begin{array}{l} -n, c-a, b; 1 \\ c, 1+b-d-n \end{array}\right], \tag{1.4}$$

$${}_3F_2\left[\begin{array}{l} a, b, c; 1 \\ d, e \end{array}\right] = \frac{\Gamma(d)\Gamma(c)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} \; {}_3F_2\left[\begin{array}{l} d-a, c-a, s; 1 \\ s+b, s+c \end{array}\right], \tag{1.5}$$

where $s = d + e - a - b - c$.

$$\begin{aligned} {}_3F_2\left[\begin{array}{l} a, b, c; 1 \\ d, e \end{array}\right] &= \frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(1-b)}{\Gamma(d-b)\Gamma(c-b)\Gamma(1+b-a)\Gamma(c)} \\ &\times \; {}_3F_2\left[\begin{array}{l} b, b-d+1, b-e+1; 1 \\ 1+b-c, 1+b-a \end{array}\right] + idem(b; c). \end{aligned} \tag{1.6}$$

"$idem(b; c)$" means that the preceding expression is repeated with b and c interchanged.

In this present paper, certain transformation formulas have been established by making use of following identity due to Verma [9].

$$\sum_{n=0}^{\infty} A_n B_n \frac{(x\omega)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(\gamma+n)_n} \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k}(\beta)_{n+k}}{k!(\gamma+2n+1)_k} B_{n+k} x^k \times \\ \times \sum_{r=0}^{n} \frac{(-n)_r(n+\gamma)_r}{r!(\alpha)_r(\beta)_r} A_r \omega^r. \tag{1.7}$$

We shall also make use of following results in our analysis.

$${}_4F_3\left[\begin{array}{l} \frac{1}{2}a, \frac{1}{2}+\frac{a}{2}, b+n, -n; 1 \\ \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}, 1+a \end{array}\right] = \frac{(b-a)_n}{(b)_n}. \tag{1.8}$$

[Slater 6; App. III (III.20)]

$${}_2F_1\left[\begin{array}{l} \alpha, \beta; 1 \\ c \end{array}\right] = \frac{\Gamma(c)\Gamma(c-\alpha-\beta)}{\Gamma(c-\alpha)\Gamma(c-\beta)}, \quad Re(c-\alpha-\beta) > 0. \tag{1.9}$$

[Slater 6; App. III (III.3)]

$${}_3F_2\left[\begin{array}{l} \frac{a}{3}, 1+a+n, -n; \frac{3}{4} \\ \frac{1}{2}+\frac{a}{2}, 1+\frac{a}{2} \end{array}\right] = \frac{(1)_n \left(1+\frac{a}{3}\right)_m}{(1+a)_n(1)_m}, \tag{1.10}$$

where m is the greatest integer $\leq \frac{n}{3}$ [Verma and Jain 10; (4.6) p. 1036]

$${}_3F_2\left[\begin{array}{l} \frac{a}{3}, 1+a+n, -n; \frac{3}{4} \\ \frac{a}{2}, \frac{1}{2}+\frac{a}{2} \end{array}\right] = \frac{(-1)^{n+m}(1)_n \left(1+\frac{a}{3}\right)_m}{(1+a)_n(1)_m}, \tag{1.11}$$

where m is the greatest integer $\leq \frac{n}{3}$ [Verma and Jain 10; (4.7) p. 1036]

$${}_4F_3\left[\begin{array}{l} \frac{a}{3}, 1+\frac{a}{2}, 1+a+n, -n; \frac{3}{4} \\ \frac{a}{2}, \frac{1}{2}+\frac{a}{2}, 2+\frac{a}{2} \end{array}\right] = \frac{(1)_n \left(\frac{a}{2}\right)_n \left(1+\frac{a}{3}\right)_m \left(2+\frac{a}{6}\right)_m}{(1+a)_n \left(2+\frac{a}{2}\right)_n (1)_m \left(\frac{a}{6}\right)_m}, \tag{1.12}$$

where m is the greatest integer $\leq \frac{n}{3}$ [Verma and Jain 10; (4.9) p. 1037]

2. Main Results

In this section main results have been established.

(a) Taking $B_n = 1$ and $x = 1$ in the identity (1.7) we get,

$$\sum_{n=0}^{\infty} A_n \frac{(\omega)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\gamma)(\alpha)_n(\beta)_n}{n!\Gamma(\gamma+2n)} \; {}_2F_1\left[\begin{matrix}\alpha+n, \beta+n; 1\\ \gamma+2n+1\end{matrix}\right] \times \\ \times \sum_{r=0}^{n} \frac{(-n)_r(\gamma+n)_r}{r!(\alpha)_r(\beta)_r} A_r \omega^r. \tag{2.1}$$

Summing the inner ${}_2F_1$ series by making use of (1.9) we get

$$\sum_{n=0}^{\infty} A_n \frac{(\omega)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma+2n)\Gamma(1+\gamma-\alpha-\beta)(\alpha)_n(\beta)_n\Gamma(\gamma)(\gamma)_n}{n!\Gamma(1+\gamma-\alpha+n)\Gamma(1+\gamma-\beta+n)} \times \\ \times \sum_{r=0}^{n} \frac{(-n)_r(\gamma+n)_r}{r!(\alpha)_r(\beta)_r} A_r \omega^r, \tag{2.2}$$

where $Re(1+\gamma-\alpha-\beta) > 0$.
Simplifying (2.2) we have

$$\sum_{n=0}^{\infty} A_n \frac{\omega^n}{n!} = \frac{\Gamma(1+\gamma-\alpha-\beta)\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)\Gamma(1+\gamma-\beta)} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_n(1+\gamma/2)_n(\alpha)_n(\beta)_n}{n!(1+\gamma-\alpha)_n(1+\gamma-\beta)_n\left(\frac{\gamma}{2}\right)_n} \times \\ \times \sum_{r=0}^{n} \frac{(-n)_r(\gamma+n)_r}{r!(\alpha)_r(\beta)_r} A_r \omega^r, \qquad Re(1+\gamma-\alpha-\beta) > 0. \tag{2.3}$$

Now, choosing $A_r = \dfrac{(\alpha)_r(\beta)_r\left(\frac{a}{2}\right)_r\left(\frac{1}{2}+\frac{a}{2}\right)_r}{(1+a)_r\left(\frac{\gamma}{2}\right)_r\left(\frac{1}{2}+\frac{\gamma}{2}\right)_r}$ and $\omega = 1$ in (2.3) and summing the inner series on the right hand side by making use of (1.8) we find,

$${}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{a}{2}, \frac{1}{2}+\frac{a}{2}; 1\\ \frac{\gamma}{2}, \frac{1}{2}+\frac{\gamma}{2}, 1+a\end{matrix}\right] = {}_2F_1\left[\begin{matrix}\alpha, \beta; 1\\ 1+\gamma\end{matrix}\right] \; {}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma-a, 1+\frac{\gamma}{2}; -1\\ 1+\gamma-\alpha, 1+\gamma-\beta, \frac{\gamma}{2}\end{matrix}\right], \tag{2.4}$$

Taking $A_r = \dfrac{(\alpha_r)(\beta)_r}{(c)_r}$ and $\omega = 1$ in (2.3) and summing the left hand side ${}_2F_1$ series by making use of (1.9) and inner series on the right hand side by using [Slater 6; Appendix III (III 4)] we get the summation formula,

$$
{}_5F_4\left[\begin{array}{c}\gamma, 1+\dfrac{\gamma}{2}, \alpha, \beta, 1+\gamma-c; 1\\ \dfrac{\gamma}{2}, 1+\gamma-\alpha, 1+\gamma-\beta, c\end{array}\right] = \frac{\Gamma(c)\Gamma(c-\alpha-\beta)\Gamma(1+\gamma-\alpha)\Gamma(1+\gamma-\beta)}{\Gamma(c-\alpha)\Gamma(c-\beta)\Gamma(1+\gamma)\Gamma(1+\gamma-\alpha-\beta)}, \tag{2.5}
$$

which is a known result [Slater 6; App III (III.12)].

Putting $\gamma = 1+a$, $\omega = \dfrac{3}{4}$, $A_r = \dfrac{\left(\frac{a}{3}\right)_r (\alpha)_r(\beta)_r}{\left(1+\frac{a}{2}\right)_r \left(\frac{1}{2}+\frac{a}{2}\right)_r}$ in (2.3) we get,

$$
\begin{aligned}
{}_3F_2\left[\begin{array}{c}\alpha, \beta, \dfrac{a}{3}; \dfrac{3}{4}\\ \dfrac{1}{2}+\dfrac{a}{2}, 1+\dfrac{a}{2}\end{array}\right] &= \frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)}\times\\
&\times \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(1+a)_n\left(\frac{3}{2}+\frac{a}{2}\right)_n(-1)^n}{n!(2+a-\alpha)_n(2+a-\beta)_n\left(\frac{1}{2}+\frac{a}{2}\right)_n}\\
&\times \sum_{r=0}^{\infty} \frac{(-n)_r(1+a+n)_r\left(\frac{a}{3}\right)_r}{r!\left(1+\frac{a}{2}\right)_r\left(\frac{1}{2}+\frac{a}{2}\right)_r}\left(\frac{3}{4}\right)^r.
\end{aligned} \tag{2.6}
$$

Now, summing the inner series on the right hand side of (2.6) by making use of (1.10) we get,

$$
\begin{aligned}
{}_3F_2\left[\begin{array}{c}\alpha, \beta, \dfrac{a}{3}; \dfrac{3}{4}\\ \dfrac{1}{2}+\dfrac{a}{2}, 1+\dfrac{a}{2}\end{array}\right] &= \frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)}\times\\
&\times \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n\left(\frac{3}{2}+\frac{a}{2}\right)_n\left(1+\frac{a}{3}\right)_m(-1)^n}{(2+a-\alpha)_n(2+a-\beta)_n\left(\frac{1}{2}+\frac{a}{2}\right)_n(1)_m},
\end{aligned} \tag{2.7}
$$

where m is the greatest integer $\leq \frac{n}{3}$.

Again, taking $\gamma = 1+a$, $\omega = \frac{3}{4}$, $A_r = \dfrac{\left(\frac{a}{3}\right)_r (\alpha)_r(\beta)_r}{\left(\frac{a}{2}\right)_r \left(\frac{1}{2}+\frac{a}{2}\right)_r}$ in (2.3) we get,

$$
{}_3F_2\left[\begin{array}{c}\alpha, \beta, \frac{a}{3}; \frac{3}{4} \\ \frac{1}{2}+\frac{a}{2}, \frac{a}{2}\end{array}\right] = \frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)} \times
$$
$$
\times \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(1+a)_n \left(\frac{3}{2}+\frac{a}{2}\right)_n (-1)^n}{n!(2+a-\alpha)_n(2+a-\beta)_n \left(\frac{1}{2}+\frac{a}{2}\right)_n} \times
$$
$$
\times \sum_{r=0}^{n} \frac{(-n)_r(1+a+n)_r \left(\frac{a}{3}\right)_r}{r! \left(\frac{a}{2}\right)_r \left(\frac{1}{2}+\frac{a}{2}\right)_r} \left(\frac{3}{4}\right)^r . \tag{2.8}
$$

Summing the inner series on the right hand side of (2.8) by making use of (1.11) we get,

$$
{}_3F_2\left[\begin{array}{c}\alpha, \beta, \frac{a}{3}; \frac{3}{4} \\ \frac{1}{2}+\frac{a}{2}, \frac{a}{2}\end{array}\right] = \frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)} \times
$$
$$
\times \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n \left(\frac{3}{2}+\frac{a}{2}\right)_n \left(1+\frac{a}{3}\right)_m (-1)^{2n-m}}{(2+a-\alpha)_n(2+a-\beta)_n \left(\frac{1}{2}+\frac{a}{2}\right)_n (1)_m}, \tag{2.9}
$$

where m is the greatest integer $\leq \frac{n}{3}$.

Lastly, taking $\gamma = 1+a$, $\omega = \frac{3}{4}$, $A_r = \dfrac{\left(\frac{a}{3}\right)_r \left(1+\frac{a}{2}\right)_r (\alpha)_r(\beta)_r}{\left(\frac{a}{2}\right)_r \left(\frac{1}{2}+\frac{a}{2}\right)_r \left(2+\frac{a}{2}\right)_r}$ in (2.3) we get,

$$
{}_4F_3\left[\begin{matrix}\alpha,\beta,\dfrac{a}{3},1+\dfrac{a}{2};\dfrac{3}{4}\\[2mm] \dfrac{1}{2}+\dfrac{a}{2},\dfrac{a}{2},2+\dfrac{a}{2}\end{matrix}\right]=\frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)}\times
$$

$$
\times\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(1+a)_n\left(\dfrac{3}{2}+\dfrac{a}{2}\right)_n}{n!(2+a-\alpha)_n(2+a-\beta)_n\left(\dfrac{1}{2}+\dfrac{a}{2}\right)_n}
$$

$$
\times\sum_{r=0}^{n}\frac{(-n)_r(1+a+n)_r\left(\dfrac{a}{3}\right)_r\left(1+\dfrac{a}{2}\right)_r}{r!\left(\dfrac{a}{2}\right)_r\left(\dfrac{1}{2}+\dfrac{a}{2}\right)_r\left(2+\dfrac{a}{2}\right)_r}\left(\frac{3}{4}\right)^r. \tag{2.10}
$$

Summing the inner series on the right hand side of (2.10) by making use of (1.12) we get,

$$
{}_4F_3\left[\begin{matrix}\alpha,\beta,\dfrac{a}{3},1+\dfrac{a}{2};\dfrac{3}{4}\\[2mm] \dfrac{1}{2}+\dfrac{a}{2},\dfrac{a}{2},2+\dfrac{a}{2}\end{matrix}\right]=\frac{\Gamma(2+a)\Gamma(2+a-\alpha-\beta)}{\Gamma(2+a-\alpha)\Gamma(2+a-\beta)}\times
$$

$$
\times\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\left(\dfrac{3}{2}+\dfrac{a}{2}\right)_n\left(\dfrac{a}{2}\right)_n\left(1+\dfrac{a}{3}\right)_m\left(2+\dfrac{a}{6}\right)_m(-1)^n}{(2+a-\alpha)_n(2+a-\beta)_n\left(\dfrac{1}{2}+\dfrac{a}{2}\right)_n\left(2+\dfrac{a}{2}\right)_n\left(\dfrac{a}{6}\right)_m(1)_m}, \tag{2.11}
$$

where m is the greatest integer $\leq \dfrac{n}{3}$.
Similar other results can also be scored.

Acknowledgement

The named author Akash Rawat would like to thanks Council of Scientific and Industrial Research (CSIR) for supporting a Junior Research fellowship No: 08/336(0001)/2019-EMR-I, under which this work has been done.

References

[1] Gasper, G. and Rahman, M., Basic Hypergeometric series, (Second Edition) Cambridge University Press, 2004.

[2] Singh Satya Prakash and Yadav Vijay, On Certain Transformation Formulas Involving Basic Hypergeometric Series, Proceedings of National Academy of Sciences, 90(2020), 623-627.

[3] Singh S. P., Singh S. N. and Yadav Vijay, On certain product formulas for basic hypergeometric functions, South East Asian J. of Math. & Math. Sci., Vol. 13, No. 2 (2017), 85-94.

[4] Singh Satya Prakash, Yadav Vijay and Singh Priyanka, On certain q-series identities, J. of Ramanujan Society of Math. and Math. Sc., Vol. 1, No. 2 (2012), 95-104.

[5] Singh S. N. and Singh Satya Prakash, On some more q-series identities, Italian Journal of Pure and Applied Mathematics, 35 (2015), 669-678.

[6] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, 1966.

[7] Srivastava H. M., Singh S. N., Singh S. P. and Yadav Vijay, Certain Derived WP-Bailey Pairs and Transformation Formulas for q-Hypergeometric Series, Filomat, 31, No. 14 (2017), 4619-4628.

[8] Thomae, J., Ueber die Funktionen, welchechurcs Reihon Von der Form dergestelit warden . . . , f. Reine angew, math. 87, (1879), 26-73.

[9] Verma, A., Some transformations of series with arbitrary terms, Institute Lombardo (Rend. Sc.) A 106, (1972), 342-353.

[10] Verma, A. and Jain, V. K., Some summation formulae of Basic Hypergeometric series, Indian J. Pure appl. Math., 11 (8); (1980), p. 1021-1038.